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On an atomic decomposition in Banach spaces

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ABSTRACT. An atomic decomposition is considered in Banach space. A method for constructing an atomic decomposition of Banach space, starting with atomic decomposition of subspaces is presented. Some relations between them are established. The proposed method is used in the study of the frame properties of systems of eigenfunctions and associated functions of discontinuous differential operators.

1. Introduction

One of the commonly used methods for solving differential equations is the method of separation of variables (Fourier method). This method yields the appropriate spectral problem (usually with respect to the space variables). Justification of this method requires the study frame properties (completeness, minimality, basicity, an atomic decomposition and etc.) of the systems of eigenfunctions of a spectral problem in various spaces of functions. That is why many mathematicians have been paying so much attention lately to the study of frame properties (such as completeness, minimality, basicity, atomic decomposition) of the systems of special functions, mostly eigenfunctions and associated functions of differential operators. Various methods have been developed for establishing these properties. For more information we refer the reader to [2–4, 16–19, 27, 28]. In case of discontinuous differential operator, there arise the systems of eigenfunctions that cannot be treated for frame properties by the earlier methods. To shed some light on this situation, we consider the following model spectral problem for second order discontinuous differential operator

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$$-y''(x) = \lambda y(x), \qquad x \in (-1,0) \cup (0,1), \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} & y(-1) = y(1) = 0, \\ & y(-0) = y(+0), \\ & y'(-0) - y'(+0) = \lambda m y(0). \end{aligned} \tag{1.2}$$

This spectral problem has two sets of eigenfunctions [16]:

$$u_n^{(1)}(x) = \sin \pi n x, \quad x \in [-1,1], n \in \mathbb{N},$$

and

$$\tilde{u}_n^{(2)}(x) = \begin{cases} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1,0], \\ -\sin \pi n x + O\left(\frac{1}{n}\right), & x \in [0,1], \ n \in \mathbb{N}. \end{cases}$$

Such spectral problems arise when solving the problem of a loaded string fixed at both ends with a load placed in the middle of the string by the Fourier method [1, 32]. The use of this method requires the study of basis properties of the double system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in\mathbb{N}}$ in corresponding spaces of functions (usually in the Lebesgue or Sobolev spaces). Of course, it should be started with the basis properties of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in\mathbb{N}}$, which is the principal part of the asymptotics of the system $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in\mathbb{N}}$, where

$$u_n^{(2)} = \begin{cases} \sin \pi n x, & x \in [-1,0), \\ -\sin \pi n x, & x \in [0,1]. \end{cases}$$

This is usually followed by the application of various perturbation methods. This approach is well studied (see, e.g., the articles [3–6, 14–16, 18, 19, 24, 34] and the monographs [12, 13, 20, 23, 29, 30]). On the other hand, it is not difficult to see that the principal part $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in\mathbb{N}}$ is not a standard (in other words, classical) system. It turns out that the form of the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in\mathbb{N}}$ is not special, i.e. it can be derived from the general case. The general approach to these systems allows to introduce a new approach for constructing bases with a lot of applications in the spectral theory of differential operators. It should be noted that some problems of an atomic decomposition and frames with respect to the specific systems have been previously studied in [8–11, 22, 26, 31, 34].

In this work we consider an abstract approach to the above problem. We consider a direct expansion of a Banach space with respect to subspaces. We offer a method for constructing an atomic decomposition for a space starting with atomic decomposition for subspaces.

2. Notation and necessary information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers; $L[M]$ will denote the linear span of the set M and $\overline{M}$ will stand for the closure of M; X^* will denote a space conjugate to X; $L(X_1; X_2)$ will be a space of linear bounded operators from X_1 to X_2 with $L(X;X) = L(X)$; D_T will denote a domain of the operator T and R_T will be the range of T; $KerT$ will stand for the kernel of the operator T; $< x, f >= f(x)$ will denote the value of the functional f at the point x; Banach space will be referred to as B-space; $\|\cdot\|_X$ will denote a norm in X; $\Leftrightarrow$ will mean "if and only if"; $1 : n \equiv \{1; \ldots ; n\}$; δ_{ij} will be the Kronecker symbol.

We will also use the concept of the space of coefficients. We define it as follows. Let $\vec{x} \equiv \{x_n\}_{n\in\mathbb{N}} \subset X$ be a non-degenerate system in a B-space X, i.e. $x_n \neq 0$, $\forall n \in \mathbb{N}$. Define

$$\mathscr{K}_{\vec{x}} \equiv \left\{ \{\lambda_n\}_{n\in\mathbb{N}} : \text{the series} \sum_{n=1}^{\infty} \lambda_n x_n \text{ is convergent in} X \right\}.$$

Introduce the norm in $\mathscr{K}_{\vec{x}}$:

$$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|,$$

where $\vec{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$. With respect to the usual operations of addition and multiplication by a complex number, $\mathscr{K}_{\vec{x}}$ is a B- space. Take $\forall\vec{\lambda} \in \mathscr{K}_{\vec{x}}$ and consider the operator $T : \mathscr{K}_{\vec{x}} \to X$:

$$T\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n, \qquad \vec{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}.$$

Denote by $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{K}_{\vec{x}}$ a canonical system in $\mathscr{K}_{\vec{x}}$, where $e_n = \{\delta_{nk}\}_{k\in\mathbb{N}}$. It is absolutely clear that $Te_n = x_n, \forall n \in \mathbb{N}$. The following statement is true.

