

RESULTS OF THE CHEBYSHEV TYPE INEQUALITY FOR PSEUDO-INTEGRAL

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ABSTRACT. In this paper, some results of the Chebyshev type integral inequality for the pseudo-integral are proven. The obtained results, are related to the measure of a level set of the maximum and the sum of two non-negative integrable functions. Finally, we applied our results to the case of comonotone functions.

1. INTRODUCTION

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, \infty]$ endowed with pseudo-addition \oplus and with pseudo-multiplication \odot (see [2, 3, 11, 13, 15]). Based on this structure there where developed the concepts of \oplus -measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. Pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [3, 14].

The well-known Chebyshev inequality is a part of the classical mathematical analysis (see [1]). The following inequality is the pseudo Chebyshev inequality:

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu,$$

where $f : [c, d] \rightarrow [a, b]$ is a non-negative integrable function, $g : [a, b] \rightarrow [0, \infty]$ is a continuous and increasing function, $A = [c, d]$ and

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$e \in [a, b]$. In this paper, results of the Chebyshev type integral inequality for pseudo-integral which relate the measure of a level set of the maximum and the sum of two non-negative integrable functions and their integrals are proven. Finally, we apply our results to the case of comonotone functions and s -decomposable fuzzy measures. The paper is organized as follows: Section 2 and 3 contain some of preliminaries, such as pseudo-operations, pseudo-analysis and pseudo-additive measures as well as integrals. In Section 4, we have proved some results of the Chebyshev type inequality for pseudo-integrals. Finally, a conclusion is given in Section 5.

2. PRELIMINARIES

In this section, we are going to review some well-known results of pseudo-operations, pseudo-analysis and pseudo-additive measures as well as integrals for details, we refer to [7, 10, 11, 12, 13].

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by \preceq .

Definition 2.1. The operation \oplus (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \rightarrow [a, b]$ which is commutative, nondecreasing (with respect to \preceq), associative and with a zero (natural) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b]$, $\mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation \odot (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \rightarrow [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b]$, $\mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ that \odot is a distributive pseudo-multiplication with respect to \oplus , i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring (see [7, 12]). In this paper we consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

- (a) Suppose that $x \oplus y = \sup(x, y)$, \odot is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation \sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.
- (b) Suppose that $x \oplus y = \inf(x, y)$, \odot is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the idempotent operation \inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \rightarrow [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$. If the zero element for the pseudo-addition is a , we will consider increasing generators. Then $g(a) = 0$ and $g(b) = 1$. If the zero element for the pseudo-addition is b , we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = 1$. If the generator g is increasing (respectively decreasing), then the operation \oplus induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have

- (a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$, also $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation \sup induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).
- (b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation \inf induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

3. TWO IMPORTANT CASES: GENERATED AND MAX-PLUS SEMIRINGS

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \rightarrow [0, \infty]$, Case II from Section 2. Then the pseudo-integral for a function $f : [c, d] \rightarrow [a, b]$ reduces on the g -integral [10, 11],

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

Throughout the paper, we denote $(0, \infty)$ by R^+ and $(-\infty, 0) \cup (0, \infty)$ by R^{\pm} . Now, we can easily obtain the properties listed in the following proposition.

Proposition 3.1. *Let $(X, F, \mu, \mathbb{R}_+^-, \oplus, \odot)$ be a pseudo-space and $f, g \in F$, then:*

- (1) *If $f = 0$ on A a.e., then $\int_A^{\oplus} f d\mu = 0$.*
- (2) *If $\mu(A) = 0$, then $\int_A^{\oplus} f d\mu = 0$.*
- (3) *$\int_A^{\oplus} a d\mu = a \odot \mu(A)$.*
- (4) *If $f \leq g$ on A , then $\int_A^{\oplus} f d\mu \leq \int_A^{\oplus} g d\mu$.*
- (5) *If $A \subset B$, then $\int_A^{\oplus} f d\mu \leq \int_B^{\oplus} f d\mu$.*

