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Generated topology on infinite sets by ultrafilters

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Abstract. Let X be an infinite set, equipped with a topology τ. In this paper, we studied the relationship between τ and ultrafilters on X. We can discovered, among other things, some relations of the Robinson's compactness theorem, continuity and the separation axioms. It is important also, aspects of communication between mathematical concepts.

1. Introduction

We can find a natural relation between topologies on a semigroup S and filters on S. For example, if S is a group, every left invariant topology on S is uniquely detemined by the filter of neighborhoods of the identity. It is also uniquely determined by the compact subsemigroup of βS (the Stone-$\check{C}$ech compactification of S) consisting of the ultrafilters on S which converge to the identity.

Topologies defined on S by using the algebra of βS are useful because they provide a tool for studying the algebra of βS. Also give us interesting examples in general topology. For example, for every countably infinite group S, strongly right maximal idempotents in $S^* = \beta S \setminus S$ define invariant regular topologies on S which are maximal subject to having no isolated points (See [3] section 9). Zelenuk's Theorem states that for every countable torsion-free group, S^* can contain non-trivial finite groups, see [3] or [5]. The proof of Zelenuk's Theorem depends in an essential way on a topology defined by a finite subgroup of S^*.

In this paper, we just concentrate on an infinite set X, and investigate relation between ultrafilters on X and topology on this set.
This paper is organized in four sections. In Section 2, we have introduced the Stone-Čech compactification breifly and presented some basic premises of the next sections.

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In Section 3, for a topological space X the points close to the point x, represented by x^*, are defined. Then the notions open, closed, compactness, continiuity and convergence by mean of filters are redefined. Furthurmore, we will prove Robinson's premise, with regard to the compact sets, by mean of ultrafilters close to one point, proved by nonstandard analysis methods.

In Section 4, the topology genererated by ultrafilters on X will be introduced. One of it's results is expressing seperation axioms by ultrafilters close to one point. For instance, in the topological space (X, τ), if x^*, y^* from $x, y \in X$, converging the meaning of the close points to x and y, respectively, then X is Hausdorff if and only if $x^* \cap y^* = \emptyset$. In the end, two types of topology not obtained by usual methods, will be introduced (Definition 4.5).

2. Preliminary

Let Γ be a family of sets that together with A and B contains the intersection $A \cap B$. By a filter in Γ, we mean a non-empty subfamily $\mathcal{F} \subseteq \Gamma$, satisfying the following conditions:

(F_1) $\emptyset \notin \mathcal{F}$.
(F_2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.
(F_3) If $A \in \mathcal{F}$ and $A \subseteq B \in \Gamma$, then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ in Γ is a maximal filter or an ultrafilter in Γ, if for every filter $\mathcal{A}$ in Γ that contains $\mathcal{F}$ we have $\mathcal{A} = \mathcal{F}$.

A filter-base in Γ is a non-empty family $\mathcal{G} \subseteq \Gamma$, such that $\emptyset \notin \mathcal{G}$, and if $A_1, A_2 \in \mathcal{G}$ then there exists $A_3 \in \mathcal{G}$, such that $A_3 \subseteq A_1 \cap A_2$.

One can readily see that for any filter-base $\mathcal{G}$ in Γ, the family

$$\mathcal{F}_{\mathcal{G}} = \{A \in \Gamma : \text{ there exists } B \in \mathcal{G} \text{ such that } B \subseteq A\},$$

is a filter in Γ. For a topological space (X, τ), a filter in τ is called an open-filter. A filter $\mathcal{F}$ on (X, τ) converges to a point $x \in X$ if and only if $\tau_x \subseteq \mathcal{F}$, where

$$\tau_x = \{U \subset X : x \in V \subseteq U \text{ for some } V \in \tau\},$$

is the collection of all neighborhoods of $x \in X$. In this case, the point $x \in X$ is called a limit of filter $\mathcal{F}$ and we denoted it by $x \in \lim \mathcal{F}$. A point x is called a cluster point of a filter $\mathcal{F}$ if x belongs to the closure of every member of $\mathcal{F}$. Clearly, x is a cluster point of a filter $\mathcal{F}$ if and only if every neighborhood of x intersects all members of $\mathcal{F}$. This implies, in particular, that every cluster point of an ultrafilter is a limit of this ultrafilter. It is obvious that a subset $A \subseteq X$ is closed in τ if and only if a limit of any filter containing A belongs to A. Let X_d denote X with discrete topology. If X and Y are completely regular spaces, then any continuous function $f : X \to Y$ has a unique continuous extension $f^\beta : \beta X \to \beta Y$.

