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APPROXIMATION OF FIXED POINTS FOR A CONTINUOUS REPRESENTATION OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. This paper introduces an implicit scheme for a continuous representation of nonexpansive mappings on a closed convex subset of a Hilbert space with respect to a sequence of invariant means defined on an appropriate space of bounded, continuous real valued functions of the semigroup. The main result is to prove the strong convergence of the proposed implicit scheme to the unique solution of the variational inequality on the solution of systems of equilibrium problems and the common fixed points of a sequence of nonexpansive mappings and a continuous representation of nonexpansive mappings.

1. Introduction

Let H be a Hilbert space and let $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is $G(u,u) = 0$ for every $u \in H$. The Equilibrium Problem is defined as follows:

$$\text{Find} \quad \tilde{x} \in H \quad \text{such that} \quad G(\tilde{x}, y) \geq 0 \quad \text{for all} \quad y \in H. \tag{1.1}$$

A solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by SEP(G).

Let C be a closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

Let f be an α-contraction on H (i.e. $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in H$ with $0 \leq \alpha < 1$), and A be a bounded linear operator on H. The following variational inequality problem with viscosity is of great

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interest [8].
Find x^* in C such that

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0 \quad (x\in C), \tag{1.2}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\left(\frac{1}{2}\langle Ax,x\rangle + h(x)\right),$$

where γ satisfies $\|I-A\| \leq 1-\alpha\gamma$ and h is a potential function for γf (that is $h'(x)=\gamma f(x)$).

Let C be a closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is called Lipschitzian, if there exists a nonnegative number k such that $\|Tx - Ty\| \leq k\|x-y\|$ for all $x, y \in C$.

Let $B : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality problem, denoted by $VI(C, B)$ is to find $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0,$$

for all $v \in C$. For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u-z, v-u\rangle \geq 0, \quad (v\in C),$$

if and only if $u = P_C z$. It is known that the projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,$$

for $x, y \in H$.

Recall that the following definitions:

(1) B is called v-strongly monotone, if

$$\langle Bx - By\ ,\ x-y\rangle \geq v\|x-y\|^2 \quad \text{for all} \quad x,y\in C,$$

for a constant $v > 0$. This implies that

$$\|Bx - By\| \geq v\|x-y\|,$$

that is, B is v-expansive and when $v = 1$, it is expansive.

(2) B is said to be v-cocoercive, if there exists a constant $v > 0$ such that

$$\langle Bx - By, x-y\rangle \geq v\|Bx-By\|^2 \quad \text{for all} \quad x,y\in C,$$

clearly, every v-cocoercive map B is $1/v$-Lipschitz continuous ([18]).

(3) B is called relaxed u-cocoercive, if there exists a constant $u > 0$ such that

$$\langle Bx - By, x-y\rangle \geq (-u)\|Bx-By\|^2 \quad \text{for all} \quad x,y\in C.$$

(4) B is said to be relaxed (u,v)-cocoercive, if there exist two constants $u,v>0$ such that

$$\langle Bx-By, x-y\rangle \geq (-u)\|Bx-By\|^2 + v\|x-y\|^2 \quad \text{for all} \quad x,y\in C.$$

For $u=0$, B is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps. Clearly, every v-strongly monotone map is a relaxed (u,v)-cocoercive map.

(5) A semitopological semigroup is a semigroup S with a Hausdorff topology such that for each $a\in S$, the mappings $s\to a.s$ and $s\to s.a$ from S to S are continuous.

Plubtieng and Punpaeng in [10] proved a strong convergence theorem for an implicit sequence $\{x_n\}$ obtained from the viscosity approximation method for finding a common element in SEP(G)∩Fix(T) which satisfies the variational inequality (1.2) (see also [16]):

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let G be a bifunction from $H\times H$ into $\mathbb{R}$ satisfying*

(A_1) $G(x,x)=0$ *for all* $x\in C$;
(A_2) *G is monotone, i.e.* $G(x,y)+G(y,x)\leq 0$ *for all* $x,y\in C$;
(A_3) *For all* $x,y,z\in C$,

$$\limsup_{t\to 0} G(tz+(1-t)x,y)\leq G(x,y);$$

(A_4) *For all $x\in C$, $y\mapsto G(x,y)$ is convex and lower semicontinuous.*

For $x\in H$ and $r>0$, set $S_r:H\to C$ to be the resolvent of G i.e. $S_r(x)$ is the unique $z\in C$ for which

$$G(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad (y\in C).$$

Let T be a nonexpansive mapping on H such that $\mathrm{SEP(G)}\cap\mathrm{Fix(T)}\neq\emptyset$. *Let f be a contraction of H into itself with $\alpha\in(0,1)$, and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}>0$, and $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_n=\alpha_n\gamma f(x_n)+(I-\alpha_n A)Tu_n, & (n\in\mathbb{N}),\\ G(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq 0 & (y\in H),\end{cases}$$

where $u_n=S_{r_n}x_n$, $\{r_n\}\subset(0,\infty)$ and $\alpha_n\subset[0,1]$ satisfying $\lim_{n\to\infty}\alpha_n=0$, and $\liminf_{n\to\infty} r_n>0$. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point z in $\mathrm{Fix(T)}\cap\mathrm{SEP(G)}$, *which solves the variational inequality*

$$\langle (A-\gamma f)z, z-x\rangle \leq 0, \quad x\in \mathrm{Fix(T)}\cap\mathrm{SEP(G)}.$$

The difference of this work with other works is that, we use of a semitopological semigroup S, $C(S)$ instead of $B(S)$ and a sequence of invariant means on an amenable subspace of $C(S)$ and a continuous representation of nonexpansive mappings and when S equipped by discrete topology we conclude the results for $B(S)$.