Second case is when the semiring is of the form $([a, b], \max, \odot)$, Case I(a) from Section 2. Then the pseudo-integral for a function $f : \mathbb{R} \rightarrow [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup (f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m . Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [9]. We shall denote by μ the usual Lebesgue measure on \mathbb{R} . We have

$$\begin{aligned} m(A) &= \text{ess sup}(x|x \in A) \\ &= \sup\{a|\mu(\{x|x \in A, x > a\}) > 0\}. \end{aligned}$$

We have by [9]:

Theorem 3.2. Let m be a sup-measure on $([0, \infty], \mathbb{B}([0, \infty]))$, where $\mathbb{B}([0, \infty])$ is the Borel σ -algebra on $[0, \infty]$, $m(A) = \text{ess sup}_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \rightarrow [0, \infty]$ is a continuous density. Then for any pseudo-addition \oplus with a generator g there exists a family $\{m_{\lambda}\}$ of \oplus_{λ} -measure on $([0, \infty], \mathbb{B})$, where \oplus_{λ} is generated by g^{λ} (the function g of the power λ), $\lambda \in (0, \infty)$, such that $\lim_{\lambda \rightarrow \infty} m_{\lambda} = m$.

For any continuous function $f : [0, \infty] \rightarrow [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g -integrals, [9].

Theorem 3.3. Let $([0, \infty], \sup, \odot)$ be a semiring, when \odot is generated with g , i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a, b]$. Let m be the same as in Theorem 3.1. Then there exists a family $\{m_{\lambda}\}$ of \oplus_{λ} -measures, where \oplus_{λ} is generated by g^{λ} , $\lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \rightarrow [0, \infty]$,

$$\begin{aligned} \int^{\sup} f \odot dm &= \lim_{\lambda \rightarrow \infty} \int^{\oplus_{\lambda}} f \odot dm_{\lambda} \\ &= \lim_{\lambda \rightarrow \infty} (g^{\lambda})^{-1} \left(\int g^{\lambda}(f(x)) dx \right). \end{aligned}$$

The author has proved in [2], the Chebyshev type inequality for pseudo-integral as follows:

Theorem 3.4. Let $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function, then for any non-negative integrable function $f : [c, d] \rightarrow [a, b]$ the inequality

$$(3.1) \quad \mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu$$

holds where $A = [c, d]$ and $e \in [a, b]$.

4. MAIN RESULTS

In this section, we prove a version of Markov type integral inequality for pseudo-integral and take some corollaries and results. Author proved the following theorem in [2], and here we have gotten some corollaries from this theorem.

Theorem 4.1. *If $g : [a, b] \rightarrow [0, \infty]$ is a continuous and increasing function, then for every non-negative integrable function $f : [c, d] \rightarrow [a, b]$, the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A^{\oplus} f d\mu$$

holds, where $e \in [a, b]$ and $A = [c, d]$.

We know that the fuzzy measure abandons the additivity. In what follows, we consider fuzzy measure satisfying the condition

$$(4.1) \quad \mu(A \cup B) \leq \mu(A) * \mu(B) \quad \forall A, B \in \Sigma,$$

where $* : [0, 1]^2 \rightarrow \mathbb{R}^+$ and $*$ is monotone (this means that if $x_1 \leq x_2$ and $y_1 \leq y_2$ then $x_1 * y_1 \leq x_2 * y_2$). Notice that in this case by the monotonicity of μ ,

$$\vee(\mu(A), \mu(B)) \leq \mu(A) * \mu(B),$$

Corollary 4.2. *Let $\mu : \Sigma \rightarrow [0, 1]$ be a fuzzy measure satisfying (4.1), $e \in [a, b]$ and $f, h : [c, d] \rightarrow [a, b]$ non-negative μ -measurable functions. Let $\frac{1}{e} \int_A^{\oplus} f d\mu \leq 1, \frac{1}{e} \int_A^{\oplus} h d\mu \leq 1$ and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then*

$$\mu(\{x \in A : \max(f, h)(x) \geq e\}) \leq \left(\frac{1}{e} \int_A^{\oplus} f d\mu\right) * \left(\frac{1}{e} \int_A^{\oplus} h d\mu\right),$$

for $A = [c, d]$, where $\max(f, h)(x) = \max\{f(x), h(x)\}$.

