Sahand Communications in Mathematical Analysis (SCMA) Vol. 8 No. 1 (2017), 33-41
http://scma.maragheh.ac.ir

Second dual space of little α-Lipschitz vector-valued operator algebras

Abbasali Shokri

Abstract. Let (X, d) be an infinite compact metric space, let $(B, \| \,.\, \|)$ be a unital Banach space, and take $\alpha \in (0,1)$. In this work, at the first we define the big and little α-Lipschitz vector-valued (B-valued) operator algebras, and consider the little α-lipschitz B-valued operator algebra, $lip_\alpha(X, B)$. Then we characterize its second dual space.

1. Introduction

A function f from a metric space (X, d) into $\mathbb{R}$ or $\mathbb{C}$ is called a Lipschitz function if there exists a constant $M > 0$ such that the following condition holds:

$$| f(x) - f(y) | \leq M d(x, y), \quad (x, y \in X),$$

or

$$\frac{| f(x) - f(y) |}{d(x, y)} \leq M, \quad (x, y \in X, x \neq y).$$

In this case, M is called the Lipschitz constant of function f.

The space $Lip(X, \mathbb{R})$ consisting of all Lipschitz functions from X into $\mathbb{R}$ which is a Banach space, is called Lipschitz space, which has many interesting and important properties.

Let (X, d) be an infinite compact metric space, and let $(B, \| \,.\, \|)$ be a unital Banach space over the scaler field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$. Let $C(X, B)$ be the set of all continuous B-valued operators from X into B, and for

2010 *Mathematics Subject Classification.* 47B48, 47C05, 47L50.
Key words and phrases. Second dual space, α-Lipschitz operator, Vector-valued operator.
Received: 12 March 2016, Accepted: 4 July 2016.

each $f \in C(X,B)$, define

$$\| f \|_\infty := \sup_{x\in X} \| f(x) \| .$$

For $f, g \in C(X,B)$ and $\lambda \in \mathbb{F}$, define

$$(f+g)(x) := f(x) + g(x), \qquad (\lambda f)(x) := \lambda f(x), \quad (x \in X).$$

It is easy to see that $\Big(C(X,B), \| \,.\, \|_\infty \Big)$ is a Banach space over $\mathbb{F}$.

For a constant $0 < \alpha \leq 1$, an operator $f \in C(X,B)$ is called a $\alpha-$Lipschitz $B-$valued operator if there exists a constant $M > 0$ such that the following condition holds:

$$\| f(x) - f(y) \| \leq M d^\alpha(x,y), \quad (x, y \in X),$$

or

$$\frac{\| f(x) - f(y) \|}{d^\alpha(x,y)} \leq M, \quad (x, y \in X, x \neq y).$$

Set

$$p_\alpha(f) := \sup_{x\neq y} \frac{\| f(x) - f(y) \|}{d^\alpha(x,y)}, \quad (x, y \in X),$$

which is called the α-Lipschitz constant of operator f. Define

$$Lip_\alpha(X,B) := \Big\{ f \in C(X,B) : \quad p_\alpha(f) < \infty \Big\},$$

and for $0 < \alpha < 1$, define

$$lip_\alpha(X,B) := \left\{ f \in Lip_\alpha(X,B) : \lim_{d(x,y)\to 0} \frac{\| f(x) - f(y) \|}{d^\alpha(x,y)} = 0 \right\},$$

where $x, y \in X, x \neq y$. The elements of $Lip_\alpha(X,B)$ and $lip_\alpha(X,B)$ are called big and little α-Lipschitz operators, respectively [4]. For each element f of $Lip_\alpha(X,B)$, define

$$\| f \|_\alpha := \| f \|_\infty + p_\alpha(f).$$

Cao, Zhang and Xu proved that $\Big(\mathrm{Lip}_\alpha(X,B), \| \,.\, \|_\alpha \Big)$ is a Banach space over $\mathbb{F}$ and $lip_\alpha(X,B)$ is a closed linear subspace of $\Big(Lip_\alpha(X,B), \| \,.\, \|_\alpha \Big)$ [4]. It is clear that $Lip_\alpha(X,B)$ is a linear subspace of $C(X,B)$. Sherbert [10], Weaver [11], Johnson [8], Cao and Xu [3], Honary and Mahyar [7], Abdollahi [1], Alimohammadi [2], Pavlovic [9], Ebadian [6], and others, studied some properties of Lipschitz algebras. In this paper, we will study the second dual space of $lip_\alpha(X,B)$.

