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On the cyclic homology of multiplier Hopf algebras

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Abstract. In this paper, we will study the theory of cyclic homology for regular multiplier Hopf algebras. We associate a cyclic module to a triple $(\mathcal{R}, \mathcal{H}, \mathcal{X})$ consisting of a regular multiplier Hopf algebra $\mathcal{H}$, a left $\mathcal{H}$-comodule algebra $\mathcal{R}$, and a unital left $\mathcal{H}$-module $\mathcal{X}$ which is also a unital algebra. First, we construct a paracyclic module to a triple $(\mathcal{R}, \mathcal{H}, \mathcal{X})$ and then prove the existence of a cyclic structure associated to this triple.

1. Introduction

In 2003, M. Khalkhali and B. Rangipour ([4]) a cyclic module according to a given triple $(\mathcal{R}, \mathcal{H}, (\mathcal{X})$ consisting of a Hoph algebra $\mathcal{H}$, a left $\mathcal{H}$-module algebra $\mathcal{R}$ and a left $\mathcal{H}$-module $\mathcal{X}$ and a suitably chosen group like element.

In this paper, we generalize their work to regular multiplier Hopf algebras. In fact, the aim of this work is to study the cyclic homology theory for regular multiplier Hopf algebras and provide a general framework to define and study operations in cyclic homology theory associated to a triple $(\mathcal{R}, \mathcal{H}, \mathcal{X})$ consisting of a regular multiplier Hopf algebra $\mathcal{H}$, a left $\mathcal{H}$-comodule algebra $\mathcal{R}$, and a unital left $\mathcal{H}$-module $\mathcal{X}$ which is also a unital algebra. In this paper, such triples are called left $\mathcal{H}$-triple.

Also in this paper, all vector spaces will be spaces over the complex field $\mathbb{C}$. The notation $V \otimes W$ stands for tensor product of two vector spaces V and W. For the background about Hopf algebras and multiplier Hopf algebras one can see [1, 5, 8].

Let us remind some facts and notations for the convenience of the reader.

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An algebra $\mathcal{A}$ is called non-degenerate if and only if for all $a \in \mathcal{A}$ we have

$$(\forall b \in \mathcal{A} : ab = 0) \quad \Rightarrow \quad a = 0, \qquad (\forall b \in \mathcal{A} : ba = 0) \quad \Rightarrow \quad a = 0.$$

It is clear that any unital algebra (an algebra $\mathcal{A}$ is called unital if there exists an element 1 in $\mathcal{A}$ such that, for every element $a \in \mathcal{A}$ we have $a.1 = a$ and $1.a = a$.) is non-degenerate. From Lemma $A.2$ in [8], the algebraic tensor product of two non-degenerate algebras is a non-degenerate algebra.

Recall that a multiplier of $\mathcal{A}$ is a pair (l, r) of linear maps from $\mathcal{A}$ to itself satisfying for all $a, b \in \mathcal{A}$:

$$l(ab) = l(a)b, \qquad r(ab) = ar(b), \qquad r(a)b = al(b).$$

The set of all multipliers of $\mathcal{A}$, denoted by $M(\mathcal{A})$, is a unital algebra in a natural way and called the multiplier algebra of $\mathcal{A}$. It is easy to see that $\mathcal{A}$ is a two-sided ideal in $M(\mathcal{A})$, and moreover $M(\mathcal{A}) = \mathcal{A}$ if and only if $\mathcal{A}$ is unital.

For two non-degenerate algebras $\mathcal{A}, \mathcal{B}$, a homomorphism φ from $\mathcal{A}$ to $M(\mathcal{B})$ is called non-degenerate if and only if $\varphi(\mathcal{A})\mathcal{B} = \mathcal{B} = \mathcal{B}\varphi(\mathcal{A})$, where $\varphi(\mathcal{A})\mathcal{B} = \{\varphi(a).b : a \in A, b \in B\}$ and $\varphi(a)$ and b belong to the algebra B. By Proposition $A.5$ in [8], such a non-degenerate homomorphism has a unique extension to a unital homomorphism $M(\mathcal{A}) \longrightarrow M(\mathcal{B})$.

For the following Definition 1.1, we refer to [3, 8, 9].

Definition 1.1. Assume that $\mathcal{H}$ is a non-degenerate algebra and Δ from $\mathcal{H}$ to $M(\mathcal{H} \otimes \mathcal{H})$ is a non-degenerate homomorphism. The pair $(\mathcal{H}, \Delta)$ is called a multiplier Hopf algebra if

(i) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$,
(ii) $\Delta(h)(1 \otimes k)$ and $(h \otimes 1)\Delta(k)$ are elements in $\mathcal{H} \otimes \mathcal{H}$, (we know that these elements are in $M(\mathcal{H} \otimes \mathcal{H})$ not neccessary in $\mathcal{H} \otimes \mathcal{H}$),
(iii) the linear mappings $T_1, T_2 : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$,

defined by

$$T_1(h \otimes k) = \Delta(h)(1 \otimes k), \qquad T_2(h \otimes k) = (h \otimes 1)\Delta(k),$$

are bijective, for all $h, k \in \mathcal{H}$.

Note that Δ is a non-degenerate homomorphism, so $\Delta \otimes id$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{H})$. Similarly, $id \otimes \Delta$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{H})$. The homomorphism Δ is called a comultiplication on $\mathcal{H}$.

We can consider the opposite comultiplication Δ' obtained from Δ by composing it with the flip operator on $\mathcal{H} \otimes \mathcal{H}$ (extended to $M(\mathcal{H} \otimes \mathcal{H})$).

A multiplier Hopf algebra $(\mathcal{H}, \Delta)$ is said to be regular if $(\mathcal{H}, \Delta')$ is also a multiplier Hopf algebra.

Now let us recall from [3, 8] some properties of a regular multiplier Hopf algebra. In the following Theorem, (vi) is proved by Proposition 2.2 in [3].

Theorem 1.2. *Assume that $(\mathcal{H}, \Delta)$ is a regular multiplier Hopf algebra. Then there exists a non-zero homomorphism ε from $\mathcal{H}$ to $\mathbb{C}$ and an invertible anti-homomorphism S from $\mathcal{H}$ to $\mathcal{H}$ such that for all $h, k \in \mathcal{H}$ we have*

(i) $(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$,
(ii) $m(S \otimes id)(\Delta(h)(1 \otimes k)) = \varepsilon(h)k$,
(iii) $m(id \otimes S)((k \otimes 1)\Delta(h)) = \varepsilon(h)k$,
(iv) $\varepsilon' = \varepsilon$ *and* $S' = S^{-1}$, *where* ε', S' *are the maps associated to* $(\mathcal{H}, \Delta')$,
(v) $(1 \otimes S(k))\Delta(S(h)) = (S \otimes S) \circ \tau(\Delta(h)(k \otimes 1))$,
(vi) *for all* $h_1, h_2, \ldots, h_n$ *in* $\mathcal{H}$, *there exists an element* $h \in \mathcal{H}$ *such that* $hh_i = h_i$, *for all* i,
(vii) *if* $\mathcal{H}$ *is commutative, then* $S^2 = id$.