Proof. As

$$\{x \in A : \max(f, h)(x) \geq e\} \subset \{x \in A : f(x) \geq e\} \cup \{x \in A : h(x) \geq e\},$$

and μ satisfies (4.1), Theorem 4.1, gives us

$$\begin{aligned} \mu\{x \in A : \max(f, h)(x) \geq e\} &\leq \mu\left(\{x \in A : f(x) \geq e\} \cup \{x \in A : h(x) \geq e\}\right) \\ &\leq \mu(\{x \in A : f(x) \geq e\}) * \mu(\{x \in A : h(x) \geq e\}) \\ &\leq \left(\frac{1}{e} \int_A^{\oplus} f d\mu\right) * \left(\frac{1}{e} \int_A^{\oplus} h d\mu\right). \end{aligned}$$

This completes the proof. □

Corollary 4.3. *Let $\mu : \Sigma \rightarrow [0, 1]$ be a fuzzy measure satisfying 4.1, $e \in [a, b]$ and $f, h : [c, d] \rightarrow [a, b]$ non-negative μ -measurable functions. Let $\frac{1}{e} \int_A^\oplus f d\mu \leq 1, \frac{1}{e} \int_A^\oplus h d\mu \leq 1$ and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then*

$$\mu(\{x \in A : f(x) + h(x) \geq e\}) \leq \left(\frac{2}{e} \int_A^\oplus f d\mu\right) \star \left(\frac{2}{e} \int_A^\oplus h d\mu\right),$$

for $A = [c, d]$.

Proof. As $f + h \leq 2 \max(f, h)$, by Corollary 4.2, we have

$$\begin{aligned} \mu(\{x \in A : f(x) + h(x) \geq e\}) &\leq \mu(\{x \in A : 2 \max(f, h)(x) \geq e\}) \\ &= \mu(\{x \in A : \max(f, h)(x) \geq \frac{e}{2}\}) \\ &\leq \left(\frac{2}{e} \int_A^\oplus f d\mu\right) \star \left(\frac{2}{e} \int_A^\oplus h d\mu\right). \end{aligned}$$

This completes the proof. \square

Particularly, if we take $\star : [0, 1]^2 \rightarrow \mathbb{R}^+$ as the usual addition, we obtain the following result.

Corollary 4.4. *Let $\mu : \Sigma \rightarrow [0, 1]$ be a fuzzy measure satisfying (4.1) with $\star = +$, $e \in [a, b]$ and $f, h : [c, d] \rightarrow [a, b]$ non-negative μ -measurable functions. Let $\frac{1}{e} \int_A^\oplus f d\mu \leq 1, \frac{1}{e} \int_A^\oplus h d\mu \leq 1$ and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then*

$$\begin{aligned} \mu(\{x \in A : \max(f, h)(x) \geq e\}) &\leq \frac{1}{e} \left[\int_A^\oplus f d\mu + \int_A^\oplus h d\mu \right], \\ \mu(\{x \in A : f(x) + h(x) \geq e\}) &\leq \frac{2}{e} \left[\int_A^\oplus f d\mu + \int_A^\oplus h d\mu \right], \end{aligned}$$

for $A = [c, d]$.

The following example proves that Corollary 4.4, is not true without assumption (4.1), with $\star = +$.

Example 4.5. Let X be a set with $\text{Card}\{X\} > 1$ and μ the fuzzy measure defined by

$$\mu : \mathcal{P}(X) \rightarrow [0, 1],$$

$$\mu(A) = \begin{cases} \mathbf{0}, & \text{if } A \neq X, \\ \mathbf{1}, & \text{if } A = X. \end{cases}$$

Let $A = [1, 2]$, A be a proper subset of X ($\emptyset \neq A \subsetneq X$) and consider $f, g : X \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} \mathbf{1}, & \text{if } x \in A, \\ \mathbf{0}, & \text{if } x \notin A, \end{cases}$$

$$g(x) = \begin{cases} \mathbf{0}, & \text{if } x \in X - A, \\ \mathbf{1}, & \text{if } x \notin X - A. \end{cases}$$

So $\mu\{x \in X : \max(f, g)(x) \geq 1\} = \mu(X) = 1$. As $\mu(A) = 0$, then by Proposition 3.1, $\int_A^\oplus f d\mu = 0$.