In the other words, in this paper, motivated by N. Hussain, M. L. Bami and E. Soori [5], and E. Soori [14], we introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\mathrm{SEP}(\mathcal{G})$ for a family $\mathcal{G} = \{G_k; k = 1, 2, \ldots, K\}$ of bifunctions and of the set of fixed points of a family $\{T_i\}_{i\in\mathbb{N}}$ of nonexpansive mappings from C into itself and a continuous representation $\mathcal{S} = \{T_t : t \in S\}$ of a semitopological semigroup S as nonexpansive mappings from C into itself, with respect to W-mappings and a sequence $\{\mu_n\}$ of invariant means defined on an appropriate subspace of bounded, continuous real-valued functions of the semigroup:

$$z_n = \epsilon_n \gamma f(W_n z_n) + (I - \epsilon_n A)(I - r_n B) T_{\mu_n} W_n S^K_{r_{K,n}} \cdots S^2_{r_{2,n}} S^1_{r_{1,n}} z_n \quad (n \in \mathbb{N}).$$

Our goal is to prove a result of strong convergence for an implicit scheme to approach an element $u^* \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{SEP}(\mathcal{G})$ which is the unique solution of the variational inequalities $VI(\bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{SEP}(\mathcal{G}), \mathrm{B})$, equivalently $P_{\bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)}\cap\mathrm{SEP}(\mathcal{G})}(I - \lambda B)u^* = u^*$ for each $\lambda > 0$, and we also have $P_{\bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)}\cap\mathrm{SEP}(\mathcal{G})}(I - (A - \gamma f))u^* = u^*$.

2. PRELIMINARIES

Throughout this paper H denotes a Hilbert space. Moreover, we assume that A is a bounded strongly positive operator on H with constant $\overline{\gamma}$; that is, there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad (x \in H).$$

For a map $T : H \to H$, we denote by $\mathrm{Fix(T)} := \{\mathrm{x \in H : x = Tx}\}$ the fixed point set of T. Note that if T is a nonexpansive mapping, Fix(T) is closed and convex (see [3]).

Let S be a semitopological semigroup. We denote by $B(S)$ the Banach space of all bounded real-valued functions defined on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ which consists of all bounded, continuous real-valued functions on S. For each $s \in S$ and $f \in B(S)$, we define l_s and r_s in $B(S)$ by

$$(l_s f)(t) = f(st), \qquad (r_s f)(t) = f(ts), \quad (t \in S).$$

Let X be a subspace of $C(S)$ containing 1 and let X^* be its topological dual space. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant), if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$.

Let f be a function on a semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $C(S)$ containing all the functions $t \to \langle f(t), x^*\rangle$ with $x^* \in E^*$. We know from [4] that for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\langle f_\mu, x^*\rangle = \mu_t \langle f(t), x^*\rangle$ for all $x^* \in E^*$. We denote such f_μ by $\int f(t)\, \mathrm{d}\mu(t)$. Moreover, if μ is a mean on X then from [6], $\int f(t)\, \mathrm{d}\mu(t) \in \overline{\mathrm{co}}\{f(t) : t \in S\}$.

Let C be a nonempty closed and convex subset of H. Then, a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from C into itself is said to be a continuous representation of S as nonexpansive mappings on C into itself if $\mathcal{S}$ satisfies the following :

(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;
(2) for every $x \in C$, the mapping $s \mapsto T_s x$ from S into C is continuous;
(3) for every $s \in S$, the mapping $T_s : C \to C$ is nonexpansive.

We denote by $\mathrm{Fix}(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, that is $\mathrm{Fix}(\mathcal{S})=\{x \in C : T_s x = x,\ (s \in S)\}$.

For an equilibrium function $G : H \times H \to \mathbb{R}$, $\mathrm{SEP(G)} := \{\mathrm{x} \in \mathrm{H} : \mathrm{G(x, y)} \geq 0, (\mathrm{y} \in \mathrm{H})\}$ is the set of solutions of the related equilibrium problem.

Let C be a closed convex subset of a Hilbert space H. Let P_C be the projection of H onto C. Then the projection operator P_C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following Lemma characterizes the projection P_C:

Lemma 2.1 ([15])**.** *Let C be a closed convex subset of a real Hilbert space H, $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if it satisfies the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad (z \in C).$$

Lemma 2.2 ([7])**.** *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$ Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

The following result generalizes Theorem 3.3.3 of [15].

Theorem 2.3. *Let S be a semitopological semigroup such that $C(S)$ has an invariant mean μ and let C be a closed convex subset of a Hilbert space H. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on C into itself and suppose* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$*. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$, then the followings hold:*

(i) $T_\mu T_s = T_s T_\mu = T_\mu$ *for all* $s \in S$*;*

(ii) T_μ *is a nonexpansive retraction of C onto* $\mathrm{Fix}(\mathcal{S})$*, i.e.,*

$$\|T_\mu x - T_\mu y\| \leq \|x - y\| \quad \text{for all} \quad x, y \in C \quad \text{and} \quad T_\mu^2 = T_\mu;$$

(iii) $T_\mu x \in \overline{\mathrm{co}}\{T_s x : s \in S\}$ *for each* $x \in C$*;*

(iv) $T_\mu x = x$ *for each* $x \in \mathrm{Fix}(\mathcal{S})$*.*

Proof. For proving (i)-(iii), see the proof of Theorem 3.3.3 of [15].
(iv) is clear, since for every $x \in \mathrm{Fix}(\mathcal{S})$, $T_s x = x$ for all $s \in S$. Thus $\overline{\mathrm{co}}\{T_s x : s \in S\} = \{x\}$. Hence by (iii), $T_\mu x = x$ for each $x \in \mathrm{Fix}(\mathcal{S})$. □

Theorem 2.4 ([2])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and $G : C \times C \to \mathbb{R}$ satisfy,*

(A_1) $G(x, x) = 0$ *for all* $x \in C$;

(A_2) G *is monotone, i.e.* $G(x, y) + G(y, x) \leq 0$ *for all* $x, y \in C$*;*

(A_3) *for all* $x, y, z \in C$,

$$\liminf_{t \to 0} G(tz + (1 - t)x, y) \leq G(x, y);$$

(A_4) *for all* $x \in C$, $y \mapsto G(x, y)$ *is convex and lower semicontinuous.*

For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) := \left\{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, (y \in C) \right\},$$

then S_r is well defined and the followings are valid:

(i) S_r *is single-valued;*

(ii) S_r *is firmly nonexpansive, i.e.* $\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle$, *for all* $x, y \in H$;

(iii) $\mathrm{Fix}\, S_r = \mathrm{SEP}(G)$;

(iv) $\mathrm{SEP}(G)$; is closed and convex.