Statement 2.1. *Space of coefficients $\mathscr{K}_{\vec{x}}$ is a B-space with the canonical basis $\{e_n\}_{n\in\mathbb{N}}$. Moreover, the system $\vec{x}$ forms a basis for $X \Leftrightarrow T$ performs an isomorphism between $\mathscr{K}_{\vec{x}}$ and X.*

Let's recall some concepts and facts from the frame theory. First, let us give a definition of atomic decomposition in Banach spaces.

Definition 2.2. Let X be a B-space and $\mathscr{K}$ be a B- space of the sequences of scalars. Let $\{f_k\}_{k\in\mathbb{N}} \subset X, \{g_k\}_{k\in\mathbb{N}} \subset X^*$. Then $\left(\{g_k\}_{k\in N};\right.$ $\left.\{f_k\}_{k\in\mathbb{N}}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$, if:

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}, \quad \forall f \in X;$

(ii) $\exists A, B > 0 {:} A\|f\|_X \leq \left\|\{g_k(f)\}_{k\in N}\right\|_{\mathscr{K}} \leq B\|f\|_X, \quad \forall f \in X;$

(iii) $f = \sum_{k=1}^{\infty} g_k(f) f_k, \quad \forall f \in X.$

The concept of the frame is a generalization of the concept of an atomic decomposition.

Definition 2.3. Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars. Let $\{g_k\}_{k\in N} \subset X^*$ and $S \in L(\mathscr{K};X)$. Then $\left(\{g_k\}_{k\in N}; S\right)$ forms a Banach frame for X with respect to $\mathscr{K}$, if:

(i) $\{g_k(f)\}_{k\in N} \in \mathscr{K}, \quad \forall f \in X;$

(ii) $\exists A, B > 0 {:} A\|f\|_X \leq \left\|\{g_k(f)\}_{k\in N}\right\|_{\mathscr{K}} \leq B\|f\|_X, \quad \forall f \in X;$

(iii) $S\left[\{g_k(f)\}_{k\in N}\right] = f, \quad \forall f \in X.$

The following proposition is true.

Proposition 2.4. *Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars with canonical basis $\{\delta_n\}_{n\in\mathbb{N}}$. Let $\{g_k\}_{k\in N} \subset X^*$ and $S \in L(\mathscr{K};X)$. Then the following statements are equivalent:*

(i) *$\left(\{g_k\}_{k\in N}; S\right)$ is a Banach frame for X with respect to $\mathscr{K}$;*

(ii) *$\left(\{g_k\}_{k\in N}; \{S(\delta_k)\}_{k\in N}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$.*

More information about the above facts can be found in [7, 12–14, 20, 29, 30, 33].

In the sequel, we will use the following construction and some obvious facts. Let the following direct sum hold

$$X = X_1 \oplus \cdots \oplus X_m,$$

where X_i, $i = \overline{1,m}$, are some B-spaces. For convenience, we will represent the elements of the space X in the form of a vector

$$x \in X \quad \Leftrightarrow \quad x = (x_1, x_2, \ldots, x_m),$$

where $x_k \in X_k, k = \overline{1,m}$. The norm in X will be defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^{m} \|x_i\|_{X_i}^2}.$$

Then we have $X^* = X_1^* \oplus \cdots \oplus X_m^*$ (see [23]), and for $\vartheta \in X^*$ and $x \in X$ it holds

$$< x, \vartheta >= \sum_{i=1}^{m} < x_i, \vartheta_i >,$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_m)$ and

$$\|\vartheta\|_{X^*} = \sqrt{\sum_{i=1}^{m} \|\vartheta_i\|^2_{X_i^*}}.$$

Let some system $\left\{u_n^{(i)}\right\}_{n\in\mathbb{N}} \subset X_i$ be given for every $i \in 1:m$. Consider the following system in the space X :

$$u_{in}^0 = \left(\underbrace{0, \ldots, 0, u_n^{(i)}}_{i}, 0, \ldots, 0\right), \quad i = \overline{1, m};\ n \in \mathbb{N}.$$

Let the pair $\left(\left\{u_{in}^0\right\}_{i=\overline{1,m};n\in\mathbb{N}}; \{\vartheta_{in}\}_{i=\overline{1,m};n\in\mathbb{N}}\right)$ be an atomic decomposition of X with respect to the space of coefficients $\mathscr{K}$, i.e. $\forall x \in X$ has a decomposition of the form

$$x = \sum_{i=1}^{m} \sum_{n=1}^{\infty} \vartheta_{in}(x)\, u_{in}^0, \tag{2.1}$$

moreover, the following inequality holds

$$A \left\|\{\vartheta_{in}(x)\}\right\|_{\mathscr{K}} \le \|x\|_X \le B \left\|\{\vartheta_{in}(x)\}\right\|_{\mathscr{K}}. \tag{2.2}$$