2. Preliminaries

In this section and section 3, we use (X, d) to denote an infinite compact metric space, $(B, \| \,.\, \|)$ a unital Banach space over the scalar filed $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ with unit $\mathbf{e}$, $\sigma(B)$ the spectrum of B, $\Big(lip_\alpha(X,B)\Big)^*$ the dual space and $\Big(lip_\alpha(X,B)\Big)^{**}$ the second dual space of$lip_\alpha(X,B)$, respectively.

In this section, we need the following lemma.

Lemma 2.1. *Let $x \in X$ and $\Lambda \in \sigma(B)$ be arbitrary and fix. Then for constant $0 < \alpha < 1$, the following map is continuous:*

$$h_x : lip_\alpha(X,B) \longrightarrow \mathbb{C},$$
$$h_x(f) = \langle f, h_x \rangle := \Lambda\big(f(x)\big).$$

Proof. Let $f, g \in lip_\alpha(X,B)$ be arbitrary such that $f \to g$ (pointwise). Then we have

$$\begin{aligned}\Big|h_x(f) - h_x(g)\Big| &= \Big|\Lambda\big(f(x)\big) - \Lambda\big(g(x)\big)\Big| \\ &= \Big|\Lambda\big(f(x) - g(x)\big)\Big| \\ &\leq \| \Lambda \| \| f(x) - g(x) \| \\ &< \varepsilon.\end{aligned}$$

So the map h_x is continuous for every $x \in X$. $\square$

Corollary 2.2. *For any $x \in X$, we have $h_x \in \Big(lip_\alpha(X,B)\Big)^*$, where h_x is defined in Lemma 2.1.*

Definition 2.3. For every $\phi \in \Big(lip_\alpha(X,B)\Big)^{**}$, we define the map

$$\tau : \Big(lip_\alpha(X,B)\Big)^{**} \longrightarrow Lip_\alpha(X,B),$$
$$\phi \mapsto \tau(\phi),$$

where $\tau(\phi) : X \longrightarrow B$ defined by

$$\tau(\phi)(x) := \begin{cases} \phi(x), & \phi \in lip_\alpha(X,B), \\ \langle h_x, \phi \rangle \mathbf{e}, & \phi \in \Big(lip_\alpha(X,B)\Big)^{**} \backslash lip_\alpha(X,B), \end{cases}$$

where h_x is defined in Lemma 2.1.

Obviously, the first criterion is well-defined. The second criterion by the Lemma 2.1 and the following theorem is well-defined.

Theorem 2.4. *For every $\phi \in \Big(lip_\alpha(X,B)\Big)^{**}$, the map $\tau(\phi) : X \longrightarrow B$, defined by Definition 2.3, is continuous and $\tau(\phi) \in Lip_\alpha(X,B)$.*

Proof. Since $\phi \in \Big(lip_\alpha(X,B)\Big)^{**}$ is continuous, and for any $x \in X$, h_x is continuous (by Lemma 2.1), $\tau(\phi)$ is continuous.

Now we prove that $\tau(\phi) \in Lip_\alpha(X,B)$. For this, suppose that $\phi \in \Big(lip_\alpha(X,B)\Big)^{**}$ is arbitrary.