Theorem 2.5 ([1])**.** *Let $\{r_n\} \subset (0, \infty)$ be a sequence converging to $r > 0$. For a bifunction $G : H \times H \to \mathbb{R}$, satisfying conditions (A_1)-(A_4), define S_r and S_{r_n} for $n \in \mathbb{N}$ as in* Theorem 2.4, then for every $x \in H$, we have

$$\lim_n \|S_{r_n} - S_r\| = 0.$$

Definition 2.6. A vector space X is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$,

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \quad (y \in X, y \neq x).$$

Note that every Hilbert space satisfies the Opial's condition (see [9] and [11]).

Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Following [13], for any $n \geq 1$, we define a mapping W_n of C into itself as follows,

$$
\begin{aligned}
&U_{n,n+1} := I,\\
&U_{n,n} := \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\quad\vdots\\
&U_{n,k} := \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&\quad\vdots\\
&U_{n,2} := \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n := U_{n,1} := \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned}
\tag{2.1}
$$

The following result holds for the mappings W_n.

Theorem 2.7 ([13]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)} \neq \emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$. Then*

(1) *W_n is nonexpansive and $\mathrm{Fix(W_n)} = \bigcap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Fix(T_i)}$ for each $n \geq 1$,*

(2) *for each $x \in C$ and for each positive integer j, the limit $\lim_{n\to\infty} U_{n,j}x$ exists.*

(3) *The mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} \quad (\mathrm{x \in C}),$$

is a nonexpansive mapping satisfying $\mathrm{Fix(W)} = \bigcap_{\mathrm{i=1}}^{\infty} \mathrm{Fix(T_i)}$, and it is called the W-mapping generated by $\{T_i\}_{i\in\mathbb{N}}$, and $\{\lambda_i\}_{i\in\mathbb{N}}$.

(4) *If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

3. MAIN RESULTS

In this section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and a continuous representation.

Theorem 3.1. *Let S be a semitopological semigroup and let C be a nonempty closed convex subset of a Hilbert space H. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings of C into itself. Let X be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a sequence of invariant means on X. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$ for every $i \in \mathbb{N}$, and $\mathcal{G} = \{G_k : k = 1, 2, \dots K\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, such that $\mathrm{SEP}(\mathcal{G}) \subseteq \mathrm{Fix}(\mathcal{S})$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ such that $\|A\| \leq 1$ and let B be an η-Lipschitzian, relaxed (u, v)-cocoercive mapping from C into H, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}_{k=1}^{K}$, $\{r_n\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $r_n > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq c < 1$ for some c, and γ is a real number such that $0 < 3\gamma < \overline{\gamma}$. For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.1), for every $k \in \{1, 2, \dots, K\}$ and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in* Theorem 2.4. Assume that,

(i) for every $k \in \{1, 2, \dots, K\}$, the function G_k satisfies $(\mathrm{A}_1) - (\mathrm{A}_4)$ of Theorem 2.4,

(ii) $\lim_n \epsilon_n = 0$,

(iii) for every $k \in \{1, 2, \dots, K\}$, $\liminf_n r_{k,n}$ exists and is a positive real number,

(iv) $\{r_n\} \subset [0, b]$ for some b with $0 \leq b \leq \frac{2(v-u\eta^2)}{\eta^2}$, $v > u\eta^2$,

$$\lim_n r_n = 0, \lim_n \frac{r_n}{\epsilon_n} = 0,$$

(v) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(\mathrm{T_n}) \cap \mathrm{SEP}(\mathcal{G}) \neq \emptyset$,

(vi) $\mathrm{Fix}(\mathrm{P}_{\mathfrak{F}}(\mathrm{I} - \lambda\mathrm{B})) \cap \mathrm{Fix}(\mathrm{P}_{\mathfrak{F}}(\mathrm{I} - (\mathrm{A} - \gamma\mathrm{f}))) \neq \emptyset$, for each $\lambda > 0$.

Let $\{z_n\}$ be the sequence generated by

$$z_n = \epsilon_n \gamma f(W_n z_n) + (I - \epsilon_n A)(I - r_n B) T_{\mu_n} W_n S^K_{r_{K,n}} \cdots S^2_{r_{2,n}} S^1_{r_{1,n}} z_n \quad (n \in \mathbb{N}), \tag{3.1}$$

then $\{z_n\}$ strongly converges to $u^* \in \mathfrak{F}$ which is:

(i) the unique solution of the variational inequalities $VI(\mathfrak{F}, B)$, equivalently $P_{\mathfrak{F}}(I - \lambda B)u^* = u^*$ for each $\lambda > 0$,

(ii) the unique solution of the variational inequality:

$$\langle (A-\gamma f)u^*, x-u^*\rangle \geq 0 \quad (x \in \mathfrak{F}),$$

or equivalently,

$$u^* = P_{\mathfrak{F}}(I-(A-\gamma f))u^*,$$

(iii) the unique solution of the minimization problem

$$\min_{x\in\mathfrak{F}} \frac{1}{2}\langle Ax, x\rangle + h(x),$$

where h is a potential function for γf.

Proof. Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \leq \min\left\{\|A\|^{-1}, \frac{1}{\bar{\gamma}}\right\}$. We show that $\langle (I-\epsilon_n A)x\ ,\ x\rangle \geq 0$, for all $x \in H$. We may assume that $\|x\| = 1$, so we have

$$\langle (I-\epsilon_n A)x\ ,\ x\rangle = 1 - \epsilon_n \langle Ax\ ,\ x\rangle \geq 1 - \epsilon_n\|A\| \geq 0.$$

By Lemma 2.2, we have

$$\|I-\epsilon_n A\| \leq 1-\epsilon_n\bar{\gamma}.$$

By proposition 2 in [12], there exists a unique element $u^* \in \mathfrak{F}$ such that $VI(\mathfrak{F}, B) = \{u^*\}$, equivalently $P_{\mathfrak{F}}(I-\lambda B)u^* = u^*$ for each $\lambda > 0$, hence, by condition (vi),

$$P_{\mathfrak{F}}(I-(A-\gamma f))u^* = u^*.$$

We show that $I - r_n B$ is nonexpansive. Indeed, from the relaxed (u,v)-cocoercive and η-Lipschitzian definition on B and condition (iv), we have

$$\begin{aligned}
\|(I-r_nB)x-(I-r_nB)y\|^2 &= \|(x-y)-r_n(Bx-By)\|^2\\
&= \|x-y\|^2 - 2r_n\langle x-y, Bx-By\rangle\\
&\quad + r_n^2\|Bx-By\|^2\\
&\leq \|x-y\|^2 - 2r_n\Big[-u\|Bx-By\|^2\\
&\quad + v\|x-y\|^2\Big] + r_n^2\|Bx-By\|^2\\
&\leq \|x-y\|^2 + 2r_n\eta^2u\|x-y\|^2\\
&\quad - 2r_nv\|x-y\|^2 + \eta^2r_n^2\|x-y\|^2\\
&= (1+2r_n\eta^2u - 2r_nv + \eta^2r_n^2)\|x-y\|^2\\
&\leq \|x-y\|^2,
\end{aligned}$$

which implies that the mapping $I - r_nB$ is nonexpansive.