Lemma 2.1. *With the above notations, we have:*

(a) *Let $f : X_d \to Y_d$ be a function. Then for each $p \in \beta X_d$ we have,*

$$f^\beta(p) = \{A \subseteq X_d : f^{-1}(A) \in p\}.$$

In particular, if $A \in p$, then $f(A) \in f^\beta(p)$.

(b) *Let $f, g : X_d \to Y_d$ be functions and $p \in \beta X_d$ satisfy $f^\beta(p) = g^\beta(p)$. Then for each $A \in p$, $\{x \in X_d : f(x) \in g(A)\} \in p$.*

Proof. See Lemma 3.30 and 3.31 in [3]. □

Now we review the definition of partition regularity and a theorem that connect it with ultrafilters.

Definition 2.2. Let $\mathcal{P}(X)$ is the set of all subsets of a set X, and $\mathcal{R}$ be a nonempty set of subsets of $\mathcal{P}(X)$. $\mathcal{R}$ is called partition regular if whenever $\mathcal{F}$ is a finite set of $\mathcal{P}(X)$ and $\cup\mathcal{F} \in \mathcal{R}$, there exist $A \in \mathcal{F}$ and $B \in \mathcal{R}$, such that $B \subseteq A$.

Theorem 2.3. *Let $\mathcal{R} \subseteq \mathcal{P}(X)$ be a nonempty set and assume $\emptyset \notin \mathcal{R}$. Let*

$$\mathcal{R}^\uparrow = \{B \in \mathcal{P}(X) : A \subseteq B \text{ for some } A \in \mathcal{R}\}.$$

Then the following statements are equivalence.

(a) *$\mathcal{R}$ is partition regular.*

(b) *Whenever $\mathcal{A} \subseteq \mathcal{P}(X)$ has the property that every finite nonempty subfamily of $\mathcal{A}$ has an intersection which is in $\mathcal{R}^\uparrow$, there is $\mathcal{U} \in \beta G_d$ such that $\mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{R}^\uparrow$.*

(c) *Whenever $A \in \mathcal{R}$, there is $\mathcal{U} \in \beta X_d$ such that $A \in \mathcal{U} \subseteq \mathcal{R}^\uparrow$.*

Proof. [3, Theorem 3.11]. □

Example 2.4. Let (X, τ) be a topological space. Then,

(a) For all $x \in X$, the family $\mathcal{R}_x = \{A \subseteq X : x \in \text{cl}_X A\}$ is partition regular and $\mathcal{R}_x = \mathcal{R}_x^\uparrow$. Also for all $A \in \mathcal{R}_x$, by Theorem 2.3, there exists $p \in \beta X_d$ such that $A \in p \subseteq \mathcal{R}_x$. It is obvious that $\tau_x \subseteq \mathcal{R}_x$.

(b) Let F be a nonempty subset of X. Then

$$\mathcal{R}_F = \{A \subseteq X : \text{cl}_X(A) \cap F \neq \emptyset\},$$

is partition regular and $\mathcal{R}_F = \mathcal{R}_F^\uparrow$.

3. Description of topological concepts by ultrafilters

Let (X,τ) be a topological space. For $x \in X$, with respect to τ on X, we define $\widehat{x} = \{A \subseteq X : x \in A\}$, and

$$x^* = \left\{ p \in \beta X_d : x \in \bigcap_{A \in p} \mathrm{cl}_X A \right\}.$$

In fact, x^* is the collection of all ultrafilters converge to x. It is obvious that $\widehat{x} \in x^*$. We say $p \in x^*$ is a near point to x. We define $B(X) = \cup_{x \in X} x^*$ and $\infty^* = \beta X_d - B(X)$. An element $p \in B(X)$ is called a bounded ultrafilter and $p \in \infty^*$ is called an unbounded ultrafilter. For $F \subseteq X$, define

$$F^* = \{p \in \beta X_d : p \subseteq \mathcal{R}_F^{\uparrow}\}, \text{ (see Example 2.4).}$$

It is obvious that $x^* \subseteq F^*$, for each $x \in F$.