Suppose

$$\vartheta_{in} = \left(\vartheta_{in}^{(1)}; \ldots; \vartheta_{in}^{(m)}\right) \in X^*,$$

where $\vartheta_{in}^{(k)} \in X_k^*, \forall k \in 1:m$. We have $(x = (x_1, \ldots, x_m))$:

$$\vartheta_{in}(x) = \sum_{k=1}^{m} \vartheta_{in}^{(k)}(x_k), \quad i = \overline{1, m};\ n \in \mathbb{N}.$$

Take $\forall k \in 1:m$, and let

$$x_k^0 = \left(\underbrace{0; \ldots; 0; x_k}_{k}; 0; \ldots; 0\right).$$

We have

$$\vartheta_{in}\left(x_k^0\right) = \vartheta_{in}^{(k)}(x_k).$$

Then from (2.2) we obtain

$$A \left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}} \le \|x_k\|_{X_k} \le B \left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}}.$$

Paying attention to the decomposition (2.1), we obtain

$$\begin{aligned} x_k^0 &= (0,\ldots,0,x_k,0,\ldots,0) \\ &= \sum_{i=1}^{m}\left(\underbrace{0,\ldots,0,\sum_{n=1}^{\infty}\vartheta_{in}^{(k)}\left(x_k\right)u_n^{(i)}}_{i},0,\ldots,0\right) \\ &= \left(\sum_{n=1}^{\infty}\vartheta_{1n}^{(k)}\left(x_k\right)u_n^{(1)},\ldots,\sum_{n=1}^{\infty}\vartheta_{kn}^{(k)}\left(x_k\right)u_n^{(k)},\ldots,\sum_{n=1}^{\infty}\vartheta_{mn}^{(m)}\left(x_k\right)u_n^{(m)}\right) \end{aligned}$$

then

$$\sum_{n=1}^{\infty}\vartheta_{in}^{(k)}\left(x_k\right)u_n^{(i)}=\begin{cases} x_k, & i=k, \\ 0, & i\neq k. \end{cases} \tag{2.3}$$

As a result, we obtain that

$$\left(\left\{u_n^{(k)}\right\}_{n\in\mathbb{N}};\left\{\vartheta_{kn}^{(k)}\right\}_{n\in\mathbb{N}}\right), \tag{2.4}$$

is an atomic decomposition of X_k with respect to the space of coefficients $\mathscr{K}$, for every $k\in 1:m$.

Now, vice versa, let the pair (2.4) be an atomic decomposition of X_k, for every $k\in 1:m$. Assume that the relation (2.3) holds. Put

$$\vartheta_{in}=\left(\vartheta_{in}^{(1)};\ldots;\vartheta_{in}^{(m)}\right),$$

and let

$$u_{in}^0=\left(\underbrace{0;\ldots;0;u_n^{(i)}}_{i};0;\ldots;0\right),\quad i\in 1:m;n\in\mathbb{N}.$$

Take $\forall x = (x_1; \dots; x_m) \in X$, and consider

(2.5)

$$\begin{aligned}\sum_{i=1}^{m}\sum_{n=1}^{\infty}\vartheta_{in}(x)\,u_{in}^{0} &= \sum_{i=1}^{m}\sum_{n=1}^{\infty}\left(\vartheta_{in}^{(1)}(x_1)+\cdots+\vartheta_{in}^{(m)}(x_m)\right)u_{in}^{0}\\ &= \sum_{i=1}^{m}\sum_{n=1}^{\infty}\sum_{k=1}^{m}\left(\underbrace{0;\dots;0;\vartheta_{in}^{(k)}(x_k)}_{i}u_n^{(i)};0;\dots;0\right)\\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\underbrace{0;\dots;0;\sum_{n=1}^{\infty}\vartheta_{in}^{(k)}(x_k)}_{i}u_n^{(i)};0;\dots;0\right).\end{aligned}$$

Taking into account the relation (2.3) we obtain

$$\begin{aligned}\sum_{i=1}^{m}\sum_{n=1}^{\infty}\vartheta_{in}(x)\,u_{in}^{0} &= \sum_{i=1}^{m}\left(\underbrace{0;\dots;0;x_i;}_{i}0;\dots;0\right)\\ &= \sum_{i=1}^{m}x_i^{0}\\ &= x.\end{aligned} \tag{2.6}$$

Thus, the arbitrary element $x \in X$ can be expanded with respect to the system (2.4). Let us show the validity of the inequality (2.2) . We have

$$\begin{aligned}A\left\|\{\vartheta_{in}(x)\}\right\|_{\mathscr{K}} &= A\left\|\left\{\vartheta_{in}^{(1)}(x_1)+\cdots+\vartheta_{in}^{(m)}(x_m)\right\}\right\|_{\mathscr{K}}\\ &= A\left\|\sum_{k=1}^{m}\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}}.\end{aligned} \tag{2.7}$$