As usual, ε is called the counit and S the antipode of $(\mathcal{H}, \Delta)$. Note that from (vi) in Theorem 1.2 we have $\mathcal{H}^2 = \mathcal{H}$.

Throughout this paper, we fix the regular multiplier Hopf algebra $\mathcal{H}$, and we will use $m, \Delta, \varepsilon, S$ for the multiplication, the comultiplication, the counit and the antipode, respectively.

2. Comodules Of Multiplier Hopf Algebras

Let us give a brief summary of how this section is organized. We will show that the tensor product of two left $\mathcal{H}$-comodules is a left $\mathcal{H}$-comodule. Note that in the case when $\mathcal{H}$ is commutative, we show that the tensor product of two left $\mathcal{H}$-comodule algebras is a left $\mathcal{H}$-comodule algebra. The space of coinvariants of a left $\mathcal{H}$-comodule (algebra) $\mathcal{R}$ form a subalgebra of $M(\mathcal{R})$, denoted by $\mathcal{R}^{co(\mathcal{H})}$, will play an important role in this paper. In the case when $\mathcal{H}$ is commutative, then we have

$$(\mathcal{H} \otimes \mathcal{R})^{co(\mathcal{H})} \cong M(\mathcal{R}).$$

For the study of the right $\mathcal{H}$-comodules, one can refer to [9]. In this paper, we are going to work on the left $\mathcal{H}$-comodules.

Definition 2.1. A left $\mathcal{H}$-comodule is a non-degenerate algebra $\mathcal{R}$ together with an injective linear mapping

$$\beta : \mathcal{R} \to M(\mathcal{H} \otimes \mathcal{R}),$$

such that

(i) $\beta(r)(h \otimes 1) \in \mathcal{H} \otimes \mathcal{R}$ and $(h \otimes 1)\beta(r) \in \mathcal{H} \otimes \mathcal{R}$,
(ii) $(\Delta \otimes id)(\beta(r))(h \otimes 1 \otimes 1) = (id \otimes \beta)(\beta(r)(h \otimes 1))$,

for all $r \in \mathcal{R}$ and $h \in \mathcal{H}$.

If in addition β is an algebra map, then $\mathcal{R}$ is called a left $\mathcal{H}$-comodule algebra.

Note that Δ is non-degenerate, $\Delta \otimes id$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{R})$.

It is simple to see that (ii) in Definition 2.1 is equivalent to

$$(h \otimes 1 \otimes 1)(\Delta \otimes id)(\beta(r)) = (id \otimes \beta)((h \otimes 1)\beta(r)), \tag{2.1}$$

for all $r \in \mathcal{R}$ and $h \in \mathcal{H}$.

Also, if ε is non-degenerate, then $\varepsilon \otimes id$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{R})$.

For the following Lemma, see Proposition 2.3 in [9].

Lemma 2.2. *The injectivity of β in Definition 2.1 is equivalent to the counitary property, that is, $(\varepsilon \otimes id)\beta = id$.*

Example 2.3. Any regular multiplier Hopf algebra $(\mathcal{H}, \Delta)$ is a left $\mathcal{H}$-comodule algebra via $\Delta : \mathcal{H} \to M(\mathcal{H} \otimes \mathcal{H})$.

In the following example, assume that $\mathcal{H}$ is an algebraic quantum group with a non-zero right-invariant linear functional ϕ and $\widehat{\mathcal{H}}$ denotes the dual algebraic quantum group of $\mathcal{H}$. Let us give a brief overview from the dual algebraic quantum group $\widehat{\mathcal{H}}$ of $\mathcal{H}$. In the following, by $\mathcal{H}^*$ we denote the space of all linear functionals on $\mathcal{H}$. Let $k \in \mathcal{H}$ and $\omega \in \mathcal{H}^*$. Then, $(\omega \otimes id)\Delta(k)$ will be element in $M(\mathcal{H})$ such that:

$$(\omega \otimes id)\Delta(k)h = (\omega \otimes id)(\Delta(k)(1 \otimes h)),$$

$$h(\omega \otimes id)\Delta(k) = (\omega \otimes id)((1 \otimes h)\Delta(k)),$$

for all $h \in \mathcal{H}$. We will use a similar notation for $(id \otimes \omega)\Delta(k)$. Also, $k\omega$ and ωk are elements in $\mathcal{H}^*$ defined by $(k\omega)(h) = \omega(hk)$ and $(\omega k)(h) = \omega(kh)$ respectively, for all $h \in \mathcal{H}$. Then, $\widehat{\mathcal{H}}$ is a subspace of $\mathcal{H}^*$ consisting of all functionals $k\phi$, where $k \in \mathcal{H}$, for all $k \in \mathcal{H}$ that ϕk belongs to $\widehat{\mathcal{H}}$ and it is not difficult to check that $(\omega \otimes id)\Delta(k)$ and $(id \otimes \omega)\Delta(k)$ are elements in $\mathcal{H}$, for all $\omega \in \widehat{\mathcal{H}}$. Also $\widehat{\mathcal{H}}$ is a non-degenerate algebra under the definition

$$(\omega_1\omega_2)(k) = \omega_1((id \otimes \omega_2)\Delta(k)),$$

for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}$ and $k \in \mathcal{H}$ (see [3, 7, 9] for more details).

The following example shows that $\widehat{\mathcal{H}}$ is a left $\mathcal{H}$-comodule.

Example 2.4. Because $\widehat{\mathcal{H}}$ is a subset of $\mathcal{H}^*$, one can consider $\mathcal{H}\otimes\widehat{\mathcal{H}}$ in a natural way as a subspace of the linear operators on $\mathcal{H}$. We define the linear mapping $\beta:\widehat{\mathcal{H}}\to M(\mathcal{H}\otimes\widehat{\mathcal{H}})$ by

$$(\beta(\omega)(h\otimes\omega_1))(k)=(\omega\otimes id)(\Delta((id\otimes\omega_1)\Delta(k))(1\otimes h)),$$
$$((h\otimes\omega_1)\beta(\omega))(k)=(\omega\otimes id)((1\otimes h)\Delta((\omega_1\otimes id)\Delta(k))),$$

for all $h,k\in\mathcal{H}$ and $\omega,\omega_1\in\widehat{\mathcal{H}}$. We have

$$\beta(\omega)(h\otimes 1)=\sum_{i=1}^{n} q_i\otimes p_i\phi\in\mathcal{H}\otimes\widehat{\mathcal{H}},$$

