Second dual space of little $\alpha$-Lipschitz vector-valued operator algebras

Abbasali Shokri

Abstract. Let $(X,d)$ be an infinite compact metric space, let $(B,\|\cdot\|)$ be a unital Banach space, and take $\alpha \in (0,1)$. In this work, at the first we define the big and little $\alpha$-Lipschitz vector-valued (B-valued) operator algebras, and consider the little $\alpha$-lipschitz $B$-valued operator algebra, $lip_\alpha(X,B)$. Then we characterize its second dual space.

1. Introduction

A function $f$ from a metric space $(X,d)$ into $\mathbb{R}$ or $\mathbb{C}$ is called a Lipschitz function if there exists a constant $M > 0$ such that the following condition holds:

$$|f(x) - f(y)| \leq M d(x,y), \quad (x,y \in X),$$

or

$$\frac{|f(x) - f(y)|}{d(x,y)} \leq M, \quad (x,y \in X, x \neq y).$$

In this case, $M$ is called the Lipschitz constant of function $f$.

The space $\text{Lip}(X,\mathbb{R})$ consisting of all Lipschitz functions from $X$ into $\mathbb{R}$ which is a Banach space, is called Lipschitz space, which has many interesting and important properties.

Let $(X,d)$ be an infinite compact metric space, and let $(B,\|\cdot\|)$ be a unital Banach space over the scaler field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$. Let $C(X,B)$ be the set of all continuous $B$-valued operators from $X$ into $B$, and for

2010 Mathematics Subject Classification. 47B48, 47C05, 47L50.

Key words and phrases. Second dual space, $\alpha$-Lipschitz operator, Vector-valued operator.

Received: 12 March 2016, Accepted: 4 July 2016.
each $f \in C(X, B)$, define
\[
\| f \|_{\infty} := \sup_{x \in X} \| f(x) \|.
\]

For $f, g \in C(X, B)$ and $\lambda \in \mathbb{F}$, define
\[
(f + g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x), \quad (x \in X).
\]

It is easy to see that \( (C(X, B), \| \cdot \|_{\infty}) \) is a Banach space over \( \mathbb{F} \).

For a constant \( 0 < \alpha \leq 1 \), an operator \( f \in C(X, B) \) is called a \( \alpha \)-Lipschitz \( B \)-valued operator if there exists a constant \( M > 0 \) such that the following condition holds:
\[
\| f(x) - f(y) \| \leq M d^\alpha(x, y), \quad (x, y \in X),
\]
or
\[
\frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} \leq M, \quad (x, y \in X, x \neq y).
\]

Set
\[
p_\alpha(f) := \sup_{x \neq y} \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)}, \quad (x, y \in X),
\]
which is called the \( \alpha \)-Lipschitz constant of operator \( f \). Define
\[
Lip_\alpha(X, B) := \{ f \in C(X, B) : p_\alpha(f) < \infty \},
\]
and for \( 0 < \alpha < 1 \), define
\[
lip_\alpha(X, B) := \left\{ f \in Lip_\alpha(X, B) : \lim_{d(x, y) \to 0} \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} = 0 \right\},
\]
where \( x, y \in X, x \neq y \). The elements of \( Lip_\alpha(X, B) \) and \( lip_\alpha(X, B) \) are called big and little \( \alpha \)-Lipschitz operators, respectively [1]. For each element \( f \) of \( Lip_\alpha(X, B) \), define
\[
\| f \|_{\alpha} := \| f \|_{\infty} + p_\alpha(f).
\]

Cao, Zhang and Xu proved that \( (Lip_\alpha(X, B), \| \cdot \|_{\alpha}) \) is a Banach space over \( \mathbb{F} \) and \( lip_\alpha(X, B) \) is a closed linear subspace of \( (Lip_\alpha(X, B), \| \cdot \|_{\alpha}) \) [3]. It is clear that \( Lip_\alpha(X, B) \) is a linear subspace of \( C(X, B) \). Sherbert [10], Weaver [11], Johnson [8], Cao and Xu [3], Honary and Mahyar [7], Abdollahi [1], Alimohammadi [2], Pavlovic [4], Ebadian [6], and others, studied some properties of Lipschitz algebras. In this paper, we will study the second dual space of \( lip_\alpha(X, B) \).
2. Preliminaries

In this section and section 3, we use \((X,d)\) to denote an infinite compact metric space, \((B,\| \cdot \|)\) a unital Banach space over the scalar field \(\mathbb{F}(=\mathbb{R}\text{ or }\mathbb{C})\) with unit \(e\), \(\sigma(B)\) the spectrum of \(B\), \(\left(\text{lip}_\alpha(X,B)\right)^\ast\) the dual space and \(\left(\text{lip}_\alpha(X,B)\right)^{**}\) the second dual space of \(\text{lip}_\alpha(X,B)\), respectively.