A similar argument proves that $\int_A^\oplus g d\mu = 0$. This proves that the first inequality of Corollary 4.4, is not true for this example. The same example gives us that the second inequality of Corollary 4.4, is false without assumption 4.1.

Definition 4.6. Let $f, g : X \rightarrow [0, \infty)$ be two functions. We will say that f and g are comonotone if for any $x, y \in X$ they satisfy the inequality

$$\frac{f(x) - f(y)}{g(x) - g(y)} \geq 0.$$

A very important and useful property for comonotone functions f and g is that for any $p, q \geq 0$, either $f \geq p \subset g \geq q$ or $g \geq q \subset f \geq p$ (see [6]).

Notice that for comonotone functions f and g and $e \in [a, b]$ the following condition is satisfied:

$$(4.2) \quad \mu(\{f \geq e\} \cup \{g \geq e\}) = \max\{\mu(\{f \geq e\}), \mu(\{g \geq e\})\},$$

and consequently, using Corollary 4.2, and 4.3, for the particular case $a \star b = \vee(a, b)$, we can obtain the following results.

Theorem 4.7. Let $\mu : \Sigma \rightarrow [0, 1]$ be a fuzzy measure satisfying 4.1 with $\star = \vee$, $e \in [a, b]$ and $f, h : [c, d] \rightarrow [a, b]$ non-negative μ -measurable and comonotone functions and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then

$$\mu(\{x \in A : \max(f, h)(x) \geq e\}) \leq \frac{1}{e} \max \left\{ \int_A^\oplus f d\mu, \int_A^\oplus h d\mu \right\},$$

$$\mu(\{x \in A : f(x) + h(x) \geq e\}) \leq \frac{2}{e} \max \left\{ \int_A^\oplus f d\mu, \int_A^\oplus h d\mu \right\},$$

for $A = [c, d]$.

Proof. As $\{x \in A : \max(f, h)(x) \geq e\} \subseteq \{f \geq e\} \cup \{h \geq e\}$ and f, h are comonotone functions, by relation 4.2, we have

$$\begin{aligned} \mu(\{x \in A : \max(f, h)(x) \geq e\}) &\leq \mu(\{f \geq e\} \cup \{h \geq e\}) \\ &= \max\{\mu(\{f \geq e\}), \mu(\{h \geq e\})\} \\ &\leq \max\left\{\frac{1}{e} \int_A^\oplus f d\mu, \frac{1}{e} \int_A^\oplus h d\mu\right\} \\ &= \frac{1}{e} \max\left\{\int_A^\oplus f d\mu, \int_A^\oplus h d\mu\right\}. \end{aligned}$$

The second inequality is taken from the first inequality and the proof is straightforward. \square

Finally Theorem 4.7, can be easily generalized.

Corollary 4.8. *Let $\mu : \Sigma \rightarrow [0, 1]$ be a fuzzy measure satisfying 4.1 with $\star = \vee, e \in [a, b]$ and $f_i : [c, d] \rightarrow [a, b]$, ($i = 1, \dots, n$) non-negative μ -measurable in which any two of them are comonotone and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then*

$$\begin{aligned} \mu\left(\left\{x \in A : \max_{1 \leq i \leq n} (f_i)(x) \geq e\right\}\right) &\leq \frac{1}{e} \max_{1 \leq i \leq n} \left\{\int_A^\oplus f_i d\mu\right\}, \\ \mu\left(\left\{x \in A : \sum_{i=1}^n (f_i)(x) \geq e\right\}\right) &\leq \frac{n}{e} \max_{1 \leq i \leq n} \left\{\int_A^\oplus f_i d\mu\right\}, \end{aligned}$$

for $A = [c, d]$.