where $\omega=z\phi$ and

$$z\otimes h=\sum_{i=1}^{n}\Delta(p_i)(1\otimes q_i).$$

Also,

$$(h\otimes 1)\beta(\omega)=\sum_{i=1}^{n} q_i\otimes \phi p_i\in\mathcal{H}\otimes\widehat{\mathcal{H}},$$

where $\omega=\phi z$ and

$$z\otimes h=\sum_{i=1}^{n}(1\otimes q_i)\Delta(p_i).$$

We will use the formal notation to express a left $\mathcal{H}$-comodule $\mathcal{R}$. Let $\beta:\mathcal{R}\to M(\mathcal{H}\otimes\mathcal{R})$ be the structure map of $\mathcal{R}$. The condition (i) in definition 2.1 allows us for all $r\in\mathcal{R}$ and $h\in\mathcal{H}$ to write

$$\beta(r)(h\otimes 1)=\sum t_i\otimes s_i,$$

with a finite sum, where $t_i\in\mathcal{H}$ and $s_i\in\mathcal{R}$. For simplicity, in the following we use $\sum r^{(-1)}h\otimes r^{(0)}1$ for $\sum t_i\otimes s_i$. Similarly, one can write

$$(h\otimes 1)\beta(r)=\sum hr^{(-1)}\otimes 1r^{(0)},$$

for all $r\in\mathcal{R}$ and $h\in\mathcal{H}$. In particular, for all $h,k\in\mathcal{H}$, we put

$$\Delta(h)(1\otimes k)=\sum h^{(1)}1\otimes h^{(2)}k,$$
$$(1\otimes k)\Delta(h)=\sum 1h^{(1)}\otimes kh^{(2)},$$
$$\Delta(h)(k\otimes 1)=\sum h^{(1)}k\otimes h^{(2)}1,$$
$$(k\otimes 1)\Delta(h)=\sum kh^{(1)}\otimes 1h^{(2)}.$$

It is possible rewrite (ii) in Definition 2.1 as

$$\sum(r^{(-1)}p_i)^{(1)}1\otimes(r^{(-1)}p_i)^{(2)}q_i\otimes r^{(0)}1$$
$$=\sum r^{(-1)}h\otimes(r^{(0)}1)^{(-1)}k\otimes(r^{(0)}1)^{(0)}1,$$

where $h \otimes k = \sum \Delta(p_i)(1 \otimes q_i)$, for all $r \in \mathcal{R}$ and $h, k \in \mathcal{H}$.

Also, one can rewrite (2.1) as

$$\sum q_i(p_i r^{(-1)})^{(1)} \otimes 1(p_i r^{(-1)})^{(2)} \otimes 1r^{(0)} = \sum hr^{(-1)} \otimes k(1r^{(0)})^{(-1)} \otimes 1(1r^{(0)})^{(0)},$$

where $h \otimes k = \sum (q_i \otimes 1)\Delta(p_i)$, for all $r \in \mathcal{R}$ and $h, k \in \mathcal{H}$.

Take an element $h \in \mathcal{H}$ such that $\varepsilon(h) = 1$. Then $(\varepsilon \otimes id)\beta = id$ in Lemma 2.2 allows us for all $r \in \mathcal{R}$ to write

$$\sum \varepsilon(hr^{(-1)})1r^{(0)} = r = \sum \varepsilon(r^{(-1)}h)r^{(0)}1.$$

We would like prove that the tensor product of two left $\mathcal{H}$-comodules is a left $\mathcal{H}$-comodule. Moreover, in the case when $\mathcal{H}$ is commutative, we show that the tensor product of two left $\mathcal{H}$-comodule algebras is a left $\mathcal{H}$-comodule algebra.

Theorem 2.5. *Assume that $\mathcal{R}$ and $\mathcal{E}$ are two left $\mathcal{H}$-comodules via β and α, respectively. Then $\mathcal{R} \otimes \mathcal{E}$ is a left $\mathcal{H}$-comodule. Moreover, if $\mathcal{H}$ be commutative and in addition β and α be algebra maps, then $\mathcal{R} \otimes \mathcal{E}$ is a left $\mathcal{H}$-comodule algebra.*

Proof. $\mathcal{R} \otimes \mathcal{E}$ is a left $\mathcal{H}$-comodule, if there exists an injective linear mapping $\mu : \mathcal{R} \otimes \mathcal{E} \longrightarrow M(\mathcal{H} \otimes \mathcal{R} \otimes \mathcal{E})$ satisfying the properties of Definition 2.1.

For this we define the map μ as following:

$$\mu(r \otimes e)(h \otimes 1) = \sum \beta(r) \left(e^{(-1)}h \otimes 1\right) \otimes e^{(0)}1, \tag{2.2}$$

$$(h \otimes 1)\mu(r \otimes e) = \sum (T \otimes id) \left(1r^{(0)} \otimes \left(hr^{(-1)} \otimes 1\right) \alpha(e)\right), \tag{2.3}$$

Where $r \in \mathcal{R}, e \in \mathcal{E}$ and $h \in \mathcal{H}$. Here, $T : \mathcal{R} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{R}$ is the flip operator.

By definition, $\mu(\mathcal{R} \otimes \mathcal{E})(\mathcal{H} \otimes 1)$ and $(\mathcal{H} \otimes 1)\mu(\mathcal{R} \otimes \mathcal{E})$ are contained in $\mathcal{H} \otimes \mathcal{R} \otimes \mathcal{E}$. It is easy to check that $\mu(r \otimes e)$ is an element in $M(\mathcal{H} \otimes \mathcal{R} \otimes \mathcal{E})$.

To prove (ii) in Definition 2.1, assume that $h, k \in \mathcal{H}, r \in \mathcal{R}$ and $e \in \mathcal{E}$. Let

$$h \otimes k = \sum \Delta(p_i)(q_i \otimes 1),$$

for $p_i, q_i \in \mathcal{H}$. Also, let

$$\alpha(e)(h \otimes 1) = \sum t_i \otimes s_i,$$

for $t_i \in \mathcal{H}, s_i \in \mathcal{E}$. Then

$$\begin{aligned}(\Delta \otimes id)(\mu(r \otimes e))(h \otimes k \otimes 1) &= \sum (\Delta \otimes id)(\mu(r \otimes e)(p_i \otimes 1))(q_i \otimes 1 \otimes 1)\\ &= \sum (\Delta \otimes id)(\beta(r)(e^{(-1)}p_i \otimes 1))(q_i \otimes 1 \otimes 1 \otimes e^{(0)}1)\\ &= \sum (id \otimes \beta)(\beta(r)(t_i \otimes 1))(1 \otimes s_i^{(-1)}k \otimes 1) \otimes s_i^{(0)}1.\end{aligned}$$