In this section, we need the following lemma.

**Lemma 2.1.** Let \(x \in X\) and \(\Lambda \in \sigma(B)\) be arbitrary and fix. Then for constant \(0 < \alpha < 1\), the following map is continuous:

\[
h_x : \text{lip}_\alpha(X,B) \to \mathbb{C},
\]

\[
h_x(f) = \langle f, h_x \rangle := \Lambda(f(x)).
\]

**Proof.** Let \(f, g \in \text{lip}_\alpha(X,B)\) be arbitrary such that \(f \to g\) (pointwise). Then we have

\[
|h_x(f) - h_x(g)| = |\Lambda(f(x)) - \Lambda(g(x))| = |\Lambda(f(x) - g(x))| \leq \| \Lambda \| \| f(x) - g(x) \| < \varepsilon.
\]

So the map \(h_x\) is continuous for every \(x \in X\). \(\Box\)

**Corollary 2.2.** For any \(x \in X\), we have \(h_x \in \left(\text{lip}_\alpha(X,B)\right)^\ast\), where \(h_x\) is defined in Lemma 2.1.

**Definition 2.3.** For every \(\phi \in \left(\text{lip}_\alpha(X,B)\right)^{**}\), we define the map

\[
\tau : \left(\text{lip}_\alpha(X,B)\right)^{**} \to \text{Lip}_\alpha(X,B),
\]

\[
\phi \mapsto \tau(\phi),
\]

where \(\tau(\phi) : X \to B\) defined by

\[
\tau(\phi)(x) := \begin{cases} 
\phi(x), & \phi \in \text{lip}_\alpha(X,B), \\
(h_x,\phi)e, & \phi \in \left(\text{lip}_\alpha(X,B)\right)^{**}\setminus\text{lip}_\alpha(X,B),
\end{cases}
\]

where \(h_x\) is defined in Lemma 2.1.

Obviously, the first criterion is well-defined. The second criterion by the Lemma 2.1 and the following theorem is well-defined.

**Theorem 2.4.** For every \(\phi \in \left(\text{lip}_\alpha(X,B)\right)^{**}\), the map \(\tau(\phi) : X \to B\), defined by Definition 2.3, is continuous and \(\tau(\phi) \in \text{Lip}_\alpha(X,B)\).
Proof. Since $\phi \in \left( \text{lip}_\alpha(X,B) \right)^{**}$ is continuous, and for any $x \in X$, $h_x$ is continuous (by Lemma 2.1), $\tau(\phi)$ is continuous.

Now we prove that $\tau(\phi) \in \text{Lip}_\alpha(X,B)$. For this, suppose that $\phi \in \left( \text{lip}_\alpha(X,B) \right)^{**}$ is arbitrary.

Case 1. If $\phi \in \text{lip}_\alpha(X,B)$, then $\tau(\phi) = \phi$, and so $\tau(\phi) \in \text{lip}_\alpha(X,B) \subset \text{Lip}_\alpha(X,B)$.

Case 2. If $\phi \in \left( \text{lip}_\alpha(X,B) \right)^{**} \setminus \text{lip}_\alpha(X,B)$, then for every $x, y \in X$ with $x \neq y$ and $0 < \alpha < 1$, we have

$$p_\alpha(\tau(\phi)) = \sup_{x \neq y} \left\| \frac{\tau(\phi)(x) - \tau(\phi)(y)}{d^\alpha(x,y)} \right\|$$

$$= \sup_{x \neq y} \left\| \frac{\langle h_x, \phi \rangle e - \langle h_y, \phi \rangle e}{d^\alpha(x,y)} \right\|$$

$$= \sup_{x \neq y} \left\| \frac{\left( \langle h_x, \phi \rangle - \langle h_y, \phi \rangle \right) e}{d^\alpha(x,y)} \right\|$$