Case1. If $\phi \in lip_\alpha(X,B)$, then $\tau(\phi) = \phi$, and so

$$\tau(\phi) \in lip_\alpha(X,B) \subset Lip_\alpha(X,B).$$

Case2. If $\phi \in \Big(lip_\alpha(X,B)\Big)^{**} \backslash lip_\alpha(X,B)$, then for every $x, y \in X$ with $x \neq y$ and $0 < \alpha < 1$, we have

$$\begin{aligned}
p_\alpha\big(\tau(\phi)\big) &= \sup_{x\neq y} \frac{\| \tau(\phi)(x) - \tau(\phi)(y) \|}{d^\alpha(x,y)} \\
&= \sup_{x\neq y} \frac{\| \langle h_x, \phi\rangle \mathbf{e} - \langle h_y, \phi\rangle \mathbf{e} \|}{d^\alpha(x,y)} \\
&= \sup_{x\neq y} \frac{\| \big(\langle h_x, \phi\rangle - \langle h_y, \phi\rangle\big) \mathbf{e} \|}{d^\alpha(x,y)} \\
&= \| \mathbf{e} \| \sup_{x\neq y} \frac{\big|\langle h_x, \phi\rangle - \langle h_y, \phi\rangle\big|}{d^\alpha(x,y)} \\
&\leq \| \phi \|_\alpha \sup_{x\neq y} \frac{\| h_x - h_y \|}{d^\alpha(x,y)} \qquad \Big(\| \mathbf{e} \| = 1\Big) \\
&= \| \phi \|_\alpha \sup_{x\neq y, \|f\|_\alpha \leq 1} \frac{| \big(h_x - h_y\big)(f) |}{d^\alpha(x,y)} \qquad \Big(f \in lip_\alpha(X,B)\Big) \\
&= \| \phi \|_\alpha \sup_{x\neq y, \|f\|_\alpha \leq 1} \frac{\big|h_x(f) - h_y(f)\big|}{d^\alpha(x,y)} \\
&= \| \phi \|_\alpha \sup_{x\neq y, \|f\|_\alpha \leq 1} \frac{\big|\Lambda\big(f(x)\big) - \Lambda\big(f(y)\big)\big|}{d^\alpha(x,y)} \\
&= \| \phi \|_\alpha \sup_{x\neq y, \|f\|_\alpha \leq 1} \frac{\big|\Lambda\big(f(x) - f(y)\big)\big|}{d^\alpha(x,y)} \qquad \big(\Lambda \in \sigma(B)\big) \\
&\leq \| \phi \|_\alpha \| \Lambda \| \sup_{x\neq y, \|f\|_\alpha \leq 1} \frac{\| f(x) - f(y) \|}{d^\alpha(x,y)} \\
&\leq \| \phi \|_\alpha \| \Lambda \| \| f \|_\alpha \sup_{x\neq y} \frac{d(x,y)}{d^\alpha(x,y)} < \infty.
\end{aligned}$$

Then for every $\phi \in \Big(lip_\alpha(X,B)\Big)^{**}$, we have $\tau(\phi) \in Lip_\alpha(X,B)$. □

3. The Main results

In this section, we prove that the Banach algebra $\Big(lip_\alpha(X,B)\Big)^{**}$ is isometrically isomorphic to $Lip_\alpha(X,B)$.

Theorem 3.1. *The map τ defined by Definition 2.3 is an isometry.*

Proof. Let $\phi \in \big(lip_\alpha(X,B)\big)^{**}$ be arbitrary.
Case1. If $\phi \in lip_\alpha(X,B)$, then by Definition 2.3, $\tau(\phi)=\phi$. So

$$\parallel \tau(\phi) \parallel_\alpha = \parallel \phi \parallel_\alpha .$$

Case2. If $\phi \in \big(lip_\alpha(X,B)\big)^{**}\backslash lip_\alpha(X,B)$, then we have