Lemma 3.1. *Let (X,τ) be a Hausdorff topological space. Then,*

- (a) *$p \in \infty^*$ if and only if for each $x \in X$ there exists $A \in p$, such that $x \notin \mathrm{cl}_X A$.*
- (b) *If $\tau_x \subseteq p$ then $p \in x^*$.*
- (c) *Let $U \subseteq X$, then U is a neighborhood of x if and only if $U \in p$ for each $p \in x^*$.*
- (d) *Let $A \subseteq X$. Then $x \in \mathrm{cl}_X A$ if and only if $\mathrm{cl}_{\beta X_d} A \cap x^* \neq \emptyset$.*
- (e) *For each $x \in X$, $\tau_x = \bigcap x^*$ is a filter and,*

$$\mathrm{cl}_{\beta X_d} x^* = \overline{\tau}_x = \bigcap_{U \in \tau_x} \mathrm{cl}_{\beta X_d} U.$$

- (f) *Let $A \subseteq X$, then x is an interior point of A if and only if $x^* \subseteq \mathrm{cl}_{\beta X_d} A$. In particular, A is open if and only if $x^* \subseteq \mathrm{cl}_{\beta X_d} A$ for each $x \in A$.*

Proof. (a) Let $p \in \infty^*$, so $p \notin x^*$ for each $x \in X$. Hence, $x \notin \bigcap_{A \in p} \mathrm{cl}_X A$. Thus, $x \notin \mathrm{cl}_X A$ for some $A \in p$. Conversely, suppose that for each $x \in X$ there exists $A \in p$ such that $x \notin \mathrm{cl}_X A$. Hence, $p \notin B(X)$, and thus, $p \in \infty^*$.

(b) Let $U \in p$ for each $U \in \tau_x$, thus $U \cap A \neq \emptyset$ for each $U \in \tau_x$ and for each $A \in p$. This implies $x \in \mathrm{cl}_X A$ for each $A \in p$. Therefore, $p \in x^*$.

(c) Let $U \in \tau_x$ and $p \in x^*$. Since $U \cap A \neq \emptyset$ for each $A \in p$, so $U \in p$. Conversely, let $U \in \bigcap_{p \in x^*} p$ and $x \notin \mathrm{int}_X U$. Then $x \in \mathrm{cl}_X U^c$ and by Example 2.4, there exists $p \in \beta X_d$ such that $U^c \in p \subseteq \mathcal{R}$. This is a contradiction.

(d) By Theorem 2.3 and Example 2.4, there exists $p \in \beta X_d$, such that $A \in p \in x^*$. This implies that $\mathrm{cl}_{\beta X_d} A \cap x^* \neq \emptyset$. Now let $\mathrm{cl}_{\beta X_d} A \cap x^* \neq \emptyset$, so there exists $p \in x^*$ such that $p \in \mathrm{cl}_{\beta X_d} A$. Thus $A \in p \in x^*$ and so $x \in \mathrm{cl}_X A$.

(e) and (f) are obvious.

□

For every net $S = \{x_\alpha\}_{\alpha \in I}$ in a topological space X, the family $\mathcal{F}(S)$, consisting of all sets $A \subseteq X$ with the property that there exists $\alpha_\circ \in I$ such that $x_\alpha \in A$ whenever $\alpha \geq \alpha_\circ$, is a filter on X (see Theorem 1.6.12 in [1]). So $\{\{x_\alpha : \alpha > \alpha_\circ\} : \alpha_\circ \in I\}$ has the finite intersection property.

Lemma 3.2. *Let X be a topological space. Then:*

(a) *Let $\{x_\alpha\}_{\alpha \in I}$ be a net in X. If $x_\alpha \to p$ in βX_d for some $p \in x^*$, then $x_\alpha \to x$ in (X, τ).*

(b) *Let $\{x_\alpha\}_{\alpha \in I}$ be a net in X and $x_\alpha \to x$ in X. Then there exists $p \in x^*$, which is a cluster point of $\{x_\alpha\}_{\alpha \in I}$ in βX_d.*

(c) *Let $A \subseteq X$ be closed. Then we have, A is compact if and only if $\mathrm{cl}_{\beta X_d} A \cap \infty^* = \emptyset$.*

Proof. (a) Let U be an open neighborhood of x. Then $U \in p$ (Lemma 3.1), and so there exists $\beta \in I$ such that $x_\alpha \in U$ for each $\alpha > \beta$. This implies $x_\alpha \to x$ in (X, τ).

(b) $\{\{x_\gamma : \gamma > \beta\} : \beta \in I\}$ has the finite intersection property, then there exists $p \in \beta X_d$, such that $\{x_\gamma : \gamma > \beta\} \in p$ when $\beta \in I$. Since $x_\alpha \to x$ in X, thus for each open neighborhood U of x, there exists $\beta \in I$, such that $x_\gamma \in U$ when $\gamma > \beta$ and so $U \in p$. Lemma 3.1 (ii) implies $p \in x^*$. It is obvious p is cluster point of $\{x_\alpha\}_{\alpha \in I}$ in βX_d.

