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FIXED POINT RESULTS IN CONE METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. In this paper, we prove the existence of fixed point for Chatterjea type mappings under c-distance in cone metric spaces endowed with a graph. The main results extend, generalized and unified some fixed point theorems on c-distance in metric and cone metric spaces.

1. Introduction

In 2007, Huang and Zhang [7] defined cone metric space by considering an ordered Banach space instead of set of real numbers and proved some fixed point theorems of contractive type mappings on such spaces (also, see [8] and the references cited therein).

Let E be a real Banach space with the zero element θ. A subset P of E is called a cone if and only if (a) P is closed, non-empty and $P \neq \{\theta\}$; (b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ implies that $ax + by \in P$; (c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$, where $intP$ is the interior of P. The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq K\|y\|$. The least positive number satisfying the above is called the normal constant of P.

Definition 1.1 ([7]). Let X be a nonempty set. A function $d : X \times X \to E$ is called a cone metric on X and (X, d) is called a cone metric space if the following conditions hold.

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($d1$) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
($d2$) $d(x,y) = d(y,x)$ for all $x, y \in X$;
($d3$) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

For notions such as convergent and Cauchy sequences, completeness, continuity and etc in cone metric spaces and also other properties in a cone, we refer to [7, 8].

The most important graph theory approach to metric fixed point theory introduced so far is attributed to Jachymski [6]. In this approach, the underlying metric space is equipped with a directed graph and the Banach contraction is formulated in a graph language. For further works and results in metric spaces endowed with a graph, see, e.g., [1, 4, 5, 10]. We next review some basic notions of graph theory in relation to a cone metric space that we need in the sequel. For more details on the theory of graphs, see [2, 6].

Let (X, d) be a cone metric space and G be a directed graph with vertex set $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops, that is, $(x, x) \in E(G)$ for all $x \in X$. Assume further that G has no parallel edges. Then G can be denoted by the ordered pair $(V(G), E(G))$, and also it is said that the metric space (X, d) is endowed with the graph G.

The metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$ such that

$$E(G^{-1}) = \big\{(x,y) : (y,x) \in E(G)\big\} \quad \text{and} \quad E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

It should be remarked that if both (x, y) and (y, x) belong to $E(G)$, then we will face with parallel edges in the graph $\widetilde{G}$. To avoid this problem, we delete either the edge (x, y) or the edge (y, x) (but not both of them) from G and consider the graph $\widetilde{G}$ obtained from the remaining graph.

We also need a few notions about the connectivity of graphs.

Suppose that (X, d) is a metric space endowed with a graph G. If $x, y \in X$, then a finite sequence $(x_i)_{i=0}^N$ consisting of $N+1$ vertices of G is called a path in G from x to y of length N whenever $x_0 = x$, $x_N = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \dots, N$. The graph G is called connected if there exists a path in G between each two vertices of G, and weakly connected if the graph $\widetilde{G}$ is connected.

In 1996, Kada et al. [9] defined the concept of w-distance in metric spaces. After that, Cho et al. [3] introduced the concept of c-distance in a cone metric space such as follows and obtained some fixed point results (also, see [11–13]).

Definition 1.2. Let (X,d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:

(q_1) $\theta \preceq q(x,y)$ for all $x,y \in X$;

(q_2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$;

(q_3) for all $n \geq 1$ and $x \in X$, if $q(x,y_n) \preceq u$ for some $u = u_x$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Note that each w-distance in a metric space is a c-distance in cone metric space (X,d) with $E = \mathbb{R}$ and $P = [0,\infty)$. Also, each cone metric d on X with a normal cone is a c-distance q on X. Moreover, for c-distance q, $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ and $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Lemma 1.3 ([3, 13])**.** *Let (X,d) be a cone metric space, q be a c-distance on X. Also, let $\{x_n\}$ be sequences in X and $\{u_n\}$ be a sequence in P converging to θ. If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

The main purpose of this paper is to study a new type of Chatterjea contraction with respect to a c-distance in cone metric spaces endowed with a graph by standard iterative techniques and existence of the fixed point. Our main result generalizes a new type of Chatterjea fixed point theorems under c-distance in cone metric and also in cone metric spaces equipped with a partial order.