For the right hand of (ii) in Definition 2.1, we have

$$\begin{aligned}(id\otimes\mu)(\mu(r\otimes e)(h\otimes 1))(1\otimes k\otimes 1) &= \sum(id\otimes\mu)(\beta(r)(t_i\otimes 1)\otimes s_i)(1\otimes k\otimes 1)\\ &= \sum r^{(-1)}t_i\otimes\mu\left(r^{(0)}1\otimes s_i\right)(k\otimes 1)\\ &= \sum r^{(-1)}t_i\otimes\beta\left(r^{(0)}1\right)\left(s_i^{(-1)}k\otimes 1\right)\otimes s_i^{(0)}1\\ &= \sum(id\otimes\beta)(\beta(r)(t_i\otimes 1))\left(1\otimes s_i^{(-1)}k\otimes 1\right)\otimes s_i^{(0)}1.\end{aligned}$$

Now, we show that μ is injective. Take $h\in\mathcal{H}$ such that $\varepsilon(h)=1$. For all $r\in\mathcal{R}, e\in\mathcal{E}$ we have

$$\begin{aligned}(\varepsilon\otimes id)(\mu(r\otimes e)) &= (\varepsilon\otimes id)(\mu(r\otimes e)(h\otimes 1))\\ &= \sum r\varepsilon(e^{(-1)}h)\otimes e^{(0)}1 = r\otimes e.\end{aligned}$$

It follows from Lemma 2.2 that μ is injective. So, it implies that $\mathcal{R}\otimes\mathcal{E}$ is a left $\mathcal{H}$-comodule.

Finally, assume that $\mathcal{H}$ is commutative, β and α are algebra maps. To prove that μ is a homomorphism, let $r,t\in\mathcal{R}, h\in\mathcal{H}$ and $e,s\in\mathcal{E}$. Also

$$\alpha(s)(h\otimes 1)=\sum p_i\otimes d_i,$$

where $p_i\in\mathcal{H}, d_i\in\mathcal{E}$. Then

$$\begin{aligned}\mu(rt\otimes es)(h\otimes 1) &= \sum\beta(rt)\left((es)^{(-1)}h\otimes 1\right)\otimes(es)^{(0)}1\\ &= \sum\beta(r)\beta(t)\left(p_ie^{(-1)}\otimes 1\right)\otimes(1e^{(0)})d_i.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu(r\otimes e)\mu(t\otimes s)(h\otimes 1) &= \sum\mu(r\otimes e)(\beta(t)(p_i\otimes 1)\otimes d_i)\\ &= \sum\mu(r\otimes e)((p_i\otimes 1)\beta(t)\otimes d_i)\\ &= \sum\mu(r\otimes e)(p_i\otimes 1\otimes 1)(\beta(t)\otimes d_i)\\ &= \sum\beta(r)\left(p_ie^{(-1)}\otimes 1\right)\beta(t)\otimes(1e^{(0)})d_i\\ &= \sum\beta(r)\beta(t)\left(p_ie^{(-1)}\otimes 1\right)\otimes(1e^{(0)})d_i.\end{aligned}$$

□

Note that (2.2) in Theorem 2.5 is given by

$$\mu(r\otimes e)(h\otimes 1)=(\beta\otimes id)(r\otimes 1)\cdot(id\otimes T_1)(\alpha(e)(h\otimes 1)\otimes 1), \tag{2.4}$$

where $T_1:\mathcal{E}\otimes M(\mathcal{R})\longrightarrow M(\mathcal{R})\otimes\mathcal{E}$ is the flip operator.

Using the fact that $\beta(\mathcal{R})(\mathcal{H}\otimes M(\mathcal{R}))\subset\mathcal{H}\otimes\mathcal{R}$, we obtain the right hand side of (2.4) is an element in $\mathcal{H}\otimes\mathcal{R}\otimes\mathcal{E}$.

Also, (2.3) in the Theorem 2.5 is given by

$$(h\otimes 1)\mu(r\otimes e) = (T\otimes id)((T_2\otimes id)((h\otimes 1)\beta(r)\otimes 1)\cdot(1\otimes\alpha(e))), \tag{2.5}$$

for all $r\in\mathcal{R}, e\in\mathcal{E}$ and $h\in\mathcal{H}$.

Here, $T:\mathcal{R}\otimes\mathcal{H}\longrightarrow\mathcal{H}\otimes\mathcal{R}$, and $T_2:\mathcal{H}\otimes\mathcal{R}\longrightarrow\mathcal{R}\otimes\mathcal{H}$ are flip operators.

As a reult of Theorem 2.5, one can obtain the following corollary.

Corollary 2.6. *Let $\mathcal{X}$ be an arbitrary algebra with a non-degenerate product. Assume that $\mathcal{R}$ is a left $\mathcal{H}$-comodule via β. We can consider $\mathcal{X}$ to be a left $\mathcal{H}$-comodule via $x\longrightarrow 1\otimes x$. From Theorem 2.5, $\mathcal{X}\otimes\mathcal{R}$ is a left $\mathcal{H}$-comodule.*

So, we obtain $\mu:\mathcal{X}\otimes\mathcal{R}\longrightarrow M(\mathcal{H}\otimes\mathcal{X}\otimes\mathcal{R})$ by the formulas

$$\mu(x\otimes r)(h\otimes 1) = (id\otimes T)(\beta(r)(h\otimes 1)\otimes x),$$
$$(h\otimes 1)\mu(x\otimes r) = (id\otimes T)((h\otimes 1)\beta(r)\otimes x),$$

for all $x\in\mathcal{X}, r\in\mathcal{R}$ and $h\in\mathcal{H}$. Here, $T:\mathcal{R}\otimes\mathcal{X}\longrightarrow\mathcal{X}\otimes\mathcal{R}$ is the flip operator. Moreover, if β is an algebra map, then μ is an algebra map.

The proof of the following lemma is similar to the proof of Proposition 2.5 in [9].

Lemma 2.7. *Assume that $\mathcal{R}$ is a left $\mathcal{H}$-comodule algebra via β. Then the maps $D_1, D_2:\mathcal{H}\otimes\mathcal{R}\longrightarrow\mathcal{H}\otimes\mathcal{R}$, defined by*

$$D_1(h\otimes r) = \beta(r)(h\otimes 1), \qquad D_2(h\otimes r) = (h\otimes 1)\beta(r),$$

for all $h\in\mathcal{H}, r\in\mathcal{R}$, are bijective.