(c) Let $A \subseteq X$ be compact and $\mathrm{cl}_{\beta X_d} A \cap \infty^* \neq \emptyset$, so there exists a net $\{x_\alpha\}_{\alpha \in I}$ in A such that $x_\alpha \to p$ for some $p \in \mathrm{cl}_{\beta X_d} A \cap \infty^*$. Since A is compact, so there is a subnet $\{x_\beta\}_{\beta \in J}$ such that $x_\beta \to x \in A$ and so by (b), there is $q \in x^*$ such that $x_\beta \to q$. This implies $p = q$, and hence by Lemma 3.1 we have a contradiction. Conversely, let $\mathrm{cl}_{\beta X_d} A \cap \infty^* = \emptyset$ and $A \subseteq X$ be not compact. Hence, there is a net $\{x_\alpha\}_{\alpha \in I}$ in A, such that any subnet of $\{x_\alpha\}_{\alpha \in I}$ is divergent in A. Since βX_d is compact, so there is a subnet $\{x_\beta\}_{\beta \in J}$, such that $x_\beta \to p \in \mathrm{cl}_{\beta X_d} A$. Also $p \in \infty^*$, because if $p \notin \infty^*$ then $p \in y^*$ for some $y \in X$, so by (a), $x_\beta \to y$ in X. This implies that $y \in \mathrm{cl}_X A$ and this is a contradiction. Thus $p \in \mathrm{cl}_{\beta X_d}(A) \cap \infty^*$ and we have another contradiction.

□

Theorem 3.3 (Robinson's Compactness)**.** *Let (X, τ) be a topological space. Then $A \subseteq X$ is compact if and only if for every $p \in \mathrm{cl}_{\beta X_d} A$ there exists $x \in A$ such that $p \in x^*$.*

Proof. First suppose that A is compact but there is a point $p \in \mathrm{cl}_{\beta X_d} A$ such that $p \notin x^*$ for each $x \in A$. Then every $x \in A$ possesses an open neighborhood U_x, with $U_x \notin p$. Now, since A is compact, the open cover $\{U_x : x \in A\}$ of A has a finite subcover, say $\{U_{x_1}, \ldots, U_{x_n}\}$; i.e. $A \subseteq \cup_{i=1}^n U_{x_i}$. Thus

$$\mathrm{cl}_{\beta X_d} A \subseteq \mathrm{cl}_{\beta X_d}(\cup_{i=1}^n U_{x_i}) = \cup_{i=1}^n \mathrm{cl}_{\beta X_d}(U_{x_i}).$$

So $p \in \mathrm{cl}_{\beta X_d}(U_{x_i})$ for some $i \in \{1, \ldots, n\}$, which implies that $U_{x_i} \in p$, and we have a contradiction. Conversely, let $\{x_\beta\}_{\beta \in I}$ be a net in A. Since βX_d is compact space, so there is a subnet $\{x_\beta\}_{\beta \in J}$ such that $x_\beta \to p \in \mathrm{cl}_{\beta X_d} A$. Therefore there exists $x \in A$ such that $p \in x^*$. Now by Lemma 3.3(a), $x_\beta \to x$ in A. This completes the Proof. □

Theorem 3.4. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then the following statements are equivalent.*

(*a*) *$f : X \to Y$ is continuous.*
(*b*) *For each $x \in X$, $f^\beta(x^*) \subseteq (f(x))^*$, where $f^\beta : \beta X_d \to \beta Y_d$ is natural extension of $f : X_d \to Y_d$.*
(*c*) *$f^\beta : B(X) \to B(Y)$ is well defined and continuous.*

Proof. (*a*) implies (*b*): Let $f : X \to Y$ be continuous and $p \in x^*$. We must show $f^\beta(p) \in (f(x))^*$. By Lemma 2.1, we have

$$f^\beta(p) = \{A \subseteq Y : f^{-1}(A) \in p\}.$$

Now let $f^{-1}(A) \in p$ for some $A \subseteq Y$, so $x \in \mathrm{cl}_X f^{-1}(A)$. Hence, there exists a net $\{x_\alpha\}_{\alpha \in I} \subseteq f^{-1}(A)$, such that $x_\alpha \to x$ and so $f(x_\alpha) \to f(x)$. This implies $f(x) \in \mathrm{cl}_Y A$, and hence, $f^\beta(p) \in (f(x))^*$.