$$= \left\| e \right\| \sup_{x \neq y} \left\| \frac{\langle h_x, \phi \rangle - \langle h_y, \phi \rangle}{d^\alpha(x,y)} \right\|$$

$$\leq \left\| e \right\| \sup_{x \neq y} \left\| \frac{h_x - h_y}{d^\alpha(x,y)} \right\| \left( \left\| e \right\| = 1 \right)$$

$$= \left\| e \right\| \sup_{x \neq y, \left\| f \right\| \alpha \leq 1} \left\| \frac{\left( h_x - h_y \right)(f)}{d^\alpha(x,y)} \right\| \left( f \in \text{lip}_\alpha(X,B) \right)$$

$$= \left\| e \right\| \sup_{x \neq y, \left\| f \right\| \alpha \leq 1} \left\| \frac{h_x(f) - h_y(f)}{d^\alpha(x,y)} \right\|$$

$$= \left\| e \right\| \sup_{x \neq y, \left\| f \right\| \alpha \leq 1} \left\| \frac{\Lambda(f(x)) - \Lambda(f(y))}{d^\alpha(x,y)} \right\| \left( \Lambda \in \sigma(B) \right)$$

$$= \left\| e \right\| \sup_{x \neq y, \left\| f \right\| \alpha \leq 1} \left\| \frac{\Lambda(f(x) - f(y))}{d^\alpha(x,y)} \right\|$$

$$\leq \left\| e \right\| \sup_{x \neq y, \left\| f \right\| \alpha \leq 1} \left\| \frac{\Lambda(f(x) - f(y))}{d^\alpha(x,y)} \right\|$$

Then for every $\phi \in \left( \text{lip}_\alpha(X,B) \right)^{**}$, we have $\tau(\phi) \in \text{Lip}_\alpha(X,B)$. □
3. The Main results

In this section, we prove that the Banach algebra \((\text{lip}_\alpha(X, B))^{**}\) is isometrically isomorphic to \(\text{Lip}_\alpha(X, B)\).

**Theorem 3.1.** The map \(\tau\) defined by Definition \(\text{lip}_\alpha(X, B)\) is an isometry.

**Proof.** Let \(\phi \in (\text{lip}_\alpha(X, B))^{**}\) be arbitrary.

**Case 1.** If \(\phi \in \text{lip}_\alpha(X, B)\), then by Definition \(\text{lip}_\alpha(X, B)\), \(\tau(\phi) = \phi\). So

\[
\| \tau(\phi) \|_\alpha = \| \phi \|_\alpha.
\]

**Case 2.** If \(\phi \in (\text{lip}_\alpha(X, B))^{**}\) \(\setminus\text{lip}_\alpha(X, B)\), then we have

\[
\| \tau(\phi) \|_\alpha = \| \tau(\phi) \|_\infty + p_\alpha(\tau(\phi))
\]

\[
= \sup_{x \in X} \| \tau(\phi)(x) \| + \sup_{x \neq y} \frac{\| \tau(\phi)(x) - \tau(\phi)(y) \|}{d^\alpha(x, y)}
\]

\[
= \sup_{x \in X} \| \langle h_x, \phi \rangle e \| + \sup_{x \neq y} \frac{\| \langle h_x, \phi \rangle e - \langle h_y, \phi \rangle e \|}{d^\alpha(x, y)}
\]

\[
= \sup_{x \in X} \| \langle h_x, \phi \rangle e \| + \sup_{x \neq y} \frac{\| \langle h_x, \phi \rangle - \langle h_y, \phi \rangle \| e \|}{d^\alpha(x, y)}
\]

\[
= \sup_{x \in X} \| \langle h_x, \phi \rangle \| + \sup_{x \neq y} \frac{\| \langle h_x, \phi \rangle - \langle h_y, \phi \rangle \|}{d^\alpha(x, y)}
\]

\[
= \| \phi \|_\infty + p_\alpha(\phi)
\]

\[
= \| \phi \|_\alpha,
\]

where \(h_x \in (\text{lip}_\alpha(X, B))^*\). In any case, \(\tau\) is an isometry. So the proof is complete.

**Corollary 3.2.** Since the map \(\tau : (\text{lip}_\alpha(X, B))^{**} \to \text{Lip}_\alpha(X, B)\), defined by Definition \(\text{lip}_\alpha(X, B)\), is an isometry (by Theorem 3.1), \(\tau\) is a one-to-one operator.