It is clear that $\left\{\vartheta_{in}^{(k)}(x_k)\right\} \in \mathscr{K}$, since, $\vartheta_{in}\left(x_k^0\right) = \vartheta_{in}^{(k)}(x_k), \forall k \in 1:m$. Then from (2.7) it follows

$$A\left\|\left\{\vartheta_{in}(x)\right\}\right\|_{\mathscr{K}} \leq A\sum_{k=1}^{m}\left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}} \leq \sum_{k=1}^{m}\left\|x_k\right\|_{X_k} \leq \sqrt{m}\left(\sum_{k=1}^{m}\left\|x_k\right\|_{X_k}^2\right)^{\frac{1}{2}} = \sqrt{m}\left\|x\right\|_X, \tag{2.8}$$

i.e. it holds

$$\frac{A}{\sqrt{m}}\left\|\left\{\vartheta_{in}(x)\right\}\right\|_{\mathscr{K}} \leq \left\|x\right\|_X, \quad \forall x \in X. \tag{2.9}$$

Similarly we obtain

$$\left\|x\right\|_X = \left\|\sum_{k=1}^{m} x_k^0\right\|_X \leq \sum_{k=1}^{m}\left\|x_k\right\|_{X_k} \leq B\sum_{k=1}^{m}\left\|\left\{\vartheta_{in}^{(k)}(x_k)\right\}\right\|_{\mathscr{K}} = B\sum_{k=1}^{m}\left\|\left\{\vartheta_{in}\left(x_k^0\right)\right\}\right\|_{\mathscr{K}}. \tag{2.10}$$

Further, consider the following space of coefficients $\mathscr{K}^m$:

$$\mathscr{K}^m = \underbrace{\mathscr{K} \times \cdots \times \mathscr{K}}_{m},$$

with a norm

$$\left\|\vec{\lambda}_n\right\|_{\mathscr{K}^m} = \sum_{k=1}^{m}\left\|\left\{\lambda_n^{(k)}\right\}\right\|_{\mathscr{K}},$$

where $\left\{\vec{\lambda}_n\right\} = \left\{\lambda_n^{(1)};\ldots;\lambda_n^{(m)}\right\} \in \mathscr{K}^m$. Linear operations in $\mathscr{K}^m$ are defined as usual. Let us identify the element $\left\{\vartheta_{in}\left(x_k^0\right)\right\} \in \mathscr{K}$ with the element from $\mathscr{K}^m$: $\left\{\vec{\vartheta}_{in}\left(x_k^0\right)\right\} \in \mathscr{K}^m$, where

$$\left\{\vec{\vartheta}_{in}\left(x_k^0\right)\right\} = \left(\underbrace{0;\dots;0;\left\{\vartheta_{in}\left(x_k^0\right)\right\}}_{k};0;\dots;0\right).$$

Consequently

$$\begin{aligned}\left\{\vartheta_{in}\left(x\right)\right\} &= \left\{\sum_{k=1}^{m}\vartheta_{in}\left(x_k^0\right)\right\} \\ &= \sum_{k=1}^{m}\left\{\vec{\vartheta}_{in}\left(x_k^0\right)\right\} \in \mathscr{K}^m.\end{aligned}$$

Thus, the element $\left\{\vartheta_{in}\left(x\right)\right\}$ identified with the elements from $\mathscr{K}^m$:

$$\sum_{k=1}^{m}\left\{\vec{\vartheta}_{in}\left(x_k^0\right)\right\} = \left(\left\{\vartheta_{in}\left(x_1^0\right)\right\};\dots;\left\{\vartheta_{in}\left(x_m^0\right)\right\}\right) \in \mathscr{K}^m.$$

Assume

$$\left\{\vec{\vartheta}_{in}\left(x\right)\right\} = \left(\left\{\vartheta_{in}\left(x_1^0\right)\right\};\dots;\left\{\vartheta_{in}\left(x_m^0\right)\right\}\right).$$

Then from inequalities (2.9) and (2.10), we obtain

$$\begin{aligned}\frac{A}{\sqrt{m}}\left\|\left\{\vartheta_{in}\left(x\right)\right\}\right\|_{\mathscr{K}} &\le \frac{A}{\sqrt{m}}\sum_{k=1}^{m}\left\|\left\{\vartheta_{in}\left(x_k^0\right)\right\}\right\|_{\mathscr{K}} \\ &= \frac{A}{\sqrt{m}}\left\|\left\{\vec{\vartheta}_{in}\left(x\right)\right\}\right\|_{\mathscr{K}^m} \\ &\le \left\|x\right\|_X \le B\left\|\left\{\vec{\vartheta}_{in}\left(x\right)\right\}\right\|_{\mathscr{K}^m}.\end{aligned} \tag{2.11}$$