(*b*) implies (*a*): Let $f^\beta(x^*) \subseteq (f(x))^*$ for each $x \in X$ and there exists a net $\{x_\alpha\}_{\alpha \in I}$, such that $x_\alpha \to x$ for some $x \in X$ and $f(x_\alpha)$ is not convergent to $f(x)$. Since $x_\alpha \to x$, so for each open neighborhood $U \in \tau_X$ of x there exists $\beta_U \in I$, such that $x_\alpha \in U$ for each $\alpha > \beta_U$. Thus,

$$A = \{\{x_\alpha : \alpha > \beta_U\} : x \in U \in \tau_X\},$$

has the finite intersection property. Therefore, there exists an ultrafilter p contains A. It is obvious that $p \in x^*$, because for each $A \in p$ and for each open neighborhood U of x, we have $A \cap U \neq \emptyset$ so $x \in \mathrm{cl}_X A$. Since $f(x_\alpha)$ is not convergent to $f(x)$, so there exists a subnet $\{x_\beta\}$ such that $x_\beta \to x$ and for some open neighborhood U of $f(x)$, $f(x_\beta) \notin U$ for each β. Since U is an open neighborhood of $f(x)$, so for each $B \subseteq Y$, if $f(x) \in \mathrm{cl}_Y B$, then $U \cap B \neq \emptyset$. This implies $(f(x))^* \subseteq \mathrm{cl}_{\beta Y_d} U$, in particular, $U \in f^\beta(p)$ and hence $f^{-1}(U) \in p$, (see Lemmas 2.1 and 3.1). Now we have $\{x_\beta : \beta\} \cap f^{-1}(U) \neq \emptyset$, and this is a contradiction.

(a) implies (c), and (b) implies (c): Are obvious.

(c) implies (a): Let $U \in \tau_Y$, then $\mathrm{cl}_{\beta Y_d} \tau_Y \subseteq \mathrm{cl}_{\beta Y_d} U$. This implies

$$\mathrm{cl}_{\beta X_d} \tau_X \subseteq (f^\beta)^{-1}(\mathrm{cl}_{\beta Y_d} U) = \mathrm{cl}_{\beta X_d} f^{-1}(U),$$

and therefore $f^{-1}(U) \in \tau_X$.

□

Corollary 3.5. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then $f : X \to Y$ is continuous if and only if $f^\beta : B(X) \to B(Y)$ with $f^\beta(x^*) \subseteq (f(x))^*$ for each $x \in X$, is well define.*

Proof. Obvious. □

Theorem 3.6 (Open Mapping Theorem)**.** *Let (X, τ_X) and (Y, τ_Y) be two topological spaces, $f : X \to Y$ be a continuous function and x^* be a closed subset of βX_d for each $x \in X$. Then f is open if and only if $(f(x))^* \subseteq f^\beta(x^*)$ for each $x \in X$.*

Proof. Assume that f is open, and $x \in X$. Therefore,

$$\{f(U) : U \in \tau_x\},$$

is a neighborhood base at $f(x)$, and thus

$$\begin{aligned} f^\beta(x^*) &= f^\beta(\mathrm{cl}_{\beta X_d}(x^*)) \\ &= f^\beta\Big(\bigcap_{U \in \tau_x} \mathrm{cl}_{\beta X_d} U\Big) \text{ (by Lemma 3.1(e))} \\ &= \bigcap_{U \in \tau_x} f^\beta(\mathrm{cl}_{\beta X_d} U) \\ &= \bigcap_{U \in \tau_x} (\mathrm{cl}_{\beta Y_d} f(U)) \text{ (f is continuous)} \\ &\supseteq \bigcap_{V \in \tau_{f(x)}} (\mathrm{cl}_{\beta Y_d} V) \\ &= (f(x))^*. \end{aligned}$$

Conversely, assume $(f(x))^* \subseteq f^\beta(x^*)$ for each $x \in X$. To show f is open, pick $U \in \tau_X$ and $y \in f(U)$. Then $y = f(x)$ for some $x \in U$, and

$$y^* = (f(x))^* \subseteq f^\beta(x^*) \subseteq f^\beta(\mathrm{cl}_{\beta X_d} U) \subseteq \mathrm{cl}_{\beta Y_d} f(U).$$

Thus, $f(U) \in p$ for each $p \in y^*$. So $f(U)$ is a neighborhood of y, by Lemma 3.1(c). This completes the proof. □

