Sahand Communications in Mathematical Analysis (SCMA) Vol. 10 No. 1 (2018), 29-35
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2016.23127

On L^*- proximate order of meromorphic function

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Abstract. In this paper we introduce the notion of L^*-proximate order of meromorphic function and prove its existence.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M(r,f)$ corresponding to f is defined on $|z|=r$ as follows:

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

When f is meromorphic, $M(r,f)$ cannot be defined as f is not analytic throughout the complex plane. In this situation, one may introduce another function $T(r,f)$ known as Nevanlinna's characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T(r,f)=N(r,f)+m(r,f),$$

where

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log r,$$

is the pole-counting contribution, where $n(r,f)$ is the number of poles of f, including multiplicities, for $|z|\leq r$. On the other hand, the function $m(r,f)$ known as the proximity function is defined as

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,$$

where $\log^+ x=\max(\log x,0)$ for all $x\geq 0$.

2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. Meromorphic function, L^*-order, L^*- proximate order.
Received: 02 February 2016, Accepted: 13 August 2016.
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Now let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [3] defined it in the following way:

Definition 1.1 ([3])**.** A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon(>0)$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon},$$

for $r \geq r(\varepsilon)$ and uniformly for $k(\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \ .$$

Somasundaram and Thamizharasi [4] introduced the notion of L-order (L-lower order) for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order (L-lower order) for entire functions is L^*-order (L^*-lower order). Their definitions are as follows:

Definition 1.2 ([4])**.** The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

For an entire function of one complex variable, the notion of order and its type are classical in complex analysis and the growths of entire functions can be studied in terms of their orders and types. However, these concepts are inadequate for comparing the growth of those entire functions which are of same orders but of infinite types. To refine the above scale, Valiron [5] introduced the concept of a positive continuous function for an entire function having finite order called Lindelöf's proximate order which make it unnecessary to consider functions of minimal or maximal type and the existence of this function was proved by Valiron [5]. Later Shah [2] simplified the proof of its existance. Also Lahiri [1] generalised the idea of the proximate order for a meromorphic function with finite generalised order and proved the existence of it. Since the proximate order is not linked with L^*- order, therefore it seems reasonable to define suitably the L^*- proximate order. With this in view, we introduce the following definition of the L^*- proximate order:

Definition 1.3. Let f be meromorphic with finite L^*- order $\rho_f^{L^*}$. A function $\rho^{L^*}(r)$ is said to be the L^*- proximate order of f if the following conditions hold:

(i) $\rho^{L^*}(r)$ is non-negative and continuous for $r > r_0$,
(ii) $\rho^{L^*}(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho^{L^*\prime}(r+0)$ and $\rho^{L^*\prime}(r-0)$ exist,
(iii) $\lim_{r\to\infty} \rho^{L^*}(r) = \rho_f^{L^*}$,
(iv) $\lim_{r\to\infty} \left[re^{L(r)}\right] \rho^{L^*\prime}(r) \log \left[re^{L(r)}\right] = 0$,
(v) $\limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho^{L^*}(r)}} = 1$.

Further one can define the L^*- proximate lower order of a meromorphic function which is as follows:

Definition 1.4. Let f be meromorphic with finite L^*- lower order $\lambda_f^{L^*}$. A function $\lambda^{L^*}(r)$ is said to be the L^*- proximate lower order of f if the following conditions hold:

(i) $\lambda^{L^*}(r)$ is non-negative and continuous for $r > r_0$,
(ii) $\lambda^{L^*}(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda^{L^*\prime}(r+0)$ and $\lambda^{L^*\prime}(r-0)$ exist,
(iii) $\lim_{r\to\infty} \lambda^{L^*}(r) = \lambda_f^{L^*}$,
(iv) $\lim_{r\to\infty} \left[re^{L(r)}\right] \lambda^{L^*\prime}(r) \log \left[re^{L(r)}\right] = 0$,
(v) $\liminf_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\lambda^{L^*}(r)}} = 1$.

Now a question may arise about the existences of such functions stated in above definition. In the next seection we would like to establish the existence of the L^*- proximate order and the L^*-proximate lower order of meromorphic functions. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [5].

2. Theorems

In this scetion we present the main results of our paper.

Theorem 2.1. *For a meromorphic function f with finite L^*- order $\rho_f^{L^*}$, the L^*- proximate order $\rho^{L^*}(r)$ of f exists.*

Proof. Let

$$\sigma(r) = \frac{\log T(r,f)}{\log \left[re^{L(r)}\right]}.$$

Then

$$\limsup_{r\to\infty} \sigma(r) = \rho_f^{L^*}.$$

We consider two cases:

Case(I) : Let $\sigma(r) > \rho_f^{L^*}$ for at least a sequence of values of r tending to infinity.

we define

$$\phi(r) = \max_{x\geq r}\{\sigma(x)\}.$$

Clearly $\phi(r)$ exists and is non increasing.