Now, consider the following construction. Let $X = X_1 \oplus \dots \oplus X_m$, $\mathscr{K}$ be some space of coefficients. Take $\forall\vec{\vartheta} = \left(\vartheta^{(1)};\dots;\vartheta^{(m)}\right) \in X^*$, and put

$$\vec{\vartheta}\left(x\right) = \left(\vartheta^{(1)}\left(x_1\right);\dots;\vartheta^{(m)}\left(x_m\right)\right), \quad \forall x = \left(x_1;\dots;x_m\right) \in X.$$

Let $y = \left(y_1;\dots;y_m\right) \in X$. Assume

$$\vec{\vartheta}\left(x\right)y =: \left(\vartheta^{(1)}\left(x_1\right)y_1;\dots;\vartheta^{(m)}\left(x_m\right)y_m\right) \in X,$$

and introduce the following

Definition 2.5. The pair $\left(\left\{u_n\right\};\left\{\vec{\vartheta}_n\right\}\right)$ $\left(u_n \in X \wedge \vec{\vartheta}_n \in X^*\right)$ is called an atomic decomposition of X with respect to $\mathscr{K}^m$, if the following conditions are fulfilled:

(i) $\left\{\vec{\vartheta}_n\left(x\right)\right\} \in \mathscr{K}^m, \quad \forall x \in X;$

(ii) $\exists A; B > 0 :, A \left\| \left\{ \vec{\vartheta}_n(x) \right\} \right\|_{\mathscr{K}^m} \leq \|x\|_X \leq B \left\| \left\{ \vec{\vartheta}_n(x) \right\} \right\|_{\mathscr{K}^m}$;

(iii) $x = \sum_{n=1}^{\infty} \vec{\vartheta}_n(x) u_n, \quad \forall x \in X$.

From the relations (2.5) and (2.6) we obtain

$$\begin{aligned} x &= (x_1; \ldots; x_m) \\ &= \left(\sum_{n=1}^{\infty} \vartheta_{1n}^{(1)}(x_1) u_n^{(1)}; \ldots; \sum_{n=1}^{\infty} \vartheta_{mn}^{(m)}(x_m) u_n^{(m)} \right) \\ &= \sum_{n=1}^{\infty} \left(\vartheta_{1n}^{(1)}(x_1) u_n^{(1)}; \ldots; \vartheta_{mn}^{(m)}(x_m) u_n^{(m)} \right) \\ &= \sum_{n=1}^{\infty} \vec{\vartheta}_n(x) u_n; \end{aligned}$$

where $\vec{\vartheta}_n = \left(\vartheta_{1n}^{(1)}; \ldots; \vartheta_{mn}^{(m)} \right)$, $u_n = \left(u_n^{(1)}; \ldots; u_n^{(m)} \right)$. Then it follows from (2.11) that the pair$\left(\left\{ \vec{\vartheta}_n \right\}; \{u_n\} \right)$ is an atomic decomposition of X with respect to $\mathscr{K}^m$. So, the following theorem is true.

Theorem 2.6. (i) *Let the pair* $\left(\left\{ u_{in}^0 \right\}_{i=\overline{1,m}; n \in \mathbb{N}}; \{\vartheta_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}} \right)$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$, *where* $\vartheta_{in} = \left(\vartheta_{in}^{(1)}; \ldots; \vartheta_{in}^{(m)} \right) \in X^*$, $i \in 1:m; n \in \mathbb{N}$. *Then the relation* (2.3) *holds and system* (2.4) *is an atomic decomposition of* X_k *with respect to* $\mathscr{K}$.

(ii) *Let the pair* (2.4) *be an atomic decomposition of* X_k *for every* $k \in 1:m$ *with respect to* $\mathscr{K}$ *and the relation* (2.3) *holds. Then* $\left(\left\{ \vec{\vartheta}_n \right\}; \{u_n\} \right)$ *is an atomic decomposition of* X *with respect to* $\mathscr{K}^m$ *in the sense of Definition 2.5.*

3. Main results

Let the following direct sum hold

$$X = X_1 \oplus \cdots \oplus X_m,$$

where $X_k, k = \overline{1,m}$—are some B-spaces. Consider the system $\{u_{in}\}_{n \in \mathbb{N}} \subset X_i, i = \overline{1,m}$; and form

$$\vec{u}_{in} = (a_{i1} u_{1n}; a_{i2} u_{2n}; \ldots; a_{im} u_{mn}), \quad i = \overline{1,m}; n \in \mathbb{N}.$$

Let

$$A = (a_{ij}), \quad i, j = \overline{1,m}; \quad \Delta = \det A.$$

We will need the following easy-to-prove lemma.

Lemma 3.1. *Let* $(\{u_n\};\{\vartheta_n\})$ *be an atomic decomposition of* X *with respect to* $\mathscr{K}$ *and* $T \in L(X)$ *be some automorphism. Then*

$$\left(\{Tu_n\};\left\{(T^*)^{-1}\vartheta_n\right\}\right)$$

is also an atomic decomposition of X *with respect to* $\mathscr{K}$.