Theorem 3.7. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces, let $f : X \to Y$ be a continuous function, and K be a compact subset of X. Then $f(K)$ is compact.*

Proof. Pick $q \in \mathrm{cl}_{\beta Y_d} f(K)$, so there exists $p \in \mathrm{cl}_{\beta X_d} K$, such that $f^\beta(p) = q$. Since K is compact, by Theorem 3.3, there exists $x \in K$ such that $p \in x^*$. As f is continuous, $f^\beta(x^*) \subseteq (f(x))^*$, by Theorem 3.4, and hence $q = f^\beta(p) \in (f(x))^*$. So $f(K)$ is compact, by Theorem 3.3. □

4. The topology generated by ultrafilters

Let X be an arbitrary set; by generating a topology on X we mean selecting a family τ of subsets of X that satisfies conditions of axiom of topology, i.e., a family τ such that the pair (X, τ) is a topological space. We shall now present a new method of generating topology. This method was included in the definition of a neighborhood system combine with ultrafilter converges to a point.

Definition 4.1. For each $x \in X$, let x^* be a subset of βX_d such that $\widehat{x} \in x^*$. For $x \in X$, define $\tau^*_x = \bigcap_{p \in x^*} p$.

Theorem 4.2. (a) *Suppose we are given a set X and a collection $\{x^*\}_{x \in X}$ of families of subsets of βX_d. Then the collection $\{\tau^*_x\}_{x \in X}$ has the properties $(BP1) - (BP3)$ in [1]. Let τ be the family of all subsets of X that are unions of subfamilies of $\cup_{x \in X} \tau^*_x$. The family τ is a topology on X, called the topology generated by the ultrafilters system $\{x^*\}_{x \in X}$, and the collection $\{\tau^*_x\}_{x \in X}$ is a neighborhood system (see [1]) for the topological space (X, τ).*

(b) *Let (X, τ) be a topological space, and x^* be the collection of all ultrafilters near to x with respect to τ. Then τ^* the topology generated τ^* by the ultrafilters system $\{x^*\}_{x \in X}$ is finer than τ, and $\tau_x = \tau^*_x$ for each $x \in X$.*

Proof. Obvious. □

Example 4.3. (a) For each $x \in X$, let $x^* = \beta X_d$. Then the topology generated by the ultrafilters system is the anti-discrete topology, i.e. $\tau = \{X, \emptyset\}$.

(b) For each $x \in X$, let $x^* = \{\widehat{x}\}$, then the topology generated by the ultrafilters system is the discrete topology.

(c) Let X be an infinite set and pick $x_\circ \in X$. Define $x^* = \{\widehat{x}\}$ for each $x \neq x_\circ$ and $x^*_\circ = \beta X_d - \cup_{x \neq x_\circ} x^*$. Then all one-point subsets of X, except for the set $\{x_\circ\}$, are open-and-closed; the set $\{x_\circ\}$ is closed but is not open. For more details see Example 1.1.8 in [1].

Theorem 4.4 (Separation Axioms)**.** *Let (X,τ) be a topological space and $\{x^*\}_{x\in X}$ be the ultrafilter system with respect to topology τ. Then,*

(a) *(X,τ) is a $T_\circ$ space if and only if for each $x,y\in X$, $x^*\neq y^*$.*
(b) *(X,τ) is a T_1 space if and only if for each $x,y\in X$, $x^*-y^*\neq\emptyset$ and $y^*-x^*\neq\emptyset$.*
(c) *(X,τ) is a T_2 space(Hausdorff) if and only if for each $x,y\in X$, $x^*\cap y^*=\emptyset$.*
(d) *(X,τ) is a regular space if and only if X is a T_1 space and for every $x\in X$ and every closed set $F\subseteq X$ such that $x\notin F$, $x^*\cap \mathrm{cl}_{\beta X_d}(F^*)=\emptyset$.*
(e) *(X,τ) is a completely regular space then for every $x\in X$ and every closed set $F\subseteq X$ such that $x\notin F$, $\mathrm{cl}_{\beta X_d}(x^*)\cap\mathrm{cl}_{\beta X_d}(F^*)=\emptyset$.*
(f) *(X,τ) is a normal space if and only if X is a T_1 space and for every pair of disjoint closed subsets $A,B\subseteq X$, $\mathrm{cl}_{\beta X_d}(A^*)\cap\mathrm{cl}_{\beta X_d}(B^*)=\emptyset$.*