**Theorem 3.3.** The map \(\tau\) defined by Definition \(\text{lip}_\alpha(X, B)\) is onto.

**Proof.** Let \(x \in X\) and \(\Lambda \in \sigma(B)\) are arbitrary and fix. Define

\[
\psi : \text{Lip}_\alpha(X, B) \to (\text{lip}_\alpha(X, B))^{**},
\]

\[
S \mapsto \psi(S),
\]

in which, if \(S \in \text{lip}_\alpha(X, B)\), then \(\psi(S) := S\). Also if \(S \in \text{Lip}_\alpha(X, B) \setminus \text{lip}_\alpha(X, B)\), then

\[
(\psi(S))(h_x) := \Lambda ((S(x)) \quad (x \in X),
\]
where \( h_x \) is defined in Lemma 2.1. Now, let \( H \in Lip_\alpha(X,B) \) be arbitrary.

**Case 1.** If \( H \in lip_\alpha(X,B) \), then 

\[
(\tau(\psi(H)))(x) = (\tau(H))(x) = H(x).
\]

**Case 2.** If \( H \in Lip_\alpha(X,B) \setminus lip_\alpha(X,B) \), then 

\[
(\tau(\psi(H)))(x) = \langle h_x, \psi(H) \rangle e
= \left( (\psi(H))(h_x) \right) e
= \left( \Lambda(H(x)) \right) e
= H(x).
\]

Hence \( \tau(\psi(H)) = H \). So \( \tau \) is an onto map. \( \square \)

**Theorem 3.4.** The map \( \tau \) defined by Definition 2.3 is a homomorphism.

**Proof.** Let \( \phi_1, \phi_2 \in \left( lip_\alpha(X,B) \right)^{**} \) be arbitrary. We note that \( \tau \) acts on \( \phi_1 \phi_2 \), when \( \phi_1 \phi_2 \in lip_\alpha(X,B) \) or \( \phi_1 \phi_2 \in \left( lip_\alpha(X,B) \right)^{**} \setminus lip_\alpha(X,B) \).

**Case 1.** Let \( \phi_1 \phi_2 \in lip_\alpha(X,B) \). Then by Definition 2.3, \( \tau(\phi_1) = \phi_1 \) and \( \tau(\phi_2) = \phi_2 \). So \( \tau(\phi_1 \phi_2) = \phi_1 \phi_2 = \tau(\phi_1) \tau(\phi_2) \).

**Case 2.** Let \( \phi_1 \phi_2 \in \left( lip_\alpha(X,B) \right)^{**} \setminus lip_\alpha(X,B) \). Then for every \( x \in X \) we have 

\[
(\tau(\phi_1 \phi_2))(x) = \langle h_x, \phi_1 \phi_2 \rangle e
= \left( \langle h_x, \phi_1 \rangle \langle h_x, \phi_2 \rangle \right) e
= \left( \langle h_x, \phi_1 \rangle e \right) \left( \langle h_x, \phi_2 \rangle e \right)
= \left( (\tau(\phi_1))(x) \right) \left( (\tau(\phi_2))(x) \right)
= \left( (\tau(\phi_1) \tau(\phi_2))(x) \right),
\]

where \( h_x \) is defined by Lemma 2.1. So \( \tau(\phi_1 \phi_2) = \tau(\phi_1) \tau(\phi_2) \). \( \square \)

**Remark 3.5.** Let \( A \) be a Banach algebra. Now, we shall define two products on the Banach space \( A^{**} \).

The product map \( m_A : A \times A \rightarrow A \), \( (a, b) \mapsto ab; a, b \in A \), is a continuous bilinear map. So there is an extension of \( m_A \) to a continuous bilinear map \( \widetilde{m}_A : A^{**} \times A^{**} \rightarrow A^{**} \). We define \( \phi \square \psi := \widetilde{m}_A(\phi, \psi) \) \( \forall \phi, \psi \in A^{**} \). For \( a \in A \) and \( \phi \in A^{**} \), we have 

\[
a \square \phi = a \cdot \phi = \phi \cdot a,
\]

where \( \cdot \) denotes the module operation in \( A^{**} \). The above product \( \square \) on \( A^{**} \) is such that \( (A^{**}, \square) \) is a Banach algebra containing \( A \) as a closed subalgebra.
There is a second way of defining a product in $A^{**}$ to give an algebra $(A^{**}, \diamond) :$ indeed, we define $(A^{**}, \diamond) := ((A^{op})^{**}, \square)^{op}$, where $A^{op}$ is the opposite algebra to $A$; $A^{op}$ is the algebra formed by reversing the order of the product in $A$. So $\diamond$ corresponds to the extension $m_A$ of $m_A$ which:

$\tilde{m}_A : A^{**} \times A^{**} \rightarrow A^{**}$,
$\tilde{m}_A(\phi, \psi) := \tilde{m}_A(\psi, \phi)$.

