Abstract. $K$-frames as a generalization of frames were introduced by L. Găvruţa to study atomic systems on Hilbert spaces which allows, in a stable way, to reconstruct elements from the range of the bounded linear operator $K$ in a Hilbert space. Recently some generalizations of this concept are introduced and some of its difference with ordinary frames are studied. In this paper, we give a new generalization of $K$-frames. After proving some characterizations of generalized $K$-frames, new results are investigated and some new perturbation results are established. Finally, we give several characterizations of $K$-duals.

1. Introduction and Basic Definitions

Frames in Hilbert spaces were introduced by J. Duffin and A.C. Schaffer [12] in 1952. In 1986, frames were brought to life by Daubechies, Grossmann and Meyer [9]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [16], sampling [13, 14], coding and communications [22], filter bank theory [5], system modeling [11], and so on.

Frames are generalizations of orthonormal basis in Hilbert spaces. The elements of an orthonormal basis are linearly independent which allows every vector to be uniquely represented as a linear combination of the basis elements. This is very restrictive for practical problems. A frame also allows each element in the space to be written as a linear
combination of the elements in the frame, but linear independence between the frame elements is not required. For more information about frames see [6].

G-frames in a complex Hilbert space were introduced by W. Sun in [23] to deal with all existing frames as a united object and discussed some properties of them. Next many authors give new results, for instance see [24, 25].

On the other hand, a family of local atoms for a subspace \(H_0\) of a separable Hilbert space \(H\) as a family of analysis and synthesis systems with frame-like properties \(H_0\) was studied in [15] and generalized by L. Găvruta in [17] and called atomic systems. Let \(H\) be a Hilbert space and \(K \in B(H)\), the space of all bounded linear operators on \(H\). \(K\)-frames as a generalization of frames, i.e. frames are a special case of \(K\)-frame when \(K\) is the identity operator. It is proved that an atomic system for \(K\) is a \(K\)-frame and vice versa (see [17]). We refer to [17] and [26] for more information on these concepts. Recently, generalized \(K\)-frames and generalized atomic systems for operators introduced in [3] and [27]. Some characterizations and perturbation results for generalized \(K\)-frames are investigated in these papers. Also, some results about these concepts in Hilbert modules and Banach spaces can be found in [7, 8], respectively.

The paper is organized as follows. We continue this introductory section with a review of the basic definitions and results. In Section 2, some characterizations of g-\(K\)-frames are studied. In section 3, some more properties of generalized \(K\)-frames will be discussed. Many of our results are generalizations of results in [3, 26, 27]. In the next section some other perturbation results for g-\(K\)-frames are established. In the last section of this paper \(K\)-duals are studied.

Let us first recall the concepts of the atomic systems for \(K\) and \(K\)-frames. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in the Hilbert space \(H\) is called an atomic system for the bounded linear operator \(K\) on \(H\) if the following statements hold

(i) the series \(\sum_{n \in \mathbb{N}} c_n x_n\) converges for all \(c = \{c_n\}_{n \in \mathbb{N}} \in l^2\);
(ii) for any \(x \in H\), there exists \(a_x = \{a_n\}_{n \in \mathbb{N}} \in l^2\) such that \(Kx = \sum_{n \in \mathbb{N}} a_n x_n\) where \(\|a_x\|_2 \leq C\|x\|\); \(C\) is a positive constant.

Note that in (ii), the sequence \(a_x\) corresponding to \(x \in H\) is not unique, in general.

It is well-known that every bounded linear operator \(K\) on a separable Hilbert space admits an atomic system (see [17, Theorem 2]).
A sequence \( \{x_n\}_{n \in \mathbb{N}} \) is said to be a \( K \)-frame for \( H \) if there exist constants \( A, B > 0 \) such that
\[
A \|K^*x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 \leq B \|x\|^2, \quad (x \in H).
\]

In the sequel, it is assumed that \( \mathbb{J} \) is a finitely or countably index set.

For two Hilbert spaces \( H \) and \( F \) we denote by \( B(H, F) \) the collection of all bounded linear operators between \( H \) and \( F \), and we abbreviate \( B(H, H) \) by \( B(H) \). Also we denote the range of \( L \in B(H, F) \) by \( R(L) \).

Recall that if \( H \) and \( H_j, j \in \mathbb{J}, \) are Hilbert spaces then a sequence \( \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\} \) is said to be a generalized frame (or simply g-frame) for \( H \) with respect to \( \{H_j\}_{j \in \mathbb{J}} \) if there are two positive constants \( A \) and \( B \) such that
\[
A \|x\|^2 \leq \sum_{j \in \mathbb{J}} \|\Lambda_j(x)\|^2 \leq B \|x\|^2, \quad (x \in H).
\]

The constants \( A \) and \( B \) are the lower and upper frame bounds, respectively. If the right-hand side of this inequality holds, then \( \{\Lambda_j\}_{j \in \mathbb{J}} \) is said to be a g-Bessel sequence.

For a sequence of Hilbert spaces \( \{H_j\}_{j \in \mathbb{J}} \), the space
\[
\bigoplus_{j \in \mathbb{J}} H_j := \left\{ \{x_j\}_{j \in \mathbb{J}} : x_j \in H_j \text{ and } \sum_{j \in \mathbb{J}} \|x_j\|^2 < \infty \right\},
\]
with the inner product \( \langle \{x_j\}, \{y_j\} \rangle = \sum_{j \in \mathbb{J}} \langle x_j, y_j \rangle \) is a Hilbert space.

Now for a g-Bessel sequence \( \{\Lambda_j\}_{j \in \mathbb{J}} \), we denote by \( T_\Lambda : \bigoplus_{j \in \mathbb{J}} H_j \to H \), \( T_\Lambda^* \) and \( S \) the the synthesis operator, analysis operator and frame operator of \( \{\Lambda_j\}_{j \in \mathbb{J}} \), respectively, which are defined as follows
\[
T_\Lambda(x_j)_{j \in \mathbb{J}} = \sum_{j \in \mathbb{J}} \Lambda_j^*x_j, \quad Sx = \sum_{j \in \mathbb{J}} \Lambda_j^*\Lambda_jx.
\]

The following proposition is a characterization of g-Bessel sequences.

**Proposition 1.1** (\cite{19}). For a sequence \( \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\} \), the following statements are equivalent:

(i) \( \{\Lambda_j\}_{j \in \mathbb{J}} \) is a g-Bessel sequence for \( H \) with respect to \( \{H_j\}_{j \in \mathbb{J}} \);

(ii) The operator
\[
T : (\{x_j\}_{j \in \mathbb{J}}) \mapsto \sum_{j \in \mathbb{J}} \Lambda_j^*x_j,
\]
is well-defined and bounded from \( \bigoplus_{j \in \mathbb{J}} H_j \) to \( H \).
Also it is well known that if \( \{\Lambda_j\}_{j \in J} \) is a g-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bounds \( A \) and \( B \), then the g-frame operator \( S \) is an invertible bounded self-adjoint operator which satisfies
\[
A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|f\|^2,
\]
and
\[
B^{-1}\|x\|^2 \leq \langle S^{-1}x, x \rangle \leq A^{-1}\|x\|^2,
\]
for all \( x \in H \).

**Definition 1.2** ([4, 27]). A sequence \( \{\Lambda_j \in B(H, H_j) : j \in J\} \) is said to be a generalized atomic system (or simply g-atomic system) for the operator \( K \in B(H) \) if
\begin{enumerate}[(i)]  
  
  \item for every \( \{x_j\}_{j \in J} \in \bigoplus_{j \in J} H_j \), \( \sum_{j \in J} \Lambda_j^* x_j \) converges in \( H \),
  
  \item there exists \( C > 0 \) such that for all \( x \in H \), there is a sequence \( a_x = \{a_j\}_{j \in J} \in \bigoplus_{j \in J} H_j \), for which
\[
Kx = \sum_{j \in J} \Lambda_j^* a_j \quad \text{and} \quad \|a_x\| = \left( \sum_{j \in J} \|a_j\|^2 \right)^{1/2} \leq C\|x\|.
\end{enumerate}

**Definition 1.3** ([4, 27]). Suppose that \( K \in B(H) \). A sequence \( \{\Lambda_j \in B(H, H_j) : j \in J\} \) is said to be a generalized \( K \)-frame (or g-\( K \)-frame) for \( H \) with respect to \( \{H_j : j \in J\} \) if there are two positive constants \( A, B \) such that
\[
A\|K^* x\| \leq \sum_{j \in J} \|\Lambda_j x\|^2 \leq B\|x\|^2, \quad (x \in H).
\]

The following result in operator theory has an important role in our discussion.

**Theorem 1.4** ([11]). Let \( H, H_1 \) and \( H_2 \) be Hilbert spaces and \( L_1 \in B(H_1, H), \) \( L_2 \in B(H_2, H) \). The following statements are equivalent:
\begin{enumerate}[(i)]  
  
  \item \( R(L_1) \subseteq R(L_2) \);
  
  \item \( L_1 L_1^* \leq \lambda L_2 L_2^* \), for some \( \lambda \geq 0 \);
  
  \item there exists a bounded operator \( T \in B(H_1, H_2) \) such that \( L_1 = L_2 T \).
\end{enumerate}

Throughout the paper \( H, F \) and \( H_j \) for any \( j \in J \) are separable Hilbert spaces.

2. G-\( K \)-FRAMES AND G-ATOMIC SYSTEMS

First we give the following definition that a generalized atomic system in \([4, 27]\) is the special case of g-atomic systems.
**Definition 2.1.** Let $K \in B(H)$. A bounded linear operator $\Xi : H \to F$ is called a generalized atomic system (or $g$-atomic system) for $K \in B(H)$ with respect to $F$ if there exists $C > 0$ such that for all $x \in H$, there is $a_x \in F$ such that

$$Kx = \Xi^*a_x \quad \text{and} \quad \|a_x\| \leq C\|x\|.$$ 

**Theorem 2.2.** Let $K \in B(H)$. Then there exists a $g$-atomic system for $K \in B(H)$ with respect to $F$.

*Proof.* Define $\Xi : H \to F$ by $\Xi(e_j) = \delta_j$ for all $j \in J$, where $\{e_j\}_{j \in J}$ and $\{\delta_j\}_{j \in J}$ are orthonormal bases for $H$ and $F$, respectively. Then we have $\Xi \in B(H, F)$ and $\Xi^*\Xi Kx = Kx$. Therefore $\Xi$ is a $g$-atomic system for $K \in B(H)$ with respect to $F$. Note that $\|a_x\| := \|\Xi Kx\| \leq \|\Xi\|\|K\|\|x\|$. □

Now we give a new definition of generalized $K$-frames.

**Definition 2.3.** A bounded linear operator $\Xi$ from $H$ to $F$ is called a generalized $K$-frame (or g-$K$-frame) for $H$ with respect to $F$ if there are two positive constants $A, B$ such that

$$A\|K^*x\|^2 \leq \|\Xi x\|^2 \leq B\|x\|^2, \quad (x \in H).$$

Now we have the following characterization of $g$-atomic systems.

**Theorem 2.4.** Let $K \in B(H)$ and $\Xi$ be a bounded linear operator from $H$ to $F$. Then the following statements are equivalent:

(i) $\Xi : H \to F$ is a $g$-atomic system for $K \in B(H)$ with respect to $F$;

(ii) $\Xi : H \to F$ is a $g$-$K$-frame for $H$ with respect to $F$;

(iii) An operator $\Xi : H \to F$ is bounded and there exists another bounded operator $Q : H \to F$ such that $K = \Xi^*Q$;

(iv) $\Xi L$ is a $g$-atomic system for $L^*K$, where $L$ is surjective on $H$;

(v) An operator $\Xi : H \to F$ is bounded, linear and there exists another bounded operator $Q : H \to F$ such that $K^* = Q^*\Xi$;

(vi) An operator $\Xi : H \to F$ is bounded, linear and $R(K) \subseteq R(\Xi^*)$.

*Proof.* Assume that $K \in B(H)$ and $\Xi$ is the bounded linear operator from $H$ to $F$.

(i) $\Rightarrow$ (ii) For any $x \in H$ we have

$$\|K^*(x)\| = \sup_{\|y\| = 1} |\langle K^*x, y \rangle|$$

$$= \sup_{\|y\| = 1} |\langle x, Ky \rangle|,$$
by definition of g-atomic system for $K \in B(H)$ with respect to $F$, there exists $C > 0$ such that for any $y \in H$, there is $a_y \in F$, for which $||a_y|| \leq C||y||$ and $Ky = T^*a_y$. Thus

$$||K^*x|| = \sup_{||y||=1} |\langle x, \Xi^*a_y \rangle|$$

$$= \sup_{||y||=1} |\langle \Xi x, a_y \rangle|$$

$$\leq \sup_{||y||=1} ||\Xi x|| \ ||a_y||$$

$$\leq \sup_{||y||=1} C||\Xi x||.$$ 

Hence

$$\frac{1}{C^2} ||K^*x||^2 \leq ||\Xi x||^2,$$

which implies that (ii) is holds.

(ii) $\Rightarrow$ (iii) By Theorem 1.4 there exists a bounded operator $Q : H \to F$ such that $K = \Xi^*Q$.

(iii) $\Rightarrow$ (i) Take $a_x := Qx$, for every $x \in H$.

(iv) $\Rightarrow$ (i) Let $\Xi L : H \to F$ be a g-atomic system for $L^*K$, then for all $x \in H$, there exists $b_x \in F$ such that $L^*K = (\Xi L)^*b_x = L^*\Xi^*b_x$ so $L^*(Kx - \Xi^*b_x) = 0$. Due to injectivity of $L^*$, we have $Kx = \Xi^*b_x$. Therefore $\Xi$ is a g-atomic system for $K \in B(H)$ with respect to $F$. [i) $\Rightarrow$ (iv)] Let $\Xi$ be a g-atomic system for $K \in B(H)$ with respect to $F$ as in Definition 2.11, then $Kx = \Xi a_x$ and $L^*K = L^*\Xi^*a_x = (\Xi L)^*a_x$, it means that $\Xi L$ is a g-atomic system for $L^*K$.

(iii) $\Leftrightarrow$ (v) It is obvious.

(v) $\Leftrightarrow$ (vi) It follows from Theorem 4.1.

\[ \square \]

In view of Theorem 2.4, suppose that $F = \bigoplus_{j \in J} H_j$ and consider an operator $\Lambda_j : H \to H_j$ by $x \mapsto \Lambda_j x$, for all $j \in J$. Now, define $\Xi : H \to \bigoplus_{j \in J} H_j$ by $\Xi(x) = \{\Lambda_j x\}_{j \in J}$. Then we have the following corollary generalized [3, Theorem 2.5] and [27, Theorem 3.8].

**Corollary 2.5.** For $\{\Lambda_j \in B(H, H_j) : j \in J\}$, the following statements are equivalent:

(i) $\{\Lambda_j\}_{j \in J}$ is a g-atomic system for $K \in B(H)$ with respect to $\{H_j\}_{j \in J}$;

(ii) $\{\Lambda_j\}_{j \in J}$ is a g-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$;
(iii) \( \{ \Lambda_j \}_{j \in \mathbb{J}} \) is a \( g \)-Bessel sequence and there exists another \( g \)-Bessel sequence \( \{ \Delta_j \}_{j \in \mathbb{J}} \) such that for all \( x \in H \),

\[
K(x) = \sum_{j \in \mathbb{J}} \Lambda_j^* \Delta_j(x);
\]

(2.1)

(iv) \( \{ \Lambda_j L \}_{j \in \mathbb{J}} \) is a \( g \)-atomic system for \( L^* K \), where \( L \) is surjective on \( H \);

(v) \( \{ \Lambda_j \}_{j \in \mathbb{J}} \) is a \( g \)-Bessel sequence and there exists another \( \Delta_j \) Bessel sequence \( \{ \Delta_j \}_{j \in \mathbb{J}} \) such that

\[
K^*(x) = \sum_{j \in \mathbb{J}} \Delta_j^* \Lambda_j(x);
\]

(vi) \( \{ \Lambda_j \}_{j \in \mathbb{J}} \) is a \( g \)-Bessel sequence and \( R(K) \subseteq R(T) \), where \( T \) is the synthesis operator of \( \{ \Lambda_j \}_{j \in \mathbb{J}} \).

Also with the notations of the previous corollary, the following characterization of \( g \)-atomic systems is valid.

**Theorem 2.6.** \( \{ \Lambda_j \}_{j \in \mathbb{J}} \) is a \( g \)-null frame for \( H \) with respect to \( \{ H_j \}_{j \in \mathbb{J}} \) if and only if there exist \( A, B > 0 \) such that

\[
A K K^* \leq \sum_{j \in \mathbb{J}} \Lambda_j^* \Lambda_j \leq B I,
\]

where \( I \) is the identity operator on \( H \). Moreover, in this case

\[
\| K \| \leq \sqrt{\frac{B}{A}}.
\]

**Proof.** Since \( \{ \Lambda_j \}_{j \in \mathbb{J}} \) is a \( g \)-null frame for \( H \) with respect to \( \{ H_j \}_{j \in \mathbb{J}} \) then for any \( x \in H \) we have

\[
A \| K x \| ^2 \leq \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x \rangle
\]

\[
= \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Lambda_j x, x \right\rangle
\]

\[
\leq B \| x \|^2,
\]

which is equivalent to

\[
\langle A K K^* x, x \rangle \leq \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Lambda_j x, x \right\rangle \leq \langle B x, x \rangle, \quad (x \in H).
\]
This proves the first part. Furthermore $AKK^* \leq BI$ implies that $A\|K\|^2 \leq B$ and so

$$\|K\| \leq \sqrt{\frac{B}{A}}.$$ 

The following equivalent conditions for the sequence $\{\Lambda_j\}_{j \in J}$ to be a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ may be useful which can be proved by Theorem 1.4 and the Theorem 2.6.

**Proposition 2.7.** For a sequence $\{\Lambda_j \in B(H, H_j) : j \in J\}$, the following statements are equivalent:

(i) $\{\Lambda_j\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$;

(ii) The operator

$$T : (\{x_j\}_{j \in J}) \mapsto \sum_{j \in J} \Lambda_j^* x_j,$$

is well-defined and bounded from $\bigoplus_{j \in J} H_j$ to $H$ and $R(K) \subseteq R(T)$;

(iii) The operator

$$S : x \mapsto \sum_{j \in J} \Lambda_j^* \Lambda_j x,$$

is well-defined and bounded from $H$ to $H$ and there exists $A > 0$ such that $AKK^* \leq S$.

Let $\{e_{j,k} : k \in K_j\}$ be an orthonormal basis for $H_j$. For $j \in J$ and $k \in K_j$, define $F_{j,k} := (0, 0, \ldots, e_{j,k}, 0, \ldots)$. Then $\{F_{j,k} : j \in J, k \in K_j\}$ is an orthonormal basis for $\bigoplus_{j \in J} H_j$.

With these notations we state now another characterization of $g$-$K$-frames as follows.

**Theorem 2.8.** $\{\Lambda_j\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ if and only if there exists a bounded linear operator $\Theta : \bigoplus_{j \in J} H_j \to H$ such that for any $i \in J$ and $k \in K_i$,

$$\langle x, \Theta(F_{i,k}) \rangle = \langle \{\Lambda_j x\}_{j \in J}, F_{i,k} \rangle,$$

and $R(K) \subset R(\Theta)$.

**Proof.** $\{\Lambda_j\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$, so for some $A, B > 0$,

$$A\|K^* x\|^2 \leq \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x \rangle \leq B\|x\|^2, \quad (x \in H).$$
Let $T : H \to \bigoplus_{j \in \mathbb{J}} H_j$ be defined by $T(x) = \{\Lambda_j x\}_{j \in \mathbb{J}}$. Trivially $T$ is a bounded linear operator and

$$\langle x, T^* F_{i,k} \rangle = \langle Tx, F_{i,k} \rangle = \langle \{\Lambda_j x\}_{j \in \mathbb{J}}, F_{i,k} \rangle, \quad i \in \mathbb{J}, k \in \mathbb{K}_i.$$ 

Now (2.2) implies that

$$A \|K^* x\|^2 \leq \|T(x)\|^2.$$ 

Hence $AKK^* \leq \Theta \Theta^*$, where $\Theta = T^*$. Therefore by Theorem 2.8, $R(K) \subset R(\Theta)$.

Conversely, let $\langle x, \Theta(F_{i,k}) \rangle = \langle \{\Lambda_j x\}_{j \in \mathbb{J}}, F_{i,k} \rangle$, $i \in \mathbb{J}$ and $k \in \mathbb{K}_i$, where $\Theta \in B(\bigoplus_{j \in \mathbb{J}} H_j, H)$ and $R(K) \subset R(\Theta)$. We have $\Theta^*(x) = \{\Lambda_j x\}_{j \in \mathbb{J}}$. Indeed,

$$\langle \{\Lambda_j x\}_{j \in \mathbb{J}}, F_{i,k} \rangle = \langle x, \Theta(F_{i,k}) \rangle = \langle \Theta^* x, F_{i,k} \rangle, \quad i \in \mathbb{J}, k \in \mathbb{K}_i.$$ 

Obviously $\{\Lambda_j\}_{j \in \mathbb{J}}$ is a g-Bessel sequence, since

$$\sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x \rangle = \|\Theta^* x\|^2 \leq \|\Theta^*\|^2 \|x\|^2, \quad (x \in H).$$

Also $R(K) \subset R(\Theta)$ and Theorem 2.8 imply that there exists $B > 0$ such that $BK^* \leq \Theta \Theta^*$. Therefore

$$B \|K^* x\|^2 \leq \|\Theta^* x\|^2 = \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x \rangle, \quad (x \in H).$$

Recall that if $X$ and $Y$ are two Banach spaces and $Q \in B(X,Y)$, then $Q^+$ is said to be the pseudo-inverse of $Q$ if $QQ^+Q = Q$. In particular, for any $y \in R(Q)$, $QQ^+y = y$. For more details one can see [20]. An important difference of ordinary g-frames with g-K-frames is the fact that with the notations of Corollary 2.8, $\{\Lambda_j\}_{j \in \mathbb{J}}$ and $\{\Delta_j\}_{j \in \mathbb{J}}$ are not interchangeable for any $x \in R(K)$, in general. One may see Example 3.2. of [20] for details.

We are going to show that there exists another type such that $\{\Lambda_j\}_{j \in \mathbb{J}}$ and a sequence derived by $\{\Delta_j\}_{j \in \mathbb{J}}$ are interchangeable in $R(K)$.

**Theorem 2.9.** Let $\{\Lambda_j\}_{j \in \mathbb{J}}$ and $\{\Delta_j\}_{j \in \mathbb{J}}$ are as in Corollary 2.8. If the range of $K \in B(H)$ is closed and $K^+$ is the pseudo-inverse of $K$, then there exists a sequence of operators $\{\Omega_j\}_{j \in \mathbb{J}} = \{\Delta_j K^+ |_{R(K)}\}_{j \in \mathbb{J}}$ derived by $\{\Delta_j\}_{j \in \mathbb{J}}$ such that

$$x = \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x, \quad (x \in R(K)).$$
and

\[ x = \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x, \quad (x \in R(K)). \]

Proof. Existence of \( K^+ \) is guaranteed by the fact that the range of \( K \) is closed. By (2.1), for any \( x \in R(K) \) we have

\[ \langle x, x \rangle = \langle KK^+ x, x \rangle \]

\[ = \left( \sum_{j=1}^{\infty} \Lambda_j^* \Delta_j^* K^+ |_{R(K)} x, x \right) \]

\[ = \langle x, \sum_{j=1}^{\infty} (K^+ |_{R(K)})^* \Delta_j^* \Lambda_j x \rangle \]

\[ = \langle x, \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x \rangle. \]

So

\[ x = \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x, \]

for all \( x \in R(K) \). Now define an operator \( L : H \to H \) by

\[ Lx = \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x. \]

Let the upper bounds of \( \{ \Omega_j \}_{j \in \mathbb{J}}, \{ \Lambda_j \}_{j \in \mathbb{J}} \) are \( B \) and \( C \), respectively, then

\[ \|L\| = \sup_{\|x\|=1} |\langle Lx, x \rangle| \]

\[ \leq \sup_{\|x\|=1} \left( \sum_{j \in \mathbb{J}} \| \Omega_j x \|^2 \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{J}} \| \Lambda_j x \|^2 \right)^{\frac{1}{2}} \]

\[ \leq \sqrt{BC}. \]

So \( L \in B(H) \). For \( x, y \in H \), we have

\[ \langle Lx, y \rangle = \left( \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x, y \right) \]

\[ = \sum_{j=1}^{\infty} \langle \Omega_j x, \Lambda_j y \rangle \]
and
\[
\langle x, y \rangle = \langle x, \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j y \rangle = \sum_{j=1}^{\infty} \langle \Omega_j x, \Lambda_j y \rangle.
\]

So \( \langle Lx, y \rangle = \langle x, y \rangle \), for all \( x, y \in H \), which implies that \( L = I_{R(K)} \). This completes the proof. \( \square \)

Let \( \{\Omega_j\}_{j \in J} \) and \( \{\Lambda_j\}_{j \in J} \) be as in Theorem 2.8. Then it is obvious that (2.3) or (2.4) hold if and only if \( T_\Omega^* T_{\Lambda} = I_{R(K)} \), where \( T_\Lambda^* \) and \( T_\Omega^* \) are the analysis operators for \( \{\Lambda_j\}_{j \in J} \) and \( \{\Omega_j\}_{j \in J} \), respectively.

**Corollary 2.10.** If \( \{\Omega_j\}_{j \in J} \) and \( \{\Lambda_j\}_{j \in J} \) are \( g \)-\( K \)-frames for \( H \) with respect to \( \{H_j\}_{j \in J} \) and \( T_\Lambda^* \) and \( T_\Omega^* \) are analysis operator for \( \{\Lambda_j\}_{j \in J} \) and \( \{\Omega_j\}_{j \in J} \), respectively, then \( R(\Omega_j) \perp R(\Lambda_j) \) for all \( j \in J \) if and only if \( R(T_\Omega^*) \perp R(T_\Lambda^*) \).

### 3. Some more properties of \( g \)-\( K \)-frames

In this section, first using a \( g \)-atomic system and some elements of \( B(H) \), we are going to construct new \( g \)-atomic systems.

**Proposition 3.1.** Let \( K, L \in B(H) \) and \( \{\Lambda_j\}_{j \in J} \) be a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with the frame bounds \( A, B \).

(i) If \( T : H \to H \) is an isometry such that \( K^* T = TK^* \), then \( \{\Lambda_j T^* \}_{j \in J} \) is a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with the same frame bounds.

(ii) \( \{\Lambda_j L^* \}_{j \in J} \) is a \( g \)-\( L^* K \)-frame with the frame bounds \( A \) and \( B \|L\|^2 \), respectively.

(iii) For any \( n \in \mathbb{N} \), \( \{\Lambda_j (L^*)^n \}_{j \in J} \) is a \( g \)-\( L^n K \)-frame.

(iv) If \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) and \( R(L) \subseteq R(K) \), then \( \{\Lambda_j\}_{j \in J} \) is also a \( g \)-\( L \)-frame.

**Proof.** Since \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with the frame bounds \( A, B > 0 \), so
\[
A \|K^* x\|^2 \leq \sum_{j \in J} \langle \Lambda_j x, \Lambda_j x \rangle \leq B \|x\|^2, \quad (x \in H).
\]

Hence for any \( x \in H \)
\[
\sum_{j \in J} \|\Lambda_j T x\|^2 \leq B \|T x\|^2 = B \|x\|^2.
\]
On the other hand for all $x \in H$ we have

$$\sum_{j=1}^{\infty} \|A_j Tx\|^2 \geq A\|K^* Tx\|^2$$

$$= A\langle K^* Tx, K^* Tx \rangle$$

$$= A\langle TK^* x, TK^* x \rangle$$

$$= A\langle K^* x, K^* x \rangle$$

$$= A\|K^* x\|^2,$$

which proves (i).

For proving (ii), one may see that for any $x \in H$,

$$A\|(LK)^* x\|^2 = A\|K^* L^* x\|^2$$

$$\leq \sum_{j=1}^{\infty} \|A_j L^* x\|^2$$

$$\leq B\|L^* x\|^2$$

$$\leq B\|L\|^2\|x\|^2.$$

(iii) is trivial by applying (ii).

For proving (iv), if $A$ and $B$ are the $g$-$K$-frame bounds of $\{A_j\}_{j \in J}$ then by the fact that $R(L) \subseteq R(K)$ and Theorem 1.4, there exists $\lambda > 0$ such that for any $x \in H$, $\|L^* x\|^2 \leq \lambda\|K^* x\|^2$ and

$$\frac{A}{\lambda}\|L^* x\|^2 \leq A\|K^* x\|^2$$

$$\leq \sum_{j \in J} \|A_j x\|^2$$

$$\leq B\|x\|^2.$$

□

As a corollary of (iv) one can easily see that every g-frame is indeed a $g$-$K$-frame, for any $K \in B(H)$.

**Proposition 3.2.** Let $K \in B(H)$ and $\{A_j\}_{j \in J}$ be a $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with the frame bounds $A, B$, then $\{A_j K^*\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with the frame bounds $A, B\|K\|^2$.

The frame operator of $\{A_j K^*\}_{j \in J}$ is $S' = KSK^*$, where $S$ is the frame operator of $\{A_j\}_{j \in J}$. 
Proof. Since \( \{\Lambda_j\}_{j \in J} \) is a g-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), for any \( x \in H \) we have
\[
A \|K^*x\|^2 \leq \sum_{j \in J} \|\Lambda_j K^*x\|^2 \\
\leq B \|K^*x\|^2 \\
\leq B \|K\|^2 \|x\|^2.
\]
But by definition of \( S \)
\[
SK^*x = \sum_{j \in J} \Lambda_j^* \Lambda_j K^*x.
\]
Thus
\[
KSK^*x = \sum_{j \in J} K \Lambda_j^* \Lambda_j K^*x \\
= \sum_{j \in J} ( \Lambda_j K^* )^* ( \Lambda_j K^* )x.
\]
Hence \( S' = KSK^* \).

Corollary 3.3. Let \( K \in B(H) \) and \( \{\Lambda_j\}_{j \in J} \) be a g-orthonormal basis, then \( \{\Lambda_j K^*\}_{j \in J} \) is a g-K-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \).

Proposition 3.4. Suppose that \( K \in B(H) \) and \( \{\Lambda_j\}_{j \in J} \) is a g-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), then \( \{\Lambda_j S^{-1} K\}_{j \in J} \) is a g-K-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), with the frame operator \( S' = K^* S^{-1} K \), where \( S \) is the frame operator of \( \{\Lambda_j\}_{j \in J} \).

Proof. By Corollary 2.5, it is enough to show that \( \{\Lambda_j S^{-1} K\}_{j \in J} \) is a g-atomic system. If \( S \) is the frame operator of \( \{\Lambda_j S^{-1} K\}_{j \in J} \), then it is well-known that
\[
x = \sum_{j \in J} \alpha_j^* \alpha_j S^{-1} x,
\]
for all \( x \in H \). Thus
\[
Kx = \sum_{j \in J} \alpha_j^* \alpha_j S^{-1} Kx, \quad (x \in H).
\]
Trivially \( \{\Lambda_j S^{-1} K\}_{j \in J} \) is a g-Bessel sequence, since for \( x \in H \),
\[
\sum_{j \in J} \| \Lambda_j S^{-1} K x \|^2 \leq B \| S^{-1} K x \|^2 \\
\leq B \| S^{-1} \|^2 \| K \| \| x \|^2.
\]
Also
\[ SS^{-1}Kx = \sum_{j \in J} \Lambda_j^* \Lambda_j S^{-1}Kx, \quad (x \in H), \]
and so
\[ K^*S^{-1}SS^{-1}Kx = \sum_{j \in J} KS^{-1}\Lambda_j^* \Lambda_j S^{-1}Kx \]
\[ = \sum_{j \in J} (\Lambda_j S^{-1}K)^* (\Lambda_j S^{-1}K)x, \]
which implies that \( S' = K^*S^{-1}K \). \hfill \Box

**Corollary 3.5.** If \( K \in B(H) \) and \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-orthonormal basis, then \( \{\Lambda_j S^{-1}K\}_{j \in J} \) is a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \).

**Proposition 3.6.** If \( L \in B(H) \), \( R(K) \subset R(L^*) \) and \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), then \( \{\Lambda_j L\}_{j \in J} \) is a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with the frame operator \( S' = L^*SL \), where \( S \) is the frame operator of \( \{\Lambda_j\}_{j \in J} \).

**Proof.** By the facts that \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) and \( R(K) \subset R(L^*) \), we may find positive real numbers \( A, B > 0 \) such that
\[ A\|K^*x\|^2 \leq A\|Lx\|^2 \leq \sum_{j \in J} \|\Lambda_j Lx\|^2 \leq B\|Lx\|^2 \leq B\|L\|^2\|x\|^2, \quad (x \in H). \]

The proof of \( S' = L^*SL \) is similar to the proof of (3.1). \hfill \Box

If \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-\( K_1 \)-frame, \( \{\Omega_j\}_{j \in J} \) is a \( g \)-\( K_2 \)-frame and \( R(K_1) \subset R(K_2) \), then applying Theorem 1.4 one may easily see that \( \{\Omega_j\}_{j \in J} \) is a \( g \)-\( K_1 \)-frame.

### 4. Some perturbation results

In this section, a perturbation result for generalized atomic systems is investigated. A version of the following theorem for Hilbert \( C^* \)-modules can be seen in [25].

**Theorem 4.1.** Assume that \( K \in B(H) \). Let \( \{\Lambda_j\}_{j \in J} \) be a \( g \)-\( K \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), with \( g \)-\( K \)-frame bounds \( A, B > 0 \). If there exists a constant \( M > 0 \), such that
\[ \sum_{j \in J} \|(\Lambda_j - \Theta_j)f\|^2 \leq M \min \left( \sum_{j \in J} \|\Lambda_j f\|^2, \sum_{j \in J} \|\Theta_j f\|^2 \right), \quad (4.1) \]
for any $f \in H$, then $\{\Theta_j\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$. The converse is valid for any $f \in R(K)$, when $R(K)$ is closed.

**Proof.** First suppose that (4.1) is valid. For any $f \in H$, we have

$$\left( \sum_{j \in J} \|\Lambda_j