Proof. (a) Let X be a $T_\circ$ space, so for every pair of distinct points $x,y\in X$ there exists an open set U such that $x\in U$ and $y\notin U$. Thus $x^*\subseteq \mathrm{cl}_{\beta X_d}U$ and there exists $q\in y^*$ such that $q\notin \mathrm{cl}_{\beta X_d}U$ and hence $x^*\neq y^*$. Conversely, let $x^*\neq y^*$ for each pair of distinct points $x,y\in X$. Without loss of generality, we may assume that $p\in y^*-x^*$, so there is a neighborhood U of x such that $p\notin \mathrm{cl}_{\beta X_d}U$. Notice that $y\notin U$ (Otherwise $y\in U$ implies $p\in y^*\subseteq \mathrm{cl}_{\beta X_d}U$ which is a contradiction). So there exists an open U such that $x\in U$ and $y\notin U$. Thus X is a $T_\circ$ space.

(b) Suppose that X is a T_1 space and $x\neq y$ in X. So there exist $U\in\tau_x$ and $V\in\tau_y$ such that $x\notin V$ and $y\notin U$. Thus $V\notin p$ for some $p\in x^*$ and $U\notin q$ for some $q\in y^*$. Therefore $x^*-y^*\neq\emptyset$ and $y^*-x^*\neq\emptyset$. Conversely, let for each $x,y\in X$, $x^*-y^*\neq\emptyset$ and $y^*-x^*\neq\emptyset$. By (1), $x^*-y^*\neq\emptyset$ implies that there is a neighborhood U of x such that $y\notin U$, and $y^*-x^*\neq\emptyset$ implies that there is a neighborhood V of y such that $x\notin V$. This completes the Proof.

(c) Let X be a Hausdorff space and pick $x\neq y$ in X. So there exist two open sets U and V such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. Thus $\mathrm{cl}_{\beta X_d}U\cap\mathrm{cl}_{\beta X_d}V=\emptyset$. Since $x^*\subseteq \mathrm{cl}_{\beta X_d}U$ and $y^*\subseteq \mathrm{cl}_{\beta X_d}V$, so $x^*\cap y^*=\emptyset$. Conversely, let X not be Hausdorff space, so there exist x and y in X such that $U\cap V\neq\emptyset$ for each $U\in\tau_x$ and $V\in\tau_y$. Now let

$$\mathcal{R}_y=\{A\subseteq X: y\in \mathrm{cl}_X A\},$$

so $\mathcal{R}_y$ is partition regular and $\mathcal{R}_y = \mathcal{R}^{\uparrow}$ (see Theorem 2.3). It is obvious that $\tau_x \subseteq \mathcal{R}_y$. By Theorem 2.3(b), there exists $p \in \beta X_d$ such that $\tau_x \subseteq p \subseteq \mathcal{R}_y$. Therefore $p \in x^* \cap y^*$ and this is a contradiction.

(d) Let X be regular. Let F be closed in X and $x \notin F$, then there are disjoint open sets U and V in X with $x \in U$ and $F \subseteq V$. So

$$\mathrm{cl}_{\beta X_d} U \cap \mathrm{cl}_{\beta X_d} V = \emptyset.$$

This implies that $x^* \cap \mathrm{cl}_{\beta X_d}(F^*) = \emptyset$. Conversely, let F be a closed subset of X, $x \notin F$ and let $\tau_F = \{U \in \tau : F \subseteq U\}$. If $U \cap V \neq \emptyset$ for each $U \in \tau_x$ and $V \in \tau_F$, then $\tau_F \subseteq \mathcal{R}_x$. Thus by Theorem 2.3, there exists $p \in \beta X_d$ such that $\tau_F \subseteq p \subseteq \mathcal{R}_x$, and hence $p \in x^* \cap \overline{\tau_F} = x^* \cap \mathrm{cl}_{\beta X_d}(F^*) = \emptyset$ is a contradiction.

(e) Let X be completely regular. Let F be a closed subset of X and pick $x \in X - F$. So there exists a continuous function $f : X \to [0, 1]$ such that $f|_F = 0$ and $f(x) = 1$. Hence $f^\beta : \beta X_d \to \beta[0, 1]_d$ by $f^\beta(t^*) = (f(t))^*$ for each $t \in X$ is continuous (see Theorem 3.4). Since f^β is continuous, so $(f^\beta)^{-1}(\{0\})$ and $(f^\beta)^{-1}(\{1\})$ are closed, $F^* \subseteq (f^\beta)^{-1}(\{0\})$ and $(f^\beta)^{-1}(\{0\}) \cap (f^\beta)^{-1}(\{1\}) = \emptyset$. Thus $\mathrm{cl}_{\beta X_d}(x^*) \cap \mathrm{cl}_{\beta X_d}(F^*) = \emptyset$.