$(A^{**}, \diamond)$ is a Banach algebra containing $A$ as a closed subalgebra.

In general, the two products $\square$ and $\diamond$ are distinct. Let $a \in A$, $\lambda \in A^*$ and $\phi, \psi \in A^{**}$. Then we have

$\langle \lambda, \phi \square \psi \rangle = \langle \psi, \lambda, \phi \rangle$, \hspace{1em} $\langle \lambda, \phi \diamond \psi \rangle = \langle \lambda, \phi \psi \rangle$.

The products $\square$ and $\diamond$ are the first and second Arens products, respectively, on $A^{**}$. The Banach algebra $A$ is called Arens regular if these two products coincide on $A^{**}$.

**Theorem 3.6.** Let $(X, d)$ be an infinite compact metric space, let $(B, \| \cdot \|)$ be a unital Banach space, and take $\alpha \in (0, 1)$. Then $\operatorname{lip}_\alpha(X, B)$ is Arens regular, and the Banach algebra $(\operatorname{lip}_\alpha(X, B))^{**}$ is isometrically isomorphic to $\operatorname{Lip}_\alpha(X, B)$.

**Proof.** For every $\phi \in (\operatorname{lip}_\alpha(X, B))^{**}$, we define the map

$\tau : (\operatorname{lip}_\alpha(X, B))^{**} \rightarrow \operatorname{Lip}_\alpha(X, B),$

$\phi \mapsto \tau(\phi),$

where $\tau(\phi) : X \rightarrow B$ is defined by Definition 2.3.

We note that, $\tau$, by Theorem 2.4 is well-defined, by Theorem 3.1 is an isometry, by Corollary 3.2 is one-to-one, by Theorem 3.3 is onto and by Theorem 3.4 is a homomorphism. Then $\tau$ is a isomorphism and isometry. Now we prove that $\operatorname{lip}_\alpha(X, B)$ is Arens regular.

Let $\phi, \psi \in (\operatorname{lip}_\alpha(X, B))^{**}$. Then we have $\tau(\phi), \tau(\psi) \in \operatorname{Lip}_\alpha(X, B)$. Take $F = \tau(\phi)$ and $G = \tau(\psi)$. For every $h_x \in (\operatorname{lip}_\alpha(X, B))^\ast$ ($x \in X$) and $f \in \operatorname{lip}(X, B)$, we define the operation $h_x \cdot f = h_x(f)h_x$. So for every
$x \in X$ and any $f \in \text{lip}_\alpha (X, B)$ we have
\[
\langle f, \psi \cdot h_x \rangle e = \langle h_x \cdot f, \psi \rangle e \\
= \langle h_x (f) h_x, \psi \rangle e \\
= h_x (f) \langle h_x, \psi \rangle e \\
= h_x (f) \tau (\psi) (x) \\
= G (x) h_x (f) \\
= G (x) h_x (f) e \\
= \langle f, G (x) h_x \rangle e.
\]
So $\psi \cdot h_x = G (x) h_x$ for every $x \in X$. Therefore
\[
\tau (\phi \Box \psi) (x) = \langle h_x, \phi \Box \psi \rangle e \\
= \langle \psi \cdot h_x, \phi \rangle e \\
= \langle G (x) h_x, \phi \rangle e \\
= G (x) \langle h_x, \phi \rangle e \\
= G (x) \tau (\phi) (x) \\
= G (x) F (x) \\
= (G F) (x).
\]
Then we get $\tau (\phi \Box \psi) = GF$. Similarly, we obtain $\tau (\phi \Diamond \psi) = GF$. Therefore $\text{lip}_\alpha (X, B)$ is Arens regular.

Acknowledgment. The author wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

References


Department of Mathematics, Ahar Branch, Islamic Azad University, Ahar, Iran.

E-mail address: a-shokri@iau-ahar.ac.ir