We put $S_n^k := S_{r_{k,n}}^k \cdots S_{r_{2,n}}^2 S_{r_{1,n}}^1$ for every $k \in \{1, 2, \ldots, K\}$, $S_n^0 := I$ and $B_n = (I - r_nB)$.

Putting $\mu_1 = \mu$, since S is a semitopological semigroup, by Lemma 3.4.3 in [15], we have $T_{\mu_n} = T_\mu$ for all $n \in \mathbb{N}$. Therefore we have

$$z_n = \epsilon_n \gamma f(W_n z_n) + (I - \epsilon_n A) B_n T_\mu W_n S_n^K z_n, \quad (n \in \mathbb{N}).$$

We divide the proof into seven steps. **Step1**. The existence of z_n which satisfies (3.1).

Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping N_n given by

$$N_n x := \epsilon_n \gamma f(W_n x) + (I - \epsilon_n A) B_n T_\mu W_n S_n^K x \quad (x \in H),$$

is a contraction. To see this, put $\beta_n = 1 + \epsilon_n \gamma \alpha - \epsilon_n \bar{\gamma}$, then $0 \leq \beta_n < 1 (n \in \mathbb{N})$. Using Lemma 2.2, we have

$$\begin{aligned} \|N_n x - N_n y\| &\leq \epsilon_n \gamma \|f(W_n x) - f(W_n y)\| \\ &\quad + (1 - \epsilon_n \bar{\gamma}) \|B_n T_\mu W_n S_n^K x - B_n T_\mu W_n S_n^K y\| \\ &\leq \epsilon_n \gamma \alpha \|x - y\| + (1 - \epsilon_n \bar{\gamma}) \|x - y\| \\ &= (1 + \epsilon_n \gamma \alpha - \epsilon_n \bar{\gamma}) \|x - y\| \\ &= \beta_n \|x - y\|. \end{aligned}$$

Therefore, by Banach Contraction Principle [[15],p.4], there exist a unique point z_n such that $N_n z_n = z_n$. □

Step2. $\{z_n\}$ is bounded.

Proof. Let $p \in \mathfrak{F}$. Since $S_n^K p = T_\mu p = W_n p = p$, we have

$$\begin{aligned} \|z_n - p\| &= \Big\| \epsilon_n \gamma f(W_n z_n) - \epsilon_n \gamma f(W_n p) + \epsilon_n \gamma f(W_n p) \\ &\quad + (I - \epsilon_n A)(B_n T_\mu W_n S_n^K z_n - p) - \epsilon_n A p \Big\| \\ &\leq \|\epsilon_n \gamma f(W_n z_n) - \epsilon_n \gamma f(W_n p)\| + \|(I - \epsilon_n A)(B_n T_\mu W_n S_n^K z_n - p)\| \\ &\quad + \|\epsilon_n \gamma f(p) - \epsilon_n A p\| \\ &\leq \epsilon_n \gamma \alpha \|z_n - p\| + (1 - \epsilon_n \overline{\gamma}) \Big[\|B_n T_\mu W_n S_n^K z_n - B_n p\| + \|B_n p - p\| \Big] \\ &\quad + \epsilon_n \|\gamma f(p) - A p\| \\ &\leq \epsilon_n \gamma \alpha \|z_n - p\| + (1 - \epsilon_n \overline{\gamma}) \|z_n - p\| + r_n (1 - \epsilon_n \overline{\gamma}) \|B p\| \\ &\quad + \epsilon_n \|\gamma f(p) - A p\|. \end{aligned}$$

Thus,

$$\|z_n - p\| \leq \frac{r_n (1 - \epsilon_n \overline{\gamma})}{\epsilon_n (\overline{\gamma} - \alpha \gamma)} \|B p\| + \frac{1}{\overline{\gamma} - \alpha \gamma} \|\gamma f(p) - A p\|.$$

By (iv),

$$\lim_{n\to\infty} \frac{r_n(1-\epsilon_n\overline{\gamma})}{\epsilon_n(\overline{\gamma}-\alpha\gamma)} = 0,$$

therefore $\{\frac{r_n(1-\epsilon_n\overline{\gamma})}{\epsilon_n(\overline{\gamma}-\alpha\gamma)}\}_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers, hence, the sequence $\{z_n\}$ is bounded. □

Step 3. For every fixed $k \in \{1, 2, \dots, K\}$, we have

$$\lim_n \|S_n^k z_n - S_n^{k+1} z_n\| = 0.$$

Proof. Let $p \in \mathfrak{F}$ and $k \in \{1, 2, \dots, K-1\}$. Put $\rho_n = T_\mu W_n S_n^K z_n$. Since by (ii) of Theorem 2.4, $S_{r_{k+1,n}}^{k+1}$ is firmly nonexpansive, we conclude that

$$\begin{aligned}\|p - S_n^{k+1} z_n\|^2 &= \|S_{r_{k+1,n}}^{k+1} p - S_{r_{k+1,n}}^{k+1} S_n^k z_n\|^2 \\ &\le \left\langle S_{r_{k+1,n}}^{k+1} S_n^k z_n - p\,,\ S_n^k z_n - p\right\rangle \\ &= \frac{1}{2}\Big(\|S_{r_{k+1,n}}^{k+1} S_n^k z_n - p\|^2 + \|S_n^k z_n - p\|^2 \\ &\quad - \|S_n^k z_n - S_{r_{k+1,n}}^{k+1} S_n^k z_n\|^2\Big).\end{aligned}$$