Let $\mathcal{R}$ be a left $\mathcal{H}$-comodule algebra via β. From Lemma 2.7, we have $\beta(\mathcal{R})(\mathcal{H}\otimes 1) = \mathcal{H}\otimes\mathcal{R} = (\mathcal{H}\otimes 1)\beta(\mathcal{R})$. Then the map β can be uniquely extended to $\beta: M(\mathcal{R})\longrightarrow M(\mathcal{H}\otimes\mathcal{R})$. For all $d\in M(\mathcal{R})$ one can show that

$$\beta(d)\left(\sum\beta(r_i)(h_i\otimes 1)\right) = \sum\beta(dr_i)(h_i\otimes 1),$$
$$\left(\sum(h_i\otimes 1)\beta(r_i)\right)\beta(d) = \sum(h_i\otimes 1)\beta(r_i d).$$

From (ii) in Definition 2.1, for all $d\in M(\mathcal{R}), h,k\in\mathcal{H}$ and $r\in\mathcal{R}$ we obtain the following equality

$$(\Delta\otimes id)(\beta(d))(h\otimes\beta(r)(k\otimes 1)) = (id\otimes\beta)(\beta(d)(h\otimes r))(1\otimes k\otimes 1).$$

Definition 2.8. Let $\mathcal{R}$ be a left $\mathcal{H}$-comodule via β. Assume that $\mathcal{R}$ is unital or β is an algebra map. An element $d\in M(\mathcal{R})$ is called a coinvariant element if

$$\beta(d) = 1\otimes d.$$

The set of all coinvariants of $M(\mathcal{R})$ is denoted by $\mathcal{R}^{co(\mathcal{H})}$.

In the following theorem, let $\mathcal{R}$ be a left $\mathcal{H}$-comodule algebra via β, and consider $\mathcal{H}$ as a left $\mathcal{H}$-comodule algebra via Δ.

If $\mathcal{H}$ is commutative, then Theorem 2.5 shows that $\mathcal{H}\otimes\mathcal{R}$ is a left $\mathcal{H}$-comodule algebra. Also, Theorem 1.2 shows that $S^2 = id$. We need this property in the proof of the following theorem.

Theorem 2.9. *Assume that $\mathcal{H}$ is commutative. Then*

$$(\mathcal{H}\otimes\mathcal{R})^{co(\mathcal{H})} \cong M(\mathcal{R}).$$

Proof. By using the Theorem 2.5, $\mathcal{H}\otimes\mathcal{R}$ is a left $\mathcal{H}$-comodule algebra via

$$\mu(h\otimes r)(k\otimes 1) = \sum \Delta(h)\left(r^{(-1)}k\otimes 1\right)\otimes r^{(0)}1.$$

Define $\theta : M(\mathcal{R}) \longrightarrow (\mathcal{H}\otimes\mathcal{R})^{co(\mathcal{H})}$ by

$$\theta(d)(h\otimes r) = (S\otimes id)(\beta(d)(S(h)\otimes r)),$$

$$(h\otimes r)\theta(d) = (S\otimes id)((S(h)\otimes r)\beta(d)).$$

It is easy to see that $\theta(d)$ is an element in $M(\mathcal{H}\otimes\mathcal{R})$.

To prove that θ is well defined, we will show that $\mu(\theta(d)) = 1\otimes\theta(d)$, for all $d\in M(\mathcal{R})$. For this, let $h,k\in\mathcal{H}$ and $r\in\mathcal{R}$. Write

$$\beta(r)(k\otimes 1) = \sum t_i\otimes s_i,$$

for $t_i\in\mathcal{H}$ and $s_i\in\mathcal{R}$. Also, write

$$S(h)\otimes t_i = \sum \Delta(b_{ij})(1\otimes a_{ij}),$$

for all i. Then

$$\mu(\theta(d))\mu(h\otimes r)(k\otimes 1) = \sum a_{ij}\otimes (S\otimes id)(\beta(d)(b_{ij}\otimes s_i)).$$

Let $c\in\mathcal{H}$ and write

$$\beta(d)(S(c)\otimes s_i) = \sum h_{ij}\otimes r_{ij},$$

for all i. Then

$$\begin{aligned}\mu(\theta(d))\mu(h\otimes r)(k\otimes 1)(1\otimes c\otimes 1) &= \sum h^{(1)}t_i\otimes (h^{(2)}1)S(h_{ij})\otimes r_{ij}\\ &= \sum h^{(1)}t_i\otimes \theta(d)(c\otimes s_i)(h^{(2)}1\otimes 1)\\ &= (1\otimes\theta(d))\mu(h\otimes r)(k\otimes 1)(1\otimes c\otimes 1).\end{aligned}$$

This shows that $\mu(\theta(d)) = 1\otimes\theta(d)$, for all $d\in M(\mathcal{R})$.

Define

$$\eta : (\mathcal{H}\otimes\mathcal{R})^{co(\mathcal{H})} \longrightarrow M(\mathcal{R}),$$

by $\eta(y) = (\varepsilon \otimes id)(y)$. To prove $\eta\theta = id$, assume that $d \in M(\mathcal{R}), r \in \mathcal{R}$. Choose $h \in \mathcal{H}$ such that $\varepsilon(h) = 1$. Then

$$\begin{aligned}\eta(\theta(d))r &= (\varepsilon \otimes id)(\theta(d)(h \otimes r))\\ &= (\varepsilon \otimes id)(\beta(d)(S(h) \otimes r))\\ &= \varepsilon(h)dr\\ &= dr.\end{aligned}$$

To prove $\theta\eta = id$, assume that $y \in (\mathcal{H}\otimes\mathcal{R})^{co(\mathcal{H})}$. Let $h \in \mathcal{H}, r \in \mathcal{R}$. From the Lemma 2.7, write $S(h) \otimes r = \sum \beta(r_i)(h_i \otimes 1)$, for $r_i \in \mathcal{R}, h_i \in \mathcal{H}$.

Take $k \in \mathcal{H}$ such that $\varepsilon(k) = 1$. Then

$$\theta(\eta(y))(h \otimes r) = \sum \varepsilon(y^{(1)}k)(S \otimes id)(\beta(y^{(2)}r_i)(h_i \otimes 1)),$$

where, $y(k \otimes r_i) = \sum y^{(1)}k \otimes y^{(2)}r_i$, for all i. Since $y \in (\mathcal{H} \otimes \mathcal{R})^{co(\mathcal{H})}$ we have for all i that

$$\begin{aligned}&\sum \Delta(y^{(1)}k)((y^{(2)}r_i)^{(-1)}h_i \otimes 1) \otimes (y^{(2)}r_i)^{(0)}1\\ &\qquad = \sum(1 \otimes y)(\Delta(k)(r_i^{(-1)}h_i \otimes 1) \otimes r_i^{(0)}1).\end{aligned}$$

Then

$$\begin{aligned}\theta(\eta(y))(h \otimes r) &= (m(S \otimes id) \otimes id)((1 \otimes y)(\Delta(k)(S(h) \otimes 1) \otimes r))\\ &= y(h \otimes r).\end{aligned}$$

This shows that $\theta(\eta(y)) = y$, for all $y \in (\mathcal{H} \otimes \mathcal{R})^{co(\mathcal{H})}$. □

3. Cyclic Homology Of Left $\mathcal{H}$-Triples

In this section, we define the concept of a left $\mathcal{H}$-triple and its cyclic homology. For some references to this section, one can see [2, 4, 6].