Proof. Note that $\{x \in A : \max_{1 \leq i \leq n} (f_i)(x) \geq e\} \subseteq \bigcup_{i=1}^n \{f_i \geq e\}$ and, as any two functions f_i, f_j ($i, j = 1, \dots, n$) are comonotone,

$$\begin{aligned} \mu\left(\left\{x \in A : \max_{1 \leq i \leq n} (f_i)(x) \geq e\right\}\right) &\leq \mu\left(\bigcup_{i=1}^n \{f_i \geq e\}\right) \\ &\leq \max_{1 \leq i \leq n} \mu(\{f_i \geq e\}), \end{aligned}$$

and now the rest of the proof follows the lines of Theorem 4.7. On the other hand, as

$$\begin{aligned} \left\{x \in A : \sum_{i=1}^n (f_i)(x) \geq e\right\} &\subseteq \left\{x \in A : n \cdot \max_{1 \leq i \leq n} (f_i)(x) \geq e\right\} \\ &= \left\{x \in A : \max_{1 \leq i \leq n} (f_i)(x) \geq \frac{e}{n}\right\}, \end{aligned}$$

and taking into account the first part of this Corollary, we obtain

$$\mu \left(\left\{ x \in A : \sum_{i=1}^n (f_i)(x) \geq e \right\} \right) \leq \frac{n}{e} \max_{1 \leq i \leq n} \left\{ \int_A^{\oplus} f_i d\mu \right\}.$$

This completes the proof. \square

Remark 4.9. In the proofs of Corollaries 4.2 and 4.3, the condition 4.1 is essential. the condition 4.1 is satisfied in a more general context. Let $S : [0, 1]^2 \rightarrow [0, 1]$ be monotone and $m : \Sigma \rightarrow [0, 1]$ be a fuzzy measure. We say that m is a S -decomposable fuzzy measure if $m(\emptyset) = 0$ and $m(A \cup B) = m(A)Sm(B)$ for $A, B \in \Sigma$ with $A \cap B = \emptyset$.

In what follows we prove that the condition 4.1 is valid for S -decomposable fuzzy measure μ . In fact,

$$\begin{aligned} \mu(A \cup B) &= \mu(A \cup (B - A)) \\ &= \mu(A)S\mu(B - A), \end{aligned}$$

and the monotonicity of s gives us

$$\begin{aligned} \mu(A \cup B) &= \mu(A)S\mu(B - A) \\ &\leq \mu(A)S\mu(B). \end{aligned}$$

Thus our Corollaries 4.2 and 4.3 are true for S -decomposable fuzzy measure. Particularly, when S is a t -conorm (see, for example [8]). Then

$$\mu(\{x \in A : f(x) + h(x) \geq e\}) \leq \left(\frac{2}{e} \int_A^{\oplus} f d\mu \right) \star \left(\frac{2}{e} \int_A^{\oplus} h d\mu \right),$$

our results are also true.

Corollary 4.10. *Let $\mu : \Sigma \rightarrow [0, 1]$ be a S -decomposable fuzzy measure, where $S : [0, 1]^2 \rightarrow [0, 1]$ is monotone, $e \in [a, b]$ and $f, h : [c, d] \rightarrow [a, b]$ non-negative μ -measurable functions. Let $\frac{1}{e} \int_A^{\oplus} f d\mu \leq 1, \frac{1}{e} \int_A^{\oplus} h d\mu \leq 1$ and $g : [a, b] \rightarrow [0, \infty]$ be a continuous and increasing function. Then for $A = [c, d]$, we have*

$$\mu(\{x \in A : \max(f, h)(x) \geq e\}) \leq \left(\frac{1}{e} \int_A^{\oplus} f d\mu \right) s \left(\frac{1}{e} \int_A^{\oplus} h d\mu \right),$$

$$\mu(\{x \in A : f(x) + h(x) \geq e\}) \leq \left(\frac{2}{e} \int_A^{\oplus} f d\mu \right) s \left(\frac{2}{e} \int_A^{\oplus} h d\mu \right).$$

5. CONCLUSION

We have proved some results of the Chebyshev type inequality for the pseudo-integral in three characteristic cases: One of them concerning the pseudo-integrals based on a fuzzy measure $\mu : \Sigma \rightarrow [0, 1]$ which satisfies the condition

$$\mu(A \cup B) \leq \mu(A) \star \mu(B) \quad \text{for } A, B \in \Sigma,$$

where $\star : [0, 1]^2 \rightarrow \mathbb{R}^+$ and \star is monotone. Another one concerns the pseudo-integrals based on a non-negative integrable and comonotone functions. Another one concerns the pseudo-integrals based on a s -decomposable fuzzy measure, where $s : [0, 1]^2 \rightarrow [0, 1]$ is monotone.

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