Therefore,

$$\|S_n^{k+1} z_n - p\|^2 \le \|z_n - p\|^2 - \|S_n^k z_n - S_n^{k+1} z_n\|^2. \tag{3.2}$$

Then by using (3.2) and the inequality

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle\,, \tag{3.3}$$

we obtain

$$\begin{aligned}\|z_n - p\|^2 &= \|\epsilon_n(\gamma f(W_n z_n) - Ap) + (I - \epsilon_n A)(B_n\rho_n - p)\|^2 \\ &\le \big(\epsilon_n\|\gamma f(W_n z_n) - Ap\| + (1 - \epsilon_n\overline{\gamma})\|B_n\rho_n - p\|\big)^2 \\ &\le \epsilon_n\|\gamma f(W_n z_n) - Ap\|^2 + \|B_n\rho_n - p\|^2 \\ &\quad + 2\epsilon_n\|\gamma f(W_n z_n) - Ap\|\|B_n\rho_n - p\| \\ &\le \epsilon_n\|\gamma f(W_n z_n) - Ap\|^2 + \|S_n^K z_n - p\|^2 + 2r_n\langle -Bp, B_n\rho_n - p\rangle \\ &\quad + 2\epsilon_n\|\gamma f(W_n z_n) - Ap\|\|B_n\rho_n - p\| \\ &\le \epsilon_n\|\gamma f(W_n z_n) - Ap\|^2 + \|S_n^{k+1} z_n - p\|^2 \\ &\quad + 2r_n\|Bp\|\|B_n\rho_n - p\| + 2\epsilon_n\|\gamma f(W_n z_n) - Ap\|\|B_n\rho_n - p\| \\ &\le \epsilon_n\|\gamma f(W_n z_n) - Ap\|^2 + (\|z_n - p\|^2 - \|S_n^k z_n - S_n^{k+1} z_n\|^2) \\ &\quad + 2r_n\|Bp\|\|B_n\rho_n - p\| + 2\epsilon_n\|\gamma f(W_n z_n) - Ap\|\|B_n\rho_n - p\|.\end{aligned}$$

That is,

$$\|S_n^k z_n - S_n^{k+1} z_n\|^2 \le \epsilon_n\|\gamma f(W_n z_n) - Ap\|^2$$

$$+ 2r_n\|Bp\|\|B_n\rho_n - p\|$$
$$+ 2\epsilon_n\|\gamma f(W_n z_n) - Ap\|\|B_n\rho_n - p\|.$$

Therefore, from (ii), (iv) and that $\{f(W_n z_n)\}$ and $\{B_n\rho_n\}$ are bounded sequences, we conclude

$$\lim_{n\to\infty} \|S_n^k z_n - S_n^{k+1} z_n\| = 0.$$

□

Step 4. $\lim_{n\to\infty} \|z_n - B_n T_\mu W_n S_n^K z_n\| = 0.$

Proof. Since $\{\gamma f(W_n z_n) - AB_n T_\mu W_n S_n^K z_n\}$ is a bounded sequence, we have

$$\begin{aligned}\lim_{n\to\infty} \|z_n - B_n T_\mu W_n S_n^K z_n\| &= \lim_{n\to\infty} \| \epsilon_n \gamma f(W_n z_n) \\ &\quad + (I - \epsilon_n A)B_n T_\mu W_n S_n^K z_n - B_n T_\mu W_n S_n^K z_n \| \\ &= \lim_{n\to\infty} \epsilon_n \|\gamma f(W_n z_n) - AB_n T_\mu W_n S_n^K z_n \| \\ &= 0.\end{aligned}$$

□

Step 5. The weak limit set of $\{z_n\}$ which is denoted by $\omega_\omega\{z_n\}$ is a subset of $\mathfrak{F}$.

Proof. Let $x^* \in \omega_\omega\{z_n\}$, and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup x^*$. We need to show that $x^* \in \mathfrak{F}$.
From Step 3, we obtain also that $S_{n_j}^k z_{n_j} \rightharpoonup x^*$, for all $k \in \{1, \ldots, K\}$.
Note that by (A_2) and given $y \in C$ and $k \in \{0, 1, \ldots, K-1\}$, we have

$$\frac{1}{r_{k+1,n}} \langle y - S_n^{k+1} z_n\, , \, S_n^{k+1} z_n - S_n^k z_n \rangle \geq G_{k+1}\left(y, S_n^{k+1} z_n\right).$$

Thus,

$$\langle y - S_{n_j}^{k+1} z_{n_j}\, , \, \frac{S_{n_j}^{k+1} z_{n_j} - S_{n_j}^k z_{n_j}}{r_{k+1,n_j}} \rangle \geq G_{k+1}\left(y, S_{n_j}^{k+1} z_{n_j}\right). \tag{3.4}$$

By condition (A_4), $G_i(y, .)$ for every i, is lower semicontinuous and convex, and thus weakly semicontinuous. Step 3 and condition $\liminf_n r_{k,n} > 0$ imply that

$$\frac{S_{n_j}^{k+1} z_{n_j} - S_{n_j}^k z_{n_j}}{r_{k+1,n_j}} \to 0$$

in norm. Therefore, letting $j \to \infty$ in 3.4 yields

$$G_{k+1}(y, x^*) \leq \lim_j G_{k+1}\left(y, S_{n_j}^{k+1} z_{n_j}\right) \leq 0,$$

for all $y \in C$ and $k \in \{0, 1, \ldots, K-1\}$. Replacing y with $y_t := ty + (1-t)x^*$ with $t \in (0,1)$, and using (A_1) and (A_4), we obtain

$$0 = G_{k+1}(y_t, y_t) \leq tG_{k+1}(y_t, y) + (1-t)G_{k+1}(y_t, x^*) \leq tG_{k+1}(y_t, y).$$

Hence $G_{k+1}(ty + (1-t)x^*, y) \geq 0$ for all $t \in (0,1)$ and $y \in C$, and $k \in \{0, 1, \ldots, K-1\}$. Letting $t \to 0^+$ and using (A_3), we conclude $G_{k+1}(x^*, y) \geq 0$. Therefore,

$$x^* \in \bigcap_{k=1}^{K} \mathrm{SEP(G_k)} = \mathrm{SEP}(\mathcal{G}). \tag{3.5}$$

Hence, by our assumption, $x^* \in \mathrm{Fix}(\mathcal{S})$.