Proof. Indeed, take $\forall y \in X$. From the expression $\left(T^{-1}y = x\right)$:

$$\left((T^*)^{-1}\vartheta_n\right)(y) = \vartheta_n\left(T^{-1}y\right) = \vartheta_n(x), \quad \forall n \in \mathbb{N},$$

it directly follows that $\left((T^*)^{-1}\vartheta_n\right) \subset X^*$. We have

$$\begin{aligned} A\left\|\left\{\left((T^*)^{-1}\vartheta_n\right)(y)\right\}\right\|_{\mathscr{K}} &\le \|x\|_X \\ &\le \left\|T^{-1}\right\|\|y\|_X \\ &\le \left\|T^{-1}\right\|\|T\|\|x\|_X \\ &\le \left\|T^{-1}\right\|\|T\| B\|\{\vartheta_n(x)\}\|_{\mathscr{K}} \\ &= B\left\|T^{-1}\right\|\|T\|\left\|\left((T^*)^{-1}\vartheta_n\right)(y)\right\|_{\mathscr{K}}. \end{aligned}$$

Consider

$$\begin{aligned} &x = \sum_{n=1}^{\infty}\vartheta_n(x)\,u_n \\ \Rightarrow \quad &Tx = y = \sum_{n=1}^{\infty}\vartheta_n(x)\,Tu_n = \sum_{n=1}^{\infty}\left((T^*)^{-1}\vartheta_n\right)(y)\,Tu_n. \end{aligned}$$

Lemma is proved.

Let $T_{ij} : X_i \to X_j$ be some operators. Consider the system

$$\sum_{i=1}^{m} a_{ij}T_{ij}x_i = y_j, \quad j = \overline{1,m}, \tag{3.1}$$

where $y_j \in X_j, j = \overline{1,m}$ are the given elements, and $x_i \in X_i, i = \overline{1,m}$ are the unknowns. Assume that the spaces $X_k, k = \overline{1,m}$, are pairwise isomorphic and T_{ij} performs a corresponding isomorphism. Besides, assume that the following conditions are satisfied:

α) $T_{ii} = I_i$, $T_{ij} = T_{ji}^{-1}$, $T_{jk}T_{ij} = T_{ik}, \forall i,j = \overline{1,m}$, where I_i is the identity operator in X_i.

Applying the operator $T_{j1} = T_{1j}^{-1}$ to the j-th equation in the system (3.1), we obtain the following system

$$\sum_{i=1}^{m} a_{ij}T_{i1}x_i = T_{j1}y_j, \quad j = \overline{1,m}.$$

Let $\tilde{x}_i = T_{i1}x_i, \tilde{y}_j = T_{j1}y_j$. It is clear that $\tilde{x}_i, \tilde{y}_j \in X_1$. As a result, we obtain the following system of linear equations in the space X_1:

$$\sum_{i=1}^{m} a_{ij}\tilde{x}_i = \tilde{y}_j, \quad j = \overline{1,m}.$$

If the determinant of this system $\Delta = \det(a_{ij}) \neq 0$, then it is clear that this system is uniquely solvable with respect to the unknowns $\tilde{x}_i$. Then the system (3.1) is also uniquely solvable.

Thus, the following lemma is true. □

Lemma 3.2. *Let the operators $T_{ij} : X_i \to X_j$ perform an isomorphism between X_i and X_j, the conditions α) be satisfied and $\Delta \neq 0$. Then the system* (3.1) *is uniquely solvable for $\forall y \in X, y = (y_1, y_m)$ and, moreover, $\exists M > 0$:*

$$\|x\|_X \leq M \|y\|_X, \tag{3.2}$$

where $x = (x_1, \ldots, x_m)$.

The following representation for the solution of the system (3.1) implies the validity of the estimate (3.2):

$$x_i = \sum_{j=1}^{m} b_{ij}T_{ji}y_j, \quad i = \overline{1,m},$$

where b_{ij} are the elements of the inverse matrix A^{-1}.

Consider the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,j=1}^m$. Let all the conditions of Lemma 3.2 be satisfied. It follows from this lemma that $KerT = \{0\}$, $R_T = X$, and the estimate (3.2) means $T \in L(X)$. Then it follows from Banach's theorem on the inverse operator that T is an automorphism in X. So the following theorem is true.

Theorem 3.3. *Let $T_{ij} \in L(X_i, X_j)$ be an isomorphism, the conditions α) be satisfied and $\Delta \neq 0$. Then the operator $T : X \to X$ defined by the matrix $(a_{ij}T_{ij})_{i,j=1}^m$ is an automorphism in $X = X_1 \oplus \cdots \oplus X_m$.*

Let $(\{u_{in}\}; \{\vartheta_{in}\})$ be an atomic decomposition of X_i with respect to $\mathscr{K}$, for every $i \in 1 : m$, respectively, moreover suppose they are pairwise isomorphic and $T_{ij} \in L(X_i; X_j)$ performs a corresponding isomorphism, i.e. $T_{ij}u_{in} = u_{jn}, \forall n \in \mathbb{N}$ (it is clear that the subspaces X_i and X_j are isomorphic). It is obvious that the space of coefficients $\mathscr{K}_i, i = \overline{1,m}$; of these systems are the same and let $\mathscr{K} = \mathscr{K}_i$. We require operators T_{ij} to satisfy the condition α). Consider the operator $T : X \to X$, defined by the matrix $(a_{ij}T_{ij})_{i,j=\overline{1,m}}$, i.e.