2. Main results

In this section, let (X,d) be a cone metric space associated with c-distance q and endowed with a directed graph G with $V(G) = X$ and $\Delta(X) \subseteq E(G)$. Throughout this section, we use X_f to denote the set of all points $x \in X$ such that $(x,fx) \in E(G)$. In other words,

$$X_f = \{x \in X : \quad (x,fx) \in E(G)\}.$$

Theorem 2.1. *Let (X,d) be a complete cone metric space associated with a c-distance q and endowed with a graph G, P be normal cone with normal constant K, and $f : X \to X$ be a mapping. Suppose that there exist mappings $\alpha,\beta,\gamma : X \to [0,1)$ such that the following conditions hold.*

(C1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$, $\gamma(fx) \leq \gamma(x)$ and $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*

(C2) *f preserves the edges of G; that is, $(x,y) \in E(G)$ implies $(fx,fy) \in E(G)$ for all $x,y \in X$;*

(C3) *for all $x, y \in X$ with $(x, y) \in E(G)$,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx),$$
$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y);$$

(C4) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *for all $y \in X$ with $y \neq fy$.*

If $X_f \neq \emptyset$, then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Proof. Let $x_0 \in X_f$. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. Since f preserves the edges of G and $(x_0, fx_0) \in E(G)$, then it follows that by induction $(x_n, x_{n+1}) \in E(G)$, where $x_n = fx_{n-1} = f^n x_0$ for all $n \in \mathbb{N}$. Consider $x = x_n$ and $y = x_{n-1}$ in (**C3**). Since $(x_{n-1}, x_n) \in E(G)$, we have

(2.1)

$$\begin{aligned} q(x_{n+1}, x_n) &= q(fx_n, fx_{n-1}) \\ &\preceq \alpha(x_n)q(x_n, x_{n-1}) + \beta(x_n)q(x_n, x_n) + \gamma(x_n)q(x_{n-1}, x_{n+1}) \\ &\preceq \alpha(x_{n-1})q(x_n, x_{n-1}) + \beta(x_{n-1})[q(x_n, x_{n+1}) + q(x_{n+1}, x_n)] \\ &\quad + \gamma(x_{n-1})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\ \ \vdots \\ &\preceq \alpha(x_0)q(x_n, x_{n-1}) + \beta(x_0)q(x_{n+1}, x_n) \\ &\quad + (\beta + \gamma)(x_0)q(x_n, x_{n+1}) + \gamma(x_0)q(x_{n-1}, x_n). \end{aligned}$$

Similarly, consider $x = x_n$ and $y = x_{n-1}$ in (**C3**). Since $(x_{n-1}, x_n) \in E(G)$, we have

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \alpha(x_0)q(x_{n-1}, x_n) + \beta(x_0)q(x_n, x_{n+1}) \\ &\quad + (\beta + \gamma)(x_0)q(x_{n+1}, x_n) + \gamma(x_0)q(x_n, x_{n-1}). \end{aligned} \tag{2.2}$$

Adding up (2.1) and (2.2). Then

$$u_n \preceq (\alpha + \gamma)(x_0)u_{n-1} + (2\beta + \gamma)(x_0)u_n,$$

with $u_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. Now, we get $u_n \preceq \lambda u_{n-1}$, where

$$\lambda = \frac{(\alpha + \gamma)(x_0)}{1 - (2\beta + \gamma)(x_0)} < 1,$$

by (**C1**). By repeating the procedure, we get $u_n \preceq \lambda^n u_0$ for all $n \in \mathbb{N}$. Thus,

$$q(x_n, x_{n+1}) \preceq u_n \preceq \lambda^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{2.3}$$

Now, let $m > n$. It follows from (2.3) and $\lambda \in [0, 1)$ that

$$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \tag{2.4}$$
$$\preceq (\lambda^n + \cdots + \lambda^{m-1})[q(x_1, x_0) + q(x_0, x_1)]$$
$$\preceq \frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)].$$

Since $\frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)]$ converges to θ, Lemma 1.3 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n = f^n x_0 \to x'$ as $n \to \infty$. Moreover, by (2.4) and (q_3), we get that

$$q(x_n, x') \preceq \frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)],$$

for all $n \geq 1$. Since P is a normal cone with normal constant K, we have

$$\|q(x_n, x_m)\| \leq K(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\|,$$

for all $m > n \geq 1$. In particular, we have

$$\|q(x_n, x_{n+1})\| \leq K(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\|, \tag{2.5}$$

$$\|q(x_n, x')\| \leq K(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\|, \tag{2.6}$$

for all $n \geq 1$. Suppose that $x' \neq fx'$. Then, by (2.5) and (2.6), we have

$$\begin{aligned} 0 &< \inf\{\|q(x, x')\| + \|q(x, fx)\| : x \in X\} \\ &\leq \inf\{\|q(x_n, x')\| + \|q(x_n, fx_n)\| : n \geq 1\} \\ &\leq \inf\Big\{K(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| \\ &\quad + K(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\| : n \geq 1\Big\} = 0, \end{aligned}$$

which is a contradiction. Hence $x' = fx'$. Now, let $fz = z$. Then, by (**C3**), we have

$$\begin{aligned} q(z, z) &= q(fz, fz) \\ &\preceq \alpha(z)q(z, z) + \beta(z)q(z, fz) + \gamma(z)q(z, fz) \\ &= (\alpha + \beta + \gamma)(z)q(z, z), \end{aligned}$$

which implies that $q(z, z) = \theta$ by (**C1**). This completes the proof. □

Corollary 2.2. *Let (X, d) be a complete cone metric space associated with a c-distance q and endowed with a graph G, and P be normal cone with normal constant K. Also, let $f : X \to X$ be a mapping such that the following conditions hold.*