By Theorem 2.7, we have, for every $z \in C$,

$$W_{n_j} z \to Wz. \tag{3.6}$$

Assume that $x^* \notin \mathrm{Fix(W)}$, then $Wx^* \neq x^*$. Since $x^* \in \mathrm{Fix}(\mathcal{S})$, by our assumption, we have $T_i x^* \in \mathrm{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$, and then $W_n x^* \in \mathrm{Fix}(\mathcal{S})$. Therefore, Since S is a semitopological semigroup, by (iv) of Theorem 2.3, $T_\mu W_n x^* = W_n x^*$. Then, from Opial's property of Hilbert spaces, (3.5), (3.6), and Step 4 we have

$$\begin{aligned}
\liminf_{j\to\infty} \|z_{n_j} - x^*\| &< \liminf_{j\to\infty} \|z_{n_j} - Wx^*\| \\
&\leq \liminf_{j\to\infty} \|z_{n_j} - B_{n_j} T_\mu W_{n_j} S_n^K z_{n_j}\| \\
&\quad + \liminf_{j\to\infty} \|B_{n_j} T_\mu W_{n_j} S_n^K z_{n_j} - B_{n_j} T_\mu W_{n_j} S_n^K x^*\| \\
&\quad + \liminf_{j\to\infty} \|B_{n_j} T_\mu W_{n_j} x^* - T_\mu W_{n_j} x^*\| \\
&\quad + \liminf_{j\to\infty} \|T_\mu W_{n_j} x^* - Wx^*\| \\
&\leq \liminf_{j\to\infty} \|z_{n_j} - B_{n_j} T_\mu W_{n_j} S_n^K z_{n_j}\| + \liminf_{j\to\infty} \|z_{n_j} - x^*\| \\
&\quad + \liminf_{j\to\infty} \|B_{n_j} W_{n_j} x^* - W_{n_j} x^*\| \\
&\quad + \liminf_{j\to\infty} \|W_{n_j} x^* - Wx^*\| \\
&= \liminf_{j\to\infty} \|z_{n_j} - B_{n_j} T_\mu W_{n_j} S_n^K z_{n_j}\| + \liminf_{j\to\infty} \|z_{n_j} - x^*\| \\
&\quad + \liminf_{j\to\infty} r_n \|B W_{n_j} x^*\| + \liminf_{j\to\infty} \|W_{n_j} x^* - Wx^*\| \\
&= \liminf_{j\to\infty} \|z_{n_j} - x^*\|,
\end{aligned}$$

Which is a contradiction. Therefore, x^* must belong to Fix(W). Hence, $x^* \in \mathrm{Fix(W)} \cap \mathrm{SEP}(\mathcal{G})$. By Theorem 2.7, we have $x^* \in (\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)}) \cap \mathrm{SEP}(\mathcal{G})$. Therefore, $x^* \in \mathfrak{F}$. □

Step 6. Let u^* be the unique solution of the variational inequality $VI(\mathfrak{F}, B)$. Then

$$\Gamma_1 := \limsup_n \langle -Bu^* \, , \, z_n - u^* \rangle \leq 0, \tag{3.7}$$

and

$$\Gamma_2 := \limsup_n \langle (\gamma f - A)u^* \, , \, z_n - u^* \rangle \leq 0. \tag{3.8}$$

Proof. Note that, from the definition of Γ_1, and the fact that $\{z_n\}$ is a bounded sequence, we can select a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with the following properties:

(i) $\lim_j \langle -Bu^*, z_{n_j} - u^* \rangle = \Gamma_1$;

(ii) z_{n_j} converge weakly to a point z_1.

By Step 5, we have $z_1 \in \mathfrak{F}$ and then
$\Gamma_1 = \lim_j \langle -Bu^*, z_{n_j} - u^* \rangle = \langle -Bu^*, z_1 - u^* \rangle \leq 0.$
Since $P_{\mathfrak{F}}(I - (A - \gamma f))u^* = u^*$, by Lemma 2.1, $\langle (\gamma f - A)u^* \, , \, z - u^* \rangle \leq 0$, for each $z \in \mathfrak{F}$. Similarly, we can select a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ with the following properties:

(i) $\lim_i \langle (\gamma f - A)u^*, z_{n_i} - u^* \rangle = \Gamma_2$;

(ii) z_{n_i} converge weakly to a point z_2.

By Step 5, we have $z_2 \in \mathfrak{F}$ and then
$\Gamma_2 = \lim_j \langle (\gamma f - A)u^*, z_{n_i} - u^* \rangle = \langle (\gamma f - A)u^*, z_2 - u^* \rangle \leq 0.$ □

Step 7. $\{z_n\}$ converges strongly to u^*.

Proof. Put $u_n = B_n T_\mu W_n S_n^K z_n$. By using the inequality (3.3), we have

$$\begin{aligned}
\|z_n - u^*\|^2 &= \big\|\epsilon_n \gamma (f(W_n z_n) - f(W_n u^*)) + (I - \epsilon_n A)(u_n - B_n u^*) \\
&\quad - \epsilon_n A u^* + \epsilon_n \gamma f(W_n u^*) + (I - \epsilon_n A)(B_n u^* - u^*)\big\|^2 \\
&\leq \big\|\epsilon_n \gamma (f(W_n z_n) - f(W_n u^*)) + (I - \epsilon_n A)(u_n - B_n u^*) \\
&\quad + \epsilon_n \big(\gamma f(u^*) - A u^*\big)\big\|^2 + 2\langle (I - \epsilon_n A)(B_n u^* - u^*) \, , \, z_n - u^* \rangle \\
&\leq \big(\epsilon_n \gamma \|f(W_n z_n) - f(W_n u^*)\|^2 + (1 - \epsilon_n \bar{\gamma})\|u_n - B_n u^*\|^2 \\
&\quad + 2\epsilon_n \gamma (1 - \epsilon_n \bar{\gamma})\|u_n - B_n u^*\| \|f(W_n z_n) - f(W_n u^*)\|\big) \\
&\quad + 2\epsilon_n \langle \gamma f(u^*) - A u^* \, , \, z_n - u^* \rangle \\
&\quad + 2\langle (I - \epsilon_n A)(B_n u^* - u^*) \, , \, z_n - u^* \rangle \\
&\leq \epsilon_n \gamma \|z_n - u^*\|^2 + (1 - \epsilon_n \bar{\gamma})\|z_n - u^*\|^2 + 2\epsilon_n \gamma (1 - \epsilon_n \bar{\gamma})\|z_n - u^*\|^2 \\
&\quad + 2\epsilon_n \langle \gamma f(u^*) - A u^* \, , \, z_n - u^* \rangle + 2 r_n \langle -Bu^* \, , \, z_n - u^* \rangle \\
&\quad + 2\epsilon_n r_n \|ABu^*\| \|z_n - u^*\|,
\end{aligned}$$