$$\begin{aligned}
\parallel \tau(\phi) \parallel_\alpha &= \parallel \tau(\phi) \parallel_\infty + p_\alpha(\tau(\phi)) \\
&= \sup_{x\in X} \big|\big|\tau(\phi)(x)\big|\big| + \sup_{x\neq y} \frac{\big|\big|\tau(\phi)(x)-\tau(\phi)(y)\big|\big|}{d^\alpha(x,y)} \\
&= \sup_{x\in X} \big|\big|\langle h_x,\phi\rangle \mathbf{e}\big|\big| + \sup_{x\neq y} \frac{\big|\big|\langle h_x,\phi\rangle \mathbf{e} - \langle h_y,\phi\rangle \mathbf{e}\big|\big|}{d^\alpha(x,y)} \\
&= \sup_{x\in X} \big|\langle h_x,\phi\rangle\big| \parallel \mathbf{e} \parallel + \sup_{x\neq y} \frac{\big|\langle h_x,\phi\rangle - \langle h_y,\phi\rangle\big| \parallel \mathbf{e} \parallel}{d^\alpha(x,y)} \\
&= \sup_{x\in X} \big|\langle h_x,\phi\rangle\big| + \sup_{x\neq y} \frac{\big|\langle h_x,\phi\rangle - \langle h_y,\phi\rangle\big|}{d^\alpha(x,y)} \qquad \big(\parallel \mathbf{e} \parallel = 1\big) \\
&= \parallel \phi \parallel_\infty + p_\alpha(\phi) \\
&= \parallel \phi \parallel_\alpha,
\end{aligned}$$

where $h_x \in \big(lip_\alpha(X,B)\big)^{*}$. In any case, τ is an isometry. So the proof is complete. $\square$

Corollary 3.2. *Since the map $\tau : \Big(lip_\alpha(X,B)\Big)^{**} \longrightarrow Lip_\alpha(X,B)$, defined by Definition 2.3, is an isometry (by Theorem 3.1), τ is a one-to-one operator.*

Theorem 3.3. *The map τ defined by Definition 2.3 is onto.*

Proof. Let $x \in X$ and $\Lambda \in \sigma(B)$ are arbitrary and fix. Define

$$\psi : Lip_\alpha(X,B) \longrightarrow \Big(lip_\alpha(X,B)\Big)^{**},$$

$$S \mapsto \psi(S),$$

in which, if $S \in lip_\alpha(X,B)$, then $\psi(S) := S$. Also if $S \in Lip_\alpha(X,B)\backslash lip_\alpha(X,B)$, then

$$\big(\psi(S)\big)(h_x) := \Lambda\big((S(x)\big) \quad (x \in X),$$

where h_x is defined in Lemma 2.1. Now, let $H \in Lip_\alpha(X,B)$ be arbitrary.

Case1. If $H \in lip_\alpha(X,B)$, then $\Big(\tau\big(\psi(H)\big)\Big)(x) = \big(\tau(H)\big)(x) = H(x)$.

Case2. If $H \in Lip_\alpha(X,B)\backslash lip_\alpha(X,B)$, then

$$\begin{aligned}\Big(\tau\big(\psi(H)\big)\Big)(x) &= \langle h_x, \psi(H)\rangle \mathbf{e}\\ &= \Big(\big(\psi(H)\big)(h_x)\Big)\mathbf{e}\\ &= \Big(\Lambda\big(H(x)\big)\Big)\mathbf{e}\\ &= H(x).\end{aligned}$$

Hence $\tau\big(\psi(H)\big) = H$. So τ is an onto map. □

Theorem 3.4. *The map τ defined by Definition 2.3 is a homomorphism.*

Proof. Let $\phi_1, \phi_2 \in \Big(lip_\alpha(X,B)\Big)^{**}$ be arbitrary. We note that τ acts on $\phi_1\phi_2$, when $\phi_1\phi_2 \in lip_\alpha(X,B)$ or $\phi_1\phi_2 \in \Big(lip_\alpha(X,B)\Big)^{**}\backslash lip_\alpha(X,B)$.

Case1. Let $\phi_1\phi_2 \in lip_\alpha(X,B)$. Then by Definition 2.3, $\tau(\phi_1) = \phi_1$ and $\tau(\phi_2) = \phi_2$. So $\tau(\phi_1\phi_2) = \phi_1\phi_2 = \tau(\phi_1)\tau(\phi_2)$.