(f) Let X be normal. Let A and B be two disjoint closed subsets of X. Then there are disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. Thus

$$\mathrm{cl}_{\beta X_d}(A^*) \cap \mathrm{cl}_{\beta X_d}(B^*) \subseteq \mathrm{cl}_{\beta X_d} U \cap \mathrm{cl}_{\beta X_d} V = \emptyset,$$

and the proof is finished. Conversely, let A and B be two disjoint closed subsets of X such that $\mathrm{cl}_{\beta X_d}(A^*) \cap \mathrm{cl}_{\beta X_d}(B^*) = \emptyset$. Define $\tau_A = \{U \in \tau : A \subseteq U\}$ and $\tau_B = \{U \in \tau : B \subseteq U\}$. If $U \cap V \neq \emptyset$ for each $U \in \tau_A$ and $V \in \tau_B$, so $\tau_B \subseteq \mathcal{R}_A$. Thus by Theorem 2.3, there exists $p \in \beta X_d$ such that $\tau_B \subseteq p \subseteq \mathcal{R}_A$, and hence $p \in A^* \cap \overline{\tau_B} = A^* \cap \mathrm{cl}_{\beta X_d}(B^*) = \emptyset$ is a contradiction. This completes the proof.

□

Definition 4.5. Let (X, τ) be a topological space. Then,

(a) (X, τ) is called F-Hausdorff (finer than Hausdorff) if and only if for every $x \neq y$ in X, $x^* \cap \overline{y^*} = \emptyset$.

(b) (X, τ) is called E-Hausdorff (extremely Hausdorff) if and only if for every $x \neq y$ in X, $\overline{x^*} \cap \overline{y^*} = \emptyset$.

Example 4.6. (a) Let X be an infinite set, A and B be disjoint subsets of $\beta X_d - X$ such that $A \cap \mathrm{cl}_{\beta X_d} B = \emptyset$, $B \cap \mathrm{cl}_{\beta X_d} A = \emptyset$ and $\mathrm{cl}_{\beta X_d} A \cap \mathrm{cl}_{\beta X_d} B \neq \emptyset$. Pick x_1 and x_2 in X, and define $x_1^* =$

$A \cup \{\widehat{x}_1\}$, $x_2^* = B \cup \{\widehat{x}_2\}$ and $x^* = \{\widehat{x}\}$ for each $x \in X - \{x_1, x_2\}$. Then $\{x^*\}_{x\in X}$ is an ultrafilter system. Let τ be the topology generated by the ultrafilter system $\{x^*\}_{x\in X}$. It is obvious that (X, τ) is F-Hausdorff, but is not E-Hausdorff, because $\{x_2\}$ is closed subset of X and $x_1 \notin \{x_2\}$ and $\overline{x_1^*} \cap \overline{x_2^*} \neq \emptyset$.

(*b*) Let X be an infinite set, A and B be disjoint subsets of $\beta X_d - X$ such that $\mathrm{cl}_{\beta X_d} A \cap \mathrm{cl}_{\beta X_d} B = \emptyset$. Pick x_1 and x_2 in X, and define $x_1^* = A \cup \{\widehat{x}_1\}$, $x_2^* = B \cup \{\widehat{x}_2\}$ and $x^* = \{\widehat{x}\}$ for each $x \in X - \{x_1, x_2\}$. Then $\{x^*\}_{x\in X}$ is an ultrafilter system. Let τ be the topology generated by the ultrafilter system $\{x^*\}_{x\in X}$, then it is obvious that X is Hausdorff. Let T be a closed subset of X, by Lemma 3.1, $x \in T$ if and only if $x^* \cap \mathrm{cl}_{\beta X_d} T \neq \emptyset$. Let $x \in X - \{x_1, x_2\}$ and T be a closed subset of X such that $x \notin T$, then $x^* \cap T^* = \emptyset$. Now let $x = x_1$ and T be a closed subset of X such that $x_1 \notin T$ so $x_1^* \cap T^* = \emptyset$. In a similar way, for $x_2 \neq x$ and closed subset T of X that $x_2 \notin T$, we have $x_2^* \cap T^* = \emptyset$. Therefore X is a regular space, by Theorem 4.4. Also X is E-Hausdorff.

(*b*) Every E-Hausdorff space is a F-Hausdorff.

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