therefore,

$$\epsilon_n(\bar{\gamma} - 3\gamma + 2\epsilon_n\gamma\bar{\gamma}))\|z_n - u^*\|^2 \le 2\epsilon_n\langle \gamma f(u^*) - Au^*\,,\, z_n - u^*\rangle + 2r_n\langle -Bu^*\,,\, z_n - u^*\rangle + 2\epsilon_n r_n\|ABu^*\|\|z_n - u^*\|,$$

from Step 2, there exists a positive number M_0 such that $\|z_n - u^*\| \le M_0$ for each $n \in \mathbb{N}$, hence, from (3.7), (3.8), (ii) and (iv), we conclude

$$\limsup_{n\to\infty}\|z_n - u^*\|^2 \le \limsup_{n\to\infty}\frac{2}{\bar{\gamma} - 3\gamma + 2\epsilon_n\gamma\bar{\gamma}}\langle \gamma f(u^*) - Au^*\,,\, z_n - u^*\rangle + \limsup_{n\to\infty}\frac{2r_n}{\epsilon_n(\bar{\gamma} - 3\gamma + 2\epsilon_n\gamma\bar{\gamma})}\langle -Bu^*\,,\, z_n - u^*\rangle + \limsup_{n\to\infty}\frac{2r_n}{\bar{\gamma} - 3\gamma + 2\epsilon_n\gamma\bar{\gamma}}\|ABu^*\|M_0 \le 0.$$

That is $z_n \to x^*$. □

□

4. Examples and applications

Theorem 4.1. *Let S be a semitopological semigroup, and let C be a nonempty closed convex subset of a Hilbert space H, and u be an arbitrary point in H. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings of C into itself. Let X be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a sequence of invariant means on X. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$ for every $i \in \mathbb{N}$, and $\mathcal{G} = \{G_k : k = 1, 2, \dots K\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, such that $\mathrm{SEP}(\mathcal{G}) \subseteq \mathrm{Fix}(\mathcal{S})$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ such that $\|A\| \le 1$, and let B be an η-Lipschitzian, relaxed (u, v)-cocoercive mapping from C into H. Moreover, let $\{r_{k,n}\}_{k=1}^K$, $\{r_n\}$, $\{\epsilon_n\}$, and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $r_n > 0$, $0 < \epsilon_n < 1$, and $0 < \lambda_n \le c < 1$ for some c, and γ is a real number such that $0 < 3\gamma < \overline{\gamma}$. For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$, and $\{\lambda_n\}$ as in (2.1), for every $k \in \{1, 2, \dots, K\}$ and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in* Theorem 2.4. Assume that,

(i) for every $k \in \{1, 2, \dots, K\}$, the function G_k satisfies $(\mathrm{A}_1)-(\mathrm{A}_4)$ of Theorem 2.4,

(ii) $\lim\limits_{n} \epsilon_n = 0$,

(iii) for every $k \in \{1, 2, \dots, K\}$, $\liminf_n r_{k,n}$ exists, and is a positive real number,

(iv) $\{r_n\} \subset [0, b]$ for some b with $0 \leq b \leq \frac{2(v-u\eta^2)}{\eta^2}$, $v > u\eta^2$, $\lim_n r_n = 0$, $\lim_n \frac{r_n}{\epsilon_n} = 0$,

(v) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(\mathrm{T_n}) \cap \mathrm{SEP}(\mathcal{G}) \neq \emptyset$,

(vi) $\mathrm{Fix}(\mathrm{P}_{\mathfrak{F}}(\mathrm{I} - \lambda\mathrm{B})) \cap \{\mathrm{x} \in \mathrm{H} : \mathrm{P}_{\mathfrak{F}}[(\mathrm{I} - \mathrm{A})\mathrm{x} + \mathrm{u}] = \mathrm{x}\} \neq \emptyset$, for each $\lambda > 0$.

Let $\{z_n\}$ be the sequence generated by

$$z_n = \epsilon_n u + (I - \epsilon_n A)(I - r_n B)T_{\mu_n} W_n S^K_{r_{K,n}} \cdots S^2_{r_{2,n}} S^1_{r_{1,n}} z_n, \quad (n \in \mathbb{N}),$$

then $\{z_n\}$ strongly converges to $u^* \in \mathfrak{F}$ which is:

(i) the unique solution of the variational inequalities $VI(\mathfrak{F}, B)$, equivalently $P_{\mathfrak{F}}(I - \lambda B)u^* = u^*$ for each $\lambda > 0$,

(ii) the unique solution of the variational inequality:

$$\langle Au^* - u, x - u^* \rangle \geq 0, \quad (x \in \mathfrak{F}),$$

or equivalently,

$$u^* = P_{\mathfrak{F}}[(I - A)u^* + u],$$

Proof. It suffices to take $f \equiv u$ and $\gamma = 1$ in Theorem 3.1. □

Theorem 4.2. *Let S be a semitopological semigroup, and H be a Hilbert space. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings of H into itself. Let X be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a sequence of invariant means on X. Let $\mathcal{G} = \{G_k : k = 1, 2, \dots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$, such that $\mathrm{SEP}(\mathcal{G}) \subseteq \mathrm{Fix}(\mathcal{S})$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ such that $\|A\| \leq 1$, and let B be an η-Lipschitzian, relaxed (u, v)-cocoercive mapping from H into H, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}_{k=1}^K$, $\{r_n\}$, $\{\epsilon_n\}$, and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $r_n > 0$, $0 < \epsilon_n < 1$, and $0 < \lambda_n \leq c < 1$ for some c, and γ is a real number such that $0 < 3\gamma < \overline{\gamma}$. For every $n \in \mathbb{N}$, let $\{\lambda_n\}$ be as in (2.1), for every $k \in \{1, 2, \dots, K\}$, and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in* Theorem 2.4. Assume that,