Case2. Let $\phi_1\phi_2 \in \Big(lip_\alpha(X,B)\Big)^{**}\backslash lip_\alpha(X,B)$. Then for every $x \in X$ we have

$$\begin{aligned}\big(\tau(\phi_1\phi_2)\big)(x) &= \langle h_x, \phi_1\phi_2\rangle \mathbf{e}\\ &= \Big(\langle h_x, \phi_1\rangle\langle h_x, \phi_2\rangle\Big)\mathbf{e}\\ &= \big(\langle h_x, \phi_1\rangle \mathbf{e}\big)\big(\langle h_x, \phi_2\rangle \mathbf{e}\big)\\ &= \Big(\big(\tau(\phi_1)\big)(x)\Big)\Big(\big(\tau(\phi_2)\big)(x)\Big)\\ &= \big(\tau(\phi_1)\tau(\phi_2)\big)(x),\end{aligned}$$

where h_x is defined by Lemma 2.1. So $\tau(\phi_1\phi_2) = \tau(\phi_1)\tau(\phi_2)$. □

Remark 3.5. Let A be a Banach algebra. Now, we shall define two products on the Banach space A^{**} [5].

The product map $m_A : A \times A \longrightarrow A$, $\big((a,b) \mapsto ab; a,b \in A\big)$, is a continuous bilinear map. So there is an extension of m_A to a continuous bilinear map $\widetilde{m}_A : A^{**} \times A^{**} \longrightarrow A^{**}$. We define $\phi \Box \psi := \widetilde{m}_A(\phi,\psi)\big(\forall \phi, \psi \in A^{**}\big)$. For $a \in A$ and $\phi \in A^{**}$, we have

$$a\Box\phi = a \cdot \phi \qquad \phi\Box a = \phi \cdot a,$$

where $\cdot$ denotes the module operation in A^{**}. The above product $\Box$ on A^{**} is such that $(A^{**}, \Box)$ is a Banach algebra containing A as a closed subalgebra.

There is a second way of defining a product in A^{**} to give an algebra $(A^{**}, \Diamond)$: indeed, we define $(A^{**}, \Diamond) := \big((A^{op})^{**}, \Box\big)^{op}$, where A^{op} is the *opposite* algebra to A; A^{op} is the algebra formed by reversing the order of the product in A. So $\Diamond$ corresponds to the extension $\widehat{m}_A$ of m_A which:

$$\widehat{m}_A : A^{**} \times A^{**} \longrightarrow A^{**},$$
$$\widehat{m}_A(\phi, \psi) := \widetilde{m}_A(\psi, \phi).$$

$(A^{**}, \Diamond)$ is a Banach algebra containing A as a closed subalgebra.

In general, the two products $\Box$ and $\Diamond$ are distinct. Let $a \in A$, $\lambda \in A^*$ and $\phi, \psi \in A^{**}$. Then we have

$$\langle \lambda, \phi \Box \psi \rangle = \langle \psi \cdot \lambda, \phi \rangle, \qquad \langle \lambda, \phi \Diamond \psi \rangle = \langle \lambda \cdot \phi, \psi \rangle.$$

The products $\Box$ and $\Diamond$ are the first and second Arens products, respectively, on A^{**}. The Banach algebra A is called Arens regular if these two products coincide on A^{**}.

Theorem 3.6. *Let (X, d) be an infinite compact metric space, let $(B, \| . \|)$ be a unital Banach space, and take $\alpha \in (0, 1)$. Then $lip_\alpha(X, B)$ is Arens regular, and the Banach algebra $\Big(lip_\alpha(X, B)\Big)^{**}$ is isometrically isomorphic to $Lip_\alpha(X, B)$.*

Proof. For every $\phi \in \Big(lip_\alpha(X, B)\Big)^{**}$, we define the map

$$\tau : \Big(lip_\alpha(X, B)\Big)^{**} \to Lip_\alpha(X, B),$$

$$\phi \mapsto \tau(\phi),$$

where $\tau(\phi) : X \longrightarrow B$ is defined by Definition 2.3.