Definition 3.1. A paracyclic module is a collection of $\mathbb{C}$-modules $C = \{C_n\}_n$, together with the face operators $\partial_i : C_n \longrightarrow C_{n-1}$, the degeneracy operators $\sigma_i : C_n \longrightarrow C_{n+1}$, and the cyclic operators $\tau_n : C_n \longrightarrow C_n$, such that the following identities hold

$$\begin{array}{ll}\partial_i\partial_j = \partial_{j-1}\partial_i, & 0 \le i < j \le n,\\ \sigma_i\sigma_j = \sigma_{j+1}\sigma_i, & 0 \le i \le j \le n,\\ \partial_i\tau_n = \tau_{n-1}\partial_{i-1}, & 1 \le i \le n,\\ \sigma_i\tau_n = \tau_{n+1}\sigma_{i-1}, & 1 \le i \le n,\\ \partial_0\tau_n = \partial_n, & \\ \sigma_0\tau_n = \tau_{n+1}^2\sigma_n, & \\ \partial_i\sigma_j = \sigma_{j-1}\partial_i, & \mathrm{i} < \mathrm{j},\\ \partial_i\sigma_j = \sigma_j\partial_{i-1}, & \mathrm{i} > \mathrm{j}+1,\\ \partial_i\sigma_j = id, & i = j \text{ or } i = j+1.\end{array}$$

If in addition we have $\tau_n^{n+1} = id$, then we have a cyclic module.

For the following definition, we refer to [3].

Definition 3.2. A left $\mathcal{H}$-module $\mathcal{X}$ is called unital if $\mathcal{H}\mathcal{X} = \mathcal{X}$.

For all $x \in \mathcal{X}$ we can write $x = \sum h_i x_i$, where $h_i \in \mathcal{H}$ and $x_i \in \mathcal{X}$. Take $h \in \mathcal{H}$ such that $hh_i = h_i$, for all i. Then, we get $hx = x$. So, for a unital left $\mathcal{H}$-module $\mathcal{X}$, for any $x \in \mathcal{X}$ there is an $h \in \mathcal{H}$ such that $hx = x$.

Let us give an example of a cyclic module. This example is important for understanding the basic ideas.

Example 3.3. Choose a unital subalgebra $\mathcal{V}$ of $M(\mathcal{H})$ with the following properties

$$\Delta(\mathcal{V}) \subset \mathcal{V} \otimes \mathcal{V}, \qquad S(\mathcal{V}) \subset \mathcal{V}.$$

We use the formal notation and write

$$\Delta(v) = \sum v^{(1)} \otimes v^{(2)},$$

for all $v \in \mathcal{V}$.

Assume that $\mathcal{X}$ is a unital left $\mathcal{H}$-module. The paracyclic module $C_\bullet(\mathcal{X}, \mathcal{V})$ is defined by $C_n(\mathcal{X}, \mathcal{V}) = \mathcal{X} \otimes \mathcal{V}^{\otimes(n)}$, with the face, degeneracy and cyclic operators defined by

$$\partial_0(x \otimes v_1 \otimes \cdots \otimes v_n) = \varepsilon(v_1)x \otimes v_2 \otimes \cdots \otimes v_n, \tag{3.1}$$

$$\partial_i(x \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes v_1 \otimes \cdots \otimes v_i v_{i+1} \otimes \cdots \otimes v_n, \tag{3.2}$$

$$\partial_n(x \otimes v_1 \otimes \cdots \otimes v_n) = v_n x \otimes v_1 \otimes \cdots \otimes v_{n-1}, \tag{3.3}$$

$$\sigma_0(x \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes 1 \otimes v_1 \otimes \cdots \otimes v_n, \tag{3.4}$$

$$\sigma_i(x \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes v_1 \otimes \cdots \otimes v_i \otimes 1 \otimes \cdots \otimes v_n, \tag{3.5}$$

and

$$\tau_n(x \otimes v_1 \otimes \cdots \otimes v_n) = \sum v_n^{(2)} x \otimes S(v_1^{(1)} \cdots v_n^{(1)}) \otimes v_1^{(2)} \otimes \cdots \otimes v_{n-1}^{(2)}. \tag{3.6}$$

By Proposition 3.3 in [3], $\mathcal{X}$ is a left $M(\mathcal{H})$-module and so ∂_n and τ_n are well defined. One can check that if $\tau_1^2 = id$, then it is a cyclic module.

Assume that $\mathcal{A}$ is an arbitrary non-degenerate algebra. Let
$M_0(\mathcal{H} \otimes \mathcal{A}) = \{d \in M(\mathcal{H} \otimes \mathcal{A}) | d(h \otimes 1) \in \mathcal{H} \otimes M(\mathcal{A}),\ \forall h \in \mathcal{H}\}$.

It is clear that $M_0(\mathcal{H} \otimes \mathcal{A})$ is a unital subalgebra of $M(\mathcal{H} \otimes \mathcal{A})$. For a unital left $\mathcal{H}$-module $\mathcal{X}$, we see that $\mathcal{X} \otimes M(\mathcal{A})$ is a left $M_0(\mathcal{H} \otimes \mathcal{A})$-module by

$$d \cdot \left(\sum h_i x_i \otimes a\right) = \sum d(h_i \otimes 1)(x_i \otimes a),$$

for all $h_i \in \mathcal{H}, d \in M_0(\mathcal{H} \otimes \mathcal{A}), x_i \in \mathcal{X}$ and $a \in M(\mathcal{A})$.

To prove that this is well defined, assume that $x_i \in \mathcal{X}$ and $h_i \in \mathcal{H}$ and that $\sum h_i x_i = 0$. Choose $h \in \mathcal{H}$ such that $hh_i = h_i$ for all i. We have for all $a \in M(\mathcal{A})$ and $d \in M_0(\mathcal{H} \otimes \mathcal{A})$ that

$$\begin{aligned}\sum d(h_i \otimes 1)(x_i \otimes a) &= \sum d(h \otimes 1)(h_i x_i \otimes a) \\ &= d(h \otimes 1)\left(\sum h_i x_i \otimes a\right) \\ &= 0.\end{aligned}$$

Definition 3.4. By a left $\mathcal{H}$-triple we mean a triple $(\mathcal{R}, \mathcal{H}, \mathcal{X})$, in which $\mathcal{R}$ is a left $\mathcal{H}$-comodule algebra and $\mathcal{X}$ is a unital left $\mathcal{H}$-module which is also a unital algebra.

Given a left $\mathcal{H}$-triple $(\mathcal{R}, \mathcal{H}, \mathcal{X})$, let $\beta : \mathcal{R} \to M(\mathcal{H} \otimes \mathcal{R})$ be the structure map of $\mathcal{R}$. Assume that $\mathcal{V}$ is a unital subalgebra of $M(\mathcal{R})$ such that

$$\beta(\mathcal{V})(\mathcal{H} \otimes 1) \subset \mathcal{H} \otimes \mathcal{V}, \qquad (\mathcal{H} \otimes 1)\beta(\mathcal{V}) \subset \mathcal{H} \otimes \mathcal{V}. \tag{3.7}$$

Here, $\beta : M(\mathcal{R}) \to M(\mathcal{H} \otimes \mathcal{R})$ is the extension of β.