Let $R_1 > e^{e^e}$ and $\sigma(R) > \rho_f^{L^*}$. Then for $r \geq R_1 > R$, we get $\sigma(r) \leq \sigma(R)$, since $\sigma(r)$ is continuous, there exists $r_1 \in [R, R_1]$ such that

$$\sigma(r_1) = \max_{R\leq x\leq R_1}\{\sigma(x)\}.$$

Clearly $r_1 > e^{e^e}$ and $\phi(r_1) = \sigma(r_1)$.Since values $r = r_1$ exists for a sequence of values of r tending to infinity.

Let $\rho^{L^*}(r_1) = \phi(r_1)$ and t_1 be the smallest integer not less than $1+r_1$ such that $\phi(r_1) > \phi(t_1)$.

We define $\rho^{L^*}(r) = \rho^{L^*}(r_1)$ for $r_1 < r \leq t_1$. Observing that $\phi(r)$ and $\rho^{L^*}(r_1) - \log^{[3]} r + \log^{[3]} t_1$ are continious functions of r, $\rho^{L^*}(r_1) - \log^{[3]} r + \log^{[3]} t_1 > \phi(t_1)$ for $r\,(> t_1)$ sufficiently close to t_1 and $\phi(r)$ is non increasing, we can define u_1 as follows:

$u_1 > t_1$,
$\rho^{L^*}(r) = \rho^{L^*}(r_1) - \log^{[3]} r + \log^{[3]} t_1$ for $t_1 \leq r \leq u_1$,
$\rho^{L^*}(r) = \phi(r)$ for $r = u_1$,
$\rho^{L^*}(r) > \phi(r)$ for $t_1 \leq r < u_1$.

Let r_2 be the smallest value of r for which $r_2 \geq u_1$ and $\phi(r_2) = \sigma(r_2)$. If $r_2 > u_1$ then let $\rho^{L^*}(r) = \phi(r)$ for $u_1 \leq r \leq r_2$. Since it can be easily shown that $\phi(r)$ is constant in $u_1 \leq r \leq r_2$, $\rho^{L^*}(r)$ is constant in $u_1 \leq r \leq r_2$. We repeat this process infinitely and obtain that $\rho^{L^*}(r)$ is differentiable in adjacent intervals. Further $\rho^{L^*\prime}(r) = 0$ or $(-r\log r\log\log r)^{-1}$ and $\rho^{L^*}(r) \geq \phi(r) \geq \sigma(r)$ for all $r \geq r_1$. Also $\rho^{L^*}(r) = \sigma(r)$ for a sequence of values of r tending to infinity and $\rho^{L^*}(r)$ is non increasing for $r \geq r_1$ and

$$\begin{aligned}\rho_f^{L^*} &= \limsup_{r\to\infty} \sigma(r)\\ &= \lim_{r\to\infty} \phi(r).\end{aligned}$$

So

$$\limsup_{r\to\infty} \rho^{L^*}(r) = \liminf_{r\to\infty} \rho^{L^*}(r) = \lim_{r\to\infty} \rho^{L^*}(r) = \rho_f^{L^*},$$

and

$$\lim_{r\to\infty} \left[re^{L(r)}\right] \rho^{L^*\prime}(r) \log\left[re^{L(r)}\right] = 0.$$

Further we have

$$T(r,f)7 = \left[re^{L(r)}\right]^{\sigma(r)} = \left[re^{L(r)}\right]^{\rho^{L^*}(r)},$$

for a sequence of values of r tending to infinity and

$$T(r,f) < \left[re^{L(r)}\right]^{\rho^{L^*}(r)},$$

for remaning r's. Therefore

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho^{L^*}(r)}} = 1.$$

Continuity of $\rho^{L^*}(r)$ for $r \geq r_1$ follows from its construction which is complete in case(I).

Case(II) : Let $\sigma(r) \leq \rho_f^{L^*}$ for all sufficiently large values of r.

In Case(II) we separate two cases:

Sub case (A) : Let $\sigma(r) = \rho_f^{L^*}$ for at least a sequence of values of r tending to infinity.

Sub case (B) : Let $\sigma(r) < \rho_f^{L^*}$ for all sufficiently large values of r.

In Sub case (A) we take $\rho^{L^*}(r) = \rho_f^{L^*}$ for all sufficiently large values of r.