We note that, τ, by Theorem 2.4 is well-defined, by Theorem 3.1 is an isometry, by Corollary 3.2 is one-to-one, by Theorem 3.3 is onto and by Theorem 3.4 is a homomorphism. Then τ is a isomorphism and isometry. Now we prove that $lip_\alpha(X, B)$ is Arens regular.

Let $\phi, \psi \in \Big(lip_\alpha(X, B)\Big)^{**}$. Then we have $\tau(\phi), \tau(\psi) \in Lip_\alpha(X, B)$. Take $F = \tau(\phi)$ and $G = \tau(\psi)$. For every $h_x \in \Big(lip_\alpha(X, B)\Big)^{*} (x \in X)$ and $f \in lip(X, B)$, we define the operation $h_x \cdot f = h_x(f) h_x$. So for every

$x \in X$ and any $f \in lip_\alpha(X,B)$ we have

$$\begin{aligned}\langle f, \psi \cdot h_x\rangle \mathbf{e} &= \langle h_x \cdot f, \psi\rangle \mathbf{e}\\ &= \langle h_x(f) h_x, \psi\rangle \mathbf{e}\\ &= h_x(f)\langle h_x, \psi\rangle \mathbf{e}\\ &= h_x(f)\tau(\psi)(x)\\ &= G(x)h_x(f)\\ &= G(x)h_x(f)\mathbf{e}\\ &= \langle f, G(x)h_x\rangle \mathbf{e}.\end{aligned}$$

So $\psi \cdot h_x = G(x)h_x$ for every $x \in X$. Therefore

$$\begin{aligned}\tau\big(\phi\Box\psi\big)(x) &= \langle h_x, \phi\Box\psi\rangle \mathbf{e}\\ &= \langle \psi \cdot h_x, \phi\rangle \mathbf{e}\\ &= \langle G(x)h_x, \phi\rangle \mathbf{e}\\ &= G(x)\langle h_x, \phi\rangle \mathbf{e}\\ &= G(x)\tau\big(\phi\big)(x)\\ &= G(x)F(x)\\ &= \big(GF\big)(x).\end{aligned}$$

Then we get $\tau(\phi\Box\psi) = GF$. Similarly, we obtain $\tau(\phi\Diamond\psi) = GF$. Therefore $lip_\alpha(X,B)$ is Arens regular. $\Box$

Acknowledgment. The author wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

References

1. A. Abdollahi, *The maximal ideal space of analutic Lipschitz algebras*, Rend. Circ. Mat. Palermo, 47 (1998), pp. 347-352.
2. D. Alimohammadi and A. Ebadian, *Hedberg's theorem in real Lipschitz algebras*, Indian J. Pure Appl. Math, 32 (2001), pp. 1479-1493.
3. H.X. Cao and Z.B. Xu, *Some properties of Lipschitz-α operators*, Acta Math. Sin., Chinese Series, 45 (2002), pp. 279-286.
4. H.X. Cao, J.H. Zhang, and Z.B. Xu, *Characterizations and Extensions of Lipschitz-α operators*, Acta Math. Sin., 22 (2006), pp. 671-678.
5. H.G. Dales, *Banach algebras and Automatic Continuty*, Clarendon Press. Oxford, (2000).
6. A. Ebadian, *Prime ideals in Lipschitz algebras of finite differentiable functions*, Honam Math. J., 22 (2000), pp. 21-30.

7. T.G. Honary and H. Mahyar, *Approximation in Lipschitz algebras*, Quaest. Math., 23 (2000), pp. 13-19.
8. J.A. Johnson, *Lipschitz spaces*, Pacific J. Math., 51 (1975), pp. 177-186.
9. B. Pavlovic, *Automatic continuity of Lipschitz algebras*, J. Funct. anal. 131 (1995), pp. 115-144.
10. D.R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math. 13 (1963), pp. 1387-1399.
11. N. Waver, *Subalgebras of little Lipschitz algebras*, Pacific J. Math., 173 (1996), pp. 283-293.

Department of Mathematics, Ahar Branch, Islamic Azad University, Ahar, Iran.

E-mail address: a-shokri@iau-ahar.ac.ir