$$Tx = y, \qquad x = (x_1; \ldots; x_m), \qquad y = (y_1; \ldots; y_m),$$

where

$$y_j = \sum_{i=1}^{m} a_{ij} T_{ij} x_i, \quad j = \overline{1,m}.$$

Let $\Delta = \det A \neq 0$. Assume

$$u_{in}^0 = \left(\underbrace{0; \ldots ; 0; u_{in}}_{i}; 0; \ldots ; 0 \right);$$

$$\vartheta_{in}^0 = \left(\underbrace{0; \ldots ; 0; \vartheta_{in}}_{i}; 0; \ldots ; 0 \right), \quad i = \overline{1,m}; n \in \mathbb{N}.$$

Take $\forall x = (x_1; \ldots ; x_m) \in X$. We have

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{n=1}^{\infty} \vartheta_{in}^0 (x)\, u_{in}^0 &= \sum_{i=1}^{m} \sum_{n=1}^{\infty} \vartheta_{in} (x_i)\, u_{in}^0 \\
&= \sum_{i=1}^{m} \left(\underbrace{0; \ldots ; 0; \sum_{n=1}^{\infty} \vartheta_{in} (x_i)\, u_{in}}_{i}; 0; \ldots ; 0 \right) \\
&= \sum_{i=1}^{m} \left(\underbrace{0; \ldots ; 0; x_i}_{i}; 0; \ldots ; 0 \right) \\
&= x.
\end{aligned}$$

So, for every $i \in 1 : m$, we have

$$\vartheta_{in}^0 (x) = \vartheta_{in} (x_i), \quad \forall \ltimes \in \mathbb{N}.$$

Then it is clear that

$$\left\{ \vartheta_{in}^0 (x) \right\}_{n \in \mathbb{N}} \in \mathscr{K}, \quad \forall i \in 1 : m,$$

and as a result

$$\left\{ \left\{ \vartheta_{in}^0 (x) \right\}_{n \in \mathbb{N}} \right\}_{i=\overline{1,m}} \in \mathscr{K}^m.$$

So,

$$\left(\left\{ \left\{ u_{in}^0 \right\}_{n \in \mathbb{N}} \right\}_{i=\overline{1,m}} ; \left\{ \left\{ \vartheta_{in}^0 \right\}_{n \in \mathbb{N}} \right\}_{i=\overline{1,m}} \right)$$

is an atomic decomposition of X with respect to $\mathscr{K}^m$. Then by Lemma 3.1 the pair

$$\left(\left\{ \left\{ T u_{in}^0 \right\}_{n \in \mathbb{N}} \right\}_{i=\overline{1,m}} ; \left\{ \left\{ (T^*)^{-1} \vartheta_{in}^0 \right\}_{n \in \mathbb{N}} \right\}_{i=\overline{1,m}} \right), \tag{3.3}$$

is also an atomic decomposition of X with respect to $\mathscr{K}^m$, where

$$\begin{aligned}\left(Tu_{in}^0\right)_j &= \sum_{k=1}^{m} a_{kj}T_{kj}u_{in}\delta_{ki}\\ &= a_{ij}T_{ij}u_{in}\\ &= a_{ij}u_{jn},\end{aligned}$$

i.e.

$$Tu_{in}^0 = \left(a_{i1}u_{1n}; a_{i2}u_{2n}; \ldots; a_{im}u_{mn}\right).$$

As a result, the following theorem is true.

Theorem 3.4. *Let the direct sum $X = X_1 \oplus \cdots \oplus X_m$ hold, the pair $\left(\{u_{in}\}_{n\in\mathbb{N}}; \{\vartheta_{in}\}_{n\in\mathbb{N}}\right)$ be an atomic decomposition of $X_i, i = \overline{1,m}$; with respect to $\mathscr{K}$, $T_{ij} \in L\left(X_i; X_j\right)$ be an isomorphism and $T_{ij}u_{in} = u_{jn}, \forall n \in \mathbb{N}$, for $i \neq j$. Let $\det\left(a_{ij}\right)_{i,j=\overline{1,m}} \neq 0$, and operators $T_{ij}; i,j = \overline{1,m}$; satisfy the condition α) and the operator $T \in L(X)$ be defined by the matrix $\left(a_{ij}T_{ij}\right)_{i,j=\overline{1,m}}$. Then the pair (3.3) is also an atomic decomposition of X with respect to $\mathscr{K}^m$.*