Example 3.5. (i) Let $\mathcal{R}$ be a left $\mathcal{H}$-comodule algebra, which is also a unital algebra. Let $\mathcal{V} = \mathcal{R}$.
(ii) Let $\mathcal{R}$ be a left $\mathcal{H}$-comodule algebra, and let $\mathcal{V} = \mathcal{R}^{co(\mathcal{H})}$.
(iii) Assume that $\mathcal{H}$ is an algebraic quantum group. From [7] there exists an invertible element $\delta \in M(\mathcal{H})$ such that

$$\Delta(\delta) = \delta \otimes \delta, \qquad S(\delta) = \delta^{-1}, \qquad \varepsilon(\delta) = 1.$$

Let $\mathcal{V}$ be subalgebra generated by δ, δ^{-1}.

Let

$$C_n(\mathcal{X}, \mathcal{V}) = \mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)}.$$

We define the following operators on $\{C_n(\mathcal{X}, \mathcal{V})\}_n$ by

$$\partial_0(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes v_0 v_1 \otimes v_2 \otimes \cdots \otimes v_n, \tag{3.8}$$

$$\partial_i(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes v_0 \otimes \cdots \otimes v_i v_{i+1} \otimes \cdots \otimes v_n, \tag{3.9}$$

$$\partial_n(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) = \beta(v_n) \cdot (x \otimes v_0) \otimes v_1 \otimes \cdots \otimes v_{n-1}, \tag{3.10}$$

$$\sigma_i(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) = x \otimes v_0 \otimes \cdots \otimes v_i \otimes 1 \otimes \cdots \otimes v_n, \tag{3.11}$$

$$\tau_n(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) = \beta(v_n) \cdot (x \otimes 1) \otimes v_0 \otimes \cdots \otimes v_{n-1}. \tag{3.12}$$

Note that (3.7) implies that $\beta(v_n) \in M_0(\mathcal{H} \otimes \mathcal{R})$.

Moreover, $\beta(v_n) \cdot (x \otimes v_0)$ and $\beta(v_n) \cdot (x \otimes 1)$ are elements in $\mathcal{X} \otimes \mathcal{V}$. Therefore, ∂_n and τ_n are well defined.

The proof of the following lemma is easy.

Lemma 3.6. *Endowed with the above operators, $\{C_n(\mathcal{X}, \mathcal{V})\}_n$ is a paracyclic module.*

From (3.7), $\mathcal{V}$ is a left $\mathcal{H}$-comodule algebra via $\beta^{(1)} : \mathcal{V} \longrightarrow M(\mathcal{H} \otimes \mathcal{V})$, defined by

$$\beta^{(1)}(v)(h \otimes t) = \beta(v)(h \otimes t), \qquad (h \otimes t)\beta^{(1)}(v) = (h \otimes t)\beta(v),$$

for all $v, t \in \mathcal{V}$ and $h \in \mathcal{H}$. From Theorem 2.5, $\mathcal{V}^{\otimes(n)}$ is a left $\mathcal{H}$-comodule.

Let $\beta^{(n)} : \mathcal{V}^{\otimes(n)} \longrightarrow M(\mathcal{H} \otimes \mathcal{V}^{\otimes(n)})$ be the structure map of $\mathcal{V}^{\otimes(n)}$. We have for all $v_i \in \mathcal{V}, h \in \mathcal{H}$ that

$$\begin{aligned}
&\beta^{(n)}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)(h \otimes 1) \\
&\quad = \sum \beta^{(n-1)}(v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}) \left(v_n^{(-1)} h \otimes 1 \right) \otimes v_n^{(0)} 1 \\
&\quad = \sum v_1^{(-1)} \left(v_2^{(-1)} \left(v_3^{(-1)} \left(\cdots \left(v_n^{(-1)} h \right) \cdots \right) \right) \right) \otimes v_1^{(0)} 1 \otimes v_2^{(0)} 1 \otimes \cdots \otimes v_n^{(0)} 1.
\end{aligned}$$

From Corollary 2.6, $\mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)}$ is a left $\mathcal{H}$-comodule. Since $\mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)}$ is unital, then Definition 2.8 allows us to write

$$C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V}) := (\mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)})^{co(\mathcal{H})}.$$

The following Lemma shows that $\tau_n^{n+1} = id$ on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$.

Lemma 3.7. *We have for all $a \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$ that*

$$\tau_n^{n+1}(a) = a.$$

Proof. Let $x \otimes y \in C_n(\mathcal{X}, \mathcal{V})$, where $x \in \mathcal{X}$ and $y \in \mathcal{V}^{\otimes(n+1)}$. Then

$$\tau_n^{n+1}(x \otimes y) = \beta^{(n+1)}(y)(k \otimes 1)(x \otimes 1), \tag{3.13}$$

where $kx = x$. If $x \otimes y \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$, then we have for all $h \in \mathcal{H}$ that

$$h \otimes x \otimes y = (id \otimes T)(\beta^{(n+1)}(y)(h \otimes 1) \otimes x), \tag{3.14}$$

where $T : \mathcal{V}^{\otimes(n+1)} \otimes \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)}$ is the flip operator. From (3.14) we have for all $h \in \mathcal{H}$ that

$$hx \otimes y = \beta^{(n+1)}(y)(h \otimes 1)(x \otimes 1). \tag{3.15}$$

If $x \otimes y \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$, then from (3.13) and (3.15), one gets $\tau_n^{n+1}(x \otimes y) = x \otimes y$ as required. $\square$

Definition 3.8. We define the $\mathcal{X}$-twisted antipode $\widetilde{S} : \mathcal{X} \otimes \mathcal{H} \longrightarrow \mathcal{X} \otimes \mathcal{H}$ by

$$\widetilde{S}(x \otimes h) = (id \otimes S)(\Delta'(h) \cdot (x \otimes 1)).$$

Here, Δ' is the opposite comultiplication. Since $\mathcal{H}$ is a regular multiplier Hopf algebra, we have for all $h \in \mathcal{H}$ that $\Delta'(h) \in M_0(\mathcal{H} \otimes \mathcal{H})$.

Also, $\Delta'(h) \cdot (x \otimes 1)$ is an element in $\mathcal{X} \otimes \mathcal{H}$.