(i) for every $k \in \{1, 2, \dots, K\}$, the function G_k satisfies $(\mathrm{A}_1) - (\mathrm{A}_4)$ of Theorem 2.4,

(ii) $\lim_n \epsilon_n = 0$ and,

(iii) for every $k \in \{1, 2, \ldots, K\}$, $\liminf_n r_{k,n}$ exists, and is a positive real number,

(iv) $\{r_n\} \subset [0, b]$ for some b with $0 \leq b \leq \frac{2(v-u\eta^2)}{\eta^2}$, $v > u\eta^2$, $\lim_n r_n = 0$, $\lim_n \frac{r_n}{\epsilon_n} = 0$,

(v) $\mathfrak{F} := \mathrm{SEP}(\mathcal{G}) \neq \emptyset$,

(vi) $\mathrm{Fix}(\mathrm{P}_{\mathfrak{F}}(\mathrm{I} - \lambda \mathrm{B})) \cap \mathrm{Fix}(\mathrm{P}_{\mathfrak{F}}(\mathrm{I} - (\mathrm{A} - \gamma \mathrm{f}))) \neq \emptyset$, for each $\lambda > 0$.

Let $\{z_n\}$ be the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)(I - r_n B) T_{\mu_n} S^K_{r_{K,n}} \cdots S^2_{r_{2,n}} S^1_{r_{1,n}} z_n, \quad (n \in \mathbb{N}),$$

then $\{z_n\}$ strongly converges to $u^* \in \mathfrak{F}$ which is:

(i) the unique solution of the variational inequalities $VI(\mathfrak{F}, B)$, equivalently $P_{\mathfrak{F}}(I - \lambda B)u^* = u^*$ for each $\lambda > 0$,

(ii) the unique solution of the variational inequality:

$$\langle (A - \gamma f)u^*, x - u^* \rangle \geq 0 \quad (x \in \mathfrak{F}),$$

or equivalently,

$$u^* = P_{\mathfrak{F}}(I - (A - \gamma f))u^*,$$

(iii) the unique solution of the minimization problem

$$\min_{x \in \mathfrak{F}} \frac{1}{2} \langle Ax, x \rangle + h(x),$$

where h is a potential function for γf.

Proof. Take $T_i = I$ for every $i \in \mathbb{N}$, and $C = H$ in Theorem 3.1. Then, we have $W_n = I$ for all $n \in \mathbb{N}$. So from Theorem 3.1, the sequences $\{z_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\mathcal{S})$. □

Theorem 4.3. *Let S be a semitopological semigroup, and let H be a Hilbert space. Let X be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y \rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a sequence of invariant means on X. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings from H into itself. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ such that $\|A\| \leq 1$, and let B be an η-Lipschitzian, relaxed (u, v)-cocoercive mapping from H into H, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}_{k=1}^K$, $\{r_n\}$, $\{\epsilon_n\}$, and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $r_n > 0$, $0 < \epsilon_n < 1$, and $0 < \lambda_n \leq c < 1$ for some c, and γ is a real* *number such that $0 < 3\gamma < \overline{\gamma}$. For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$, and $\{\lambda_n\}$ as in (2.1), for every $k \in \{1, 2, \dots, K\}$, and $n \in \mathbb{N}$. Assume that,*

(i) $\lim_n \epsilon_n = 0$,

(ii) *for every $k \in \{1, 2, \dots, K\}$, $\liminf_n r_{k,n}$ exists and is a positive real number,*

(iii) $\{r_n\} \subset [0, b]$ *for some b with* $0 \le b \le \frac{2(v-u\eta^2)}{\eta^2}$, $v > u\eta^2$, $\lim_n r_n = 0$, $\lim_n \frac{r_n}{\epsilon_n} = 0$,

(v) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \neq \emptyset$,

(vi) $\mathrm{Fix(P_{\mathfrak{F}}(I - \lambda B))} \cap \mathrm{Fix(P_{\mathfrak{F}}(I - (A - \gamma f)))} \neq \emptyset$, *for each $\lambda > 0$.*

Let $\{z_n\}$ be the sequence generated by

$$z_n = \epsilon_n \gamma f(W_n z_n) + (I - \epsilon_n A)(I - r_n B)W_n z_n, \quad (n \in \mathbb{N}),$$

then, $\{z_n\}$ strongly converges to $u^* \in \mathfrak{F}$ which is:

(i) *the unique solution of the variational inequalities $VI(\mathfrak{F}, B)$, equivalently $P_{\mathfrak{F}}(I - \lambda B)u^* = u^*$ for each $\lambda > 0$,*

(ii) *the unique solution of the variational inequality:*

$$\langle (A - \gamma f)u^*, x - u^* \rangle \ge 0, \quad (x \in \mathfrak{F}),$$

or equivalently,

$$u^* = P_{\mathfrak{F}}(I - (A - \gamma f))u^*,$$

(iii) *the unique solution of the minimization problem*

$$\min_{x\in\mathfrak{F}} \frac{1}{2}\langle Ax, x \rangle + h(x),$$

where h is a potential function for γf.

Proof. Take $G_k = 0$ for every $k \in \{1, 2, \dots, K\}$, $\mathcal{S} = \{I\}$ and $C = H$ in Theorem 3.1. Then, we have $\mathrm{Fix}(\mathcal{S}) = \mathrm{H}$ and $S^K_{r_{K,n}} \cdots S^2_{r_{2,n}} S^1_{r_{1,n}} z_n = z_n$. So from Theorem 3.1, the sequences $\{z_n\}$ converges strongly to x^*. □

Remark 4.4. Since v-strongly monotone mappings are relaxed (u, v)-cocoercive, Theorem 3.1 is valid if we replace the relaxed (u, v)-cocoercive condition on B by condition that B is an r-strongly monotone mapping.

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