(C1) *f preserves the edges of G; that is, $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;*

(C2) *There exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fy) + \gamma q(y, fx),$$
$$q(fy, fx) \preceq \alpha q(y, x) + \beta q(fy, x) + \gamma q(fx, y),$$

for all $x, y \in X$ with $(x, y) \in E(G)$;

(C3) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *for all $y \in X$ with $y \neq fy$.*

If $X_f \neq \emptyset$, then z has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Proof. In Theorem 2.1, consider $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. □

Now we present four important consequences of Theorem 2.1 where the graph G is replaced with the special graphs. Firstly, we put $G = G_0$ in Theorem 2.1, where G_0 is a complete graph whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$. Then we get following corollary.

Corollary 2.3. *Let (X, d) be a complete cone metric space associated with a c-distance q and endowed with the graph G_0, and P be normal cone with normal constant K. Suppose that there exist mapping $f : X \to X$ and mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold.*

(C1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$, $\gamma(fx) \leq \gamma(x)$ and $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*

(C2) *for all $x, y \in X$,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx),$$
$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y);$$

(C3) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *for all $y \in X$ with $y \neq fy$.*

Then f has a fixed point on X. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Secondly, suppose that $(X, \sqsubseteq)$ be a poset. Consider the poset graphs G_1 by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Since $\sqsubseteq$ is reflexive, it follows that both $E(G_1)$ contain all loops. Having done this, the following partially ordered version of Theorem 2.1 in complete cone metric spaces associated with a c-distance equipped a partial order is obtained:

Corollary 2.4. *Let $(X, \sqsubseteq)$ be a poset, d be a cone metric on X such that (X, d) is a complete cone metric space associated with a c-distance q and endowed with the graph G_1, and P be normal cone with normal* *constant K. Also, let $f : X \to X$ be a nondecreasing mapping and there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold.*

(C1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$, $\gamma(fx) \leq \gamma(x)$ and $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*

(C2) *for all $x, y \in X$ with $x \sqsubseteq y$,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx),$$
$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y);$$

(C3) *$\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.*

Then f has a fixed point on X if there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Now, suppose that $(X, \sqsubseteq)$ be a poset. Consider the poset graphs G_2 by $V(G_2) = X$ and $E(G_2) = \big\{(x, y) \in X \times X : x \sqsubseteq y \ \vee \ y \sqsubseteq x\big\}$. Then an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \sqsubseteq)$.

Corollary 2.5. *Let $(X, \sqsubseteq)$ be a poset, d be cone metric on X such that (X, d) is a complete cone metric space associated with a c-distance q and endowed with the graph G_2, and P be normal cone with normal constant K. Suppose that there exist mapping $f : X \to X$ which maps comparable elements of X onto comparable elements and mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold.*

(C1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$, $\gamma(fx) \leq \gamma(x)$ and $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*

(C2) *for all $x, y \in X$ such that x and y are comparable,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx),$$
$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y);$$

(C3) *$\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.*

Then f has a fixed point on X if there exists $x_0 \in X$ such that x_0 and fx_0 are comparable. Moreover, if $fz = z$, then $q(z, z) = \theta$.

Let $e \in int\ P$ be fixed. Two elements $x, y \in X$ are said to be e-close if $d(x, y) \preceq e$. Define the e-graph G_3 by $V(G_3) = X$ and $E(G_3) = \big\{(x, y) \in X \times X : \ d(x, y) \preceq e\big\}$. We see that $E(G_3)$ contains all loops.

Corollary 2.6. *Let (X, d) be a complete cone metric space associated with a c-distance q and endowed with the graph G_3. Also, let P be normal cone with normal constant K and $e \in intP$. Suppose that there* *exists mapping $f : X \to X$ which maps e-close elements of X onto e-close elements and mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold.*

(C1) $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$, $\gamma(fx) \leq \gamma(x)$ *and* $(\alpha + 2\beta + 2\gamma)(x) < 1$ *for all* $x \in X$*;*

(C2) *for all $x, y \in X$ such that x and y are e-close elements,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx),$$
$$q(fy, fx) \preceq \alpha(x)q(y, x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y);$$

(C3) $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *for all* $y \in X$ *with* $y \neq fy$.

Then f has a fixed point on X if there exists $x_0 \in X$ such that x_0 and fx_0 are e-close elements. Moreover, if $fz = z$, then $q(z, z) = \theta$.

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