In Sub case (B) let

$$\xi(r) = \max_{X\leq x\leq r}\{\sigma(x)\},$$

where $X > e^{e^e}$ is such that $\sigma(r) < \rho_f^{L^*}$ whenever $x \geq X$. We note that $\xi(r)$ is non decreasing and for all sufficiently large $r \geq X$, the roots of $\xi(x) = \rho_f^{L^*} + \log^{[3]} x - \log^{[3]} r$ are less than r. For a suitable large value $v_1 > X$, we define

$$\rho^{L^*}(v_1) = \rho_f^{L^*}, \qquad \rho^{L^*}(r) = \rho_f^{L^*} + \log^{[3]} r - \log^{[3]} v_1,$$

for $s_1 \leq r \leq v_1$ where $s_1 < v_1$ is such that $\xi(s_1) = \rho^{L^*}(s_1)$. In fact s_1 is given by the largest positive root of $\xi(x) = \rho_f^{L^*} + \log^{[3]} x - \log^{[3]} v_1$. If $\xi(s_1) = \sigma(s_1)$, let $\omega_1 (< s_1)$ be the upper bound of point ω at which $\xi(\omega) \neq \sigma(\omega)$ and $\omega < s_1$. Clearly at ω_1, $\xi(s_1) = \sigma(s_1)$. We define $\rho^{L^*}(r) = \xi(r)$ for $\omega_1 \leq r \leq s_1$. It is easy to show that $\xi(r)$ is constant in $\omega_1 \leq r \leq s_1$ and so $\rho(r)$ is constant in $\omega_1 \leq r \leq s_1$. If $\xi(s_1) = \sigma(s_1)$ we take $\omega_1 = s_1$.

We choose $v_2 > v_1$ suitably large and let

$$\rho^{L^*}(v_1) = \rho_f^{L^*}, \qquad \rho^{L^*}(r) = \rho_f^{L^*} + \log^{[3]} r - \log^{[3]} v_2,$$

for $s_2 \leq r \leq v_2$ where $s_2 < v_2$ is such that $\xi(s_2) = \rho^{L^*}(s_2)$. If $\xi(s_2) \neq \rho^{L^*}(s_2)$, let $\rho^{L^*}(r) = \xi(r)$ for $\omega_2 \leq r \leq s_2$, where ω_2 has the similar property as that of ω_1. As above $\rho^{L^*}(r)$ is constant in $[\omega_2, s_2]$. If $\xi(s_2) = \sigma(s_2)$ we take $\omega_2 = s_2$.

Let $\rho^{L^*}(r) = \rho^{L^*}(\omega_2) - \log^{[3]} r + \log^{[3]} \omega_2$ for $q_1 \leq r \leq \omega_2$ where $q_1 < \omega_2$ is the point of intersection of $y = \rho_f^{L^*}$ with $y = \rho^{L^*}(\omega_2) - \log^{[3]} x + \log^{[3]} \omega_2$. It is also possible to choose v_2 so large that $\mathrm{v}_1 < q_1$. Let $\rho^{L^*}(r) = \rho_f^{L^*}$ for $v_1 \leq r \leq q_1$. We repeat this process. Now we can show that for all $r \geq v_1$, $\rho_f^{L^*} \geq \rho^{L^*}(r) \geq \xi(r) \geq \sigma(r)$ and $\rho^{L^*}(r) = \sigma(r)$ for $r = \omega_1, \omega_2, \ldots$. So we obtain that

$$\limsup_{r\to\infty} \rho^{L^*}(r) = \liminf_{r\to\infty} \rho^{L^*}(r) = \lim_{r\to\infty} \rho^{L^*}(r) = \rho_f^{L^*}.$$

Since

$$T(r,f) = \left[re^{L(r)}\right]^{\sigma(r)} = \left[re^{L(r)}\right]^{\rho^{L^*}(r)},$$

for a sequence of values of r tending to infinity and

$$T(r,f) < \left[re^{L(r)}\right]^{\rho^{L^*}(r)},$$

for remaining r's, it follows that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho^{L^*}(r)}} = 1.$$

Also $\rho^{L^*}(r)$ is differentiable in adjacent intervals. Further $\rho^{L^*\prime}(r) = 0$ or $(-r \log r \log\log r)^{-1}$ and so

$$\lim_{r\to\infty} \left[re^{L(r)}\right] \rho^{L^*\prime}(r) \log \left[re^{L(r)}\right] = 0.$$

Continuity of $\rho^{L^*}(r)$ follows from its construction. This completes the proof of the theorem. □

Theorem 2.2. *For a meromorphic function f with finite L^*- lower order $\lambda_f^{L^*}$, the L^*- proximate lower order $\lambda^{L^*}(r)$ of f exists.*

The proof of Theorem 2.2 is omitted because it can be carried out in the line of Theorem 2.1.

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