Lemma 3.9. *The $\mathcal{X}$-twisted antipode $\widetilde{S}$ is invertible.*

Proof. Define $\widetilde{S}^{-1} : \mathcal{X} \otimes \mathcal{H} \longrightarrow \mathcal{X} \otimes \mathcal{H}$ by

$$\widetilde{S}^{-1}(x \otimes h) = (id \otimes S^{-1})(\Delta(h) \cdot (x \otimes 1)).$$

Take $h \in \mathcal{H}, x \in \mathcal{X}$. Since $\mathcal{H}$ is a regular multiplier Hopf algebra, so we have $\Delta(h) \in M_0(\mathcal{H} \otimes \mathcal{H})$. Also, $\Delta(h) \cdot (x \otimes 1)$ is an element in $\mathcal{X} \otimes \mathcal{H}$. Choose $e \in \mathcal{H}$ such that $ex = x$. Let

$$\Delta(h)(1 \otimes e) = \sum p_i \otimes q_i,$$

where $p_i, q_i \in \mathcal{H}$. Then

$$\begin{aligned}\widetilde{S}^{-1}(\widetilde{S}(x \otimes h) &= \sum \left(id \otimes S^{-1}\right) (\Delta(S(p_i))(q_i \otimes 1)(x \otimes 1)) \\ &= \sum S(k_i p_i^{(2)}) x \otimes 1 p_i^{(1)},\end{aligned}$$

where $S^{-1}(q_i) = k_i$, for all i. We have for all $d \in \mathcal{H}$ that

$$\begin{aligned}\widetilde{S}^{-1}(\widetilde{S}(x \otimes h))(1 \otimes d) &= \sum S\left((h^{(2)}1)^{(1)}1\right)\left((h^{(2)}1)^{(2)}e\right) x \otimes h^{(1)}d \\ &= \sum ex \otimes \varepsilon(h^{(2)}1)h^{(1)}d \\ &= (x \otimes h)(1 \otimes d).\end{aligned}$$

If we cancel d, we will get $\widetilde{S}^{-1}(\widetilde{S}(x \otimes h)) = x \otimes h$. Similarly, one can check that $\widetilde{S} \circ \widetilde{S}^{-1} = id$. □

The following lemma shows that τ_n is well defined on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$.

Lemma 3.10. *Assume that $\widetilde{S}^2 = id$, where $\widetilde{S}$ is defined in Definition 3.8. We have for all $a \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$ that*

$$\tau_n(a) \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V}).$$

Proof. Let

$$(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V}).$$

Then we have for all $h \in \mathcal{H}$ that

$$h \otimes x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n = (id \otimes T)((h \otimes 1)\beta^{(n+1)}(v_0 \otimes v_1 \otimes \cdots \otimes v_n) \otimes x),$$

where $T : \mathcal{V}^{\otimes(n+1)} \otimes \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{V}^{\otimes(n+1)}$ is the flip operator.

This implies for all $h \in \mathcal{H}$ that

$$\begin{aligned}&(id \otimes T)((h \otimes 1)\beta^{(n)}(v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n \otimes x) \\ &\quad = (id \otimes T_1)((S \otimes id)(\beta(v_n)(S^{-1}(h) \otimes 1)) \otimes x \otimes v_0 \otimes \cdots \otimes v_{n-1}),\end{aligned}$$

where $T_1 : \mathcal{V} \otimes (\mathcal{X} \otimes \mathcal{V}^{\otimes(n)}) \longrightarrow (\mathcal{X} \otimes \mathcal{V}^{\otimes(n)}) \otimes \mathcal{V}$ is the flip operator. This implies for all $h \in \mathcal{H}$ that

$$\begin{aligned}&(id \otimes T)\left(\beta^{(n)}(v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1})(h \otimes 1) \otimes v_n \otimes x\right) \\ &\quad = (id \otimes T_1)\left((S \otimes id)((S^{-1}(h) \otimes 1)\beta(v_n)) \otimes x \otimes v_0 \otimes \cdots \otimes v_{n-1}\right).\end{aligned}$$

Then we have for all $h \in \mathcal{H}$ that

$$\begin{aligned}&\sum (id \otimes T)\left(\beta^{(n+1)}(v_n^{(0)}1 \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1})(h \otimes 1) \otimes (v_n^{(-1)}k)x\right)\\ &\quad = \sum (id \otimes T)\left(\beta((1v_n^{(0)})^{(0)}1)(S(ev_n^{(-1)})h \otimes 1) \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1} \otimes ((1v_n^{(0)})^{(-1)}k)x\right),\end{aligned}$$

where $kx = x, S^{-1}(h)e = S^{-1}(h)$.

Using $\widetilde{S}^{-2} = id$ and the method used in the proof of Theorem 3.12 in [4], one obtains for all $h \in \mathcal{H}$ that

$$\begin{aligned}\sum (id \otimes T)&\left(\beta^{(n+1)}(v_n^{(0)}1 \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1})(h \otimes 1) \otimes (v_n^{(-1)}k)x\right)\\ &= h \otimes \beta(v_n) \cdot (x \otimes 1) \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1}.\end{aligned}$$

This means that $\tau_n(x \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_n) \in C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$. $\square$

Theorem 3.11. *Assume that $\widetilde{S}^2 = id$. Then $\{C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})\}_n$ endowed with simplicial and cyclic operators induced by (3.8)- (3.12) is a cyclic module.*

Proof. From Lemma 3.7, one gets $\tau_n^{n+1} = id$ on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$. It is easy to see that ∂_i, for all $i = 0, \dots, n-1$, and σ_i, for all $i = 0, \dots, n$, are well defined on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$. From Lemma 3.10, τ_n is well defined on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$. Finally, $\partial_0\tau_n = \partial_n$ implies that ∂_n is well defined on $C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V})$. $\square$

Let us give an example.

Example 3.12. Assume that $\mathcal{R}$ is an arbitrary non-degenerate algebra. We can consider $\mathcal{R}$ as a left $\mathcal{H}$-comodule algebra given by $r \longrightarrow 1 \otimes r$. Let $\mathcal{X} = \mathbb{C}$ and $\mathcal{V} = M(\mathcal{R})$. Then

$$C_n^{\mathcal{H}}(\mathcal{X}, \mathcal{V}) = M(\mathcal{R})^{\otimes(n+1)}.$$

In this case, cyclic homology is the same as cyclic homology of unital algebras (see [6]) and the simplicial and cyclic operators on $M(\mathcal{R})^{\otimes(n+1)}$ are as follows

$$\begin{aligned}\partial_0(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= v_0v_1 \otimes v_2 \otimes \cdots \otimes v_n,\\ \partial_i(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= v_0 \otimes \cdots \otimes v_iv_{i+1} \otimes \cdots \otimes v_n,\\ \partial_n(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= v_nv_0 \otimes v_1 \otimes \cdots \otimes v_{n-1},\\ \sigma_i(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= v_0 \otimes \cdots \otimes v_i \otimes 1 \otimes \cdots \otimes v_n,\\ \tau_n(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= v_n \otimes v_0 \otimes \cdots \otimes v_{n-1}.\end{aligned}$$

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