4. Application

4.1. **System of cosines.** Consider the following Cauchy problem

$$\begin{aligned}&-y''(x) + q(x)u(x) = \lambda^2 u(x), \quad x \in (0,\pi),\\ &u(0) = 1, \qquad u'(0) = \lambda,\end{aligned} \tag{4.1}$$

with parameter λ, where $q(\cdot) \in L_1(0,\pi)$ is a real function. Solution of the Cauchy problem (4.1) is denoted as $u_\lambda(x)$. This spectral problem can be understood in the sense of V.A.Ilyin [21]. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset R$ be some sequence of numbers and $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is a system of solution of the problem (4.1) corresponding it . As is known (see e.g. Marchenko V.A. [25]), the following relation holds

$$u_{\lambda_n}(x) = \cos\lambda_n x + \int_0^x K(x;t)\cos\lambda_n t dt, \tag{4.2}$$

where $K(\cdot;\cdot)$ is a continuous function on $[0,\pi]\times[0,\pi]$. Let

$$(Kf)(x) = \int_0^x K(x;t) f(t)\, dt.$$

It is clear that the operator $(I+K)$ is invertible in $L_p(0,\pi), 1<p<+\infty$, where $I: L_p \to L_p$ is an identity operator. Let

$$u_{1n}(x)=\begin{cases} u_{\lambda_n}(x), & x\in(0,\pi), \\ 0, & x\in(-\pi,0), \end{cases}$$

$$u_{2n}(x)=\begin{cases} 0, & x\in(0,\pi), \\ u_{\lambda_n}(-x), & x\in(-\pi,0), \end{cases}$$

Similarly we define

$$u_{1n}^c=\begin{cases} \cos\lambda_n x, & x\in(0,\pi), \\ 0, & x\in(-\pi,0), \end{cases}$$

$$u_{2n}^c(x)=\begin{cases} 0, & x\in(0,\pi), \\ \cos\lambda_n x, & x\in(-\pi,0). \end{cases}$$

Let

$$\omega_{1n}(x)=a_{11}u_{1n}(x)+a_{12}u_{2n}(x),$$
$$\omega_{2n}(x)=a_{21}u_{1n}(x)+a_{22}u_{2n}(x), \quad x\in(-\pi,\pi),$$

and

$$\omega_{1n}^c(x)=a_{11}u_{1n}^c(x)+a_{12}u_{2n}^c(x),$$
$$\omega_{2n}^c(x)=a_{21}u_{1n}^c(x)+a_{22}u_{2n}^c(x).$$

The following statement is true.

Statement 4.1. *Let $\Delta=a_{11}a_{22}-a_{12}a_{21}\neq 0$. Then the systems $\{\omega_{in}\}$ and $\{\omega_{in}^c\}$ are an atomic decomposition of $L_p(-\pi,\pi)$ at the same time with respect to some space of coefficients $\mathscr{K}$.*

4.2. **Basicity of the system of eigenfunctions of the problem (1.1), (1.2).** Let us apply these results in obtaining the basicity of eigenfunctions $\left\{u_n^{(1)};\tilde{u}_n^{(2)}\right\}_{n\in\mathbb{N}}$ of the problem (1.1), (1.2) in $X\equiv L_p(-\pi,\pi)$, $1<p<+\infty$. We identify the spaces $L_p(-\pi,0)$ and $L_p(0,\pi)$ with the corresponding subspaces $L_p(-\pi,\pi)$, and denote them by $X_1\equiv L_p(-\pi,0)$, $X_2\equiv L_p(0,\pi)$. We have $X=X_1\oplus X_2$. Thus, each element $f\in L_p(-\pi,\pi)$ is identified with the vector $(f_1;f_2)$, where $f_1=f/_{(-\pi,0)}$, $f_2=f/_{(0,\pi)}$ ($f/_M$ is a restriction of f on M). Assume $u_{1n}^0=(\sin nx;0)$; $u_{2n}^0=(0;\sin nx)$, $n\in\mathbb{N}$. It is clear that the systems $\left\{u_{kn}^0\right\}_{n\in\mathbb{N}}, k=1,2-$form an isomorphic bases for $X_k, k=1,2-$ respectively. We have

$$u_n^{(1)}=u_{1n}^0+u_{2n}^0, u_n^{(2)}=u_{1n}^0-u_{2n}^0, \quad \forall n\in\mathbb{N}.$$

Consequently, in this case the equalities $a_{11}=1; a_{12}=1; a_{21}=1; a_{22}=-1$, are true and as a result, $\det(a_{ij})_{i,j=1,2}\neq 0$. Then Theorem 3.4

implies that the system $\left\{u_n^{(1)}; u_n^{(2)}\right\}_{n\in\mathbb{N}}$ forms a basis for $L_p(-\pi,\pi)$. So, the following theorem is true.

Theorem 4.2. *System* $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in\mathbb{N}}$ *forms a basis for* $L_p(-\pi,\pi)$, $1 < p < +\infty$. *Moreover, with respect to the system of eigenfunctions* $\left\{u_n^{(1)}; \tilde{u}_n^{(2)}\right\}_{n\in\mathbb{N}}$ *of the problem* (1.1), (1.2) *the following properties are equivalent in* $L_p(-\pi,\pi)$*:*

(i) *completeness;*
(ii) *minimality;*
(iii) *basis property;*
(iv) *Riesz basis property for* $p = 2$.

The second part of assertion of the theorem follows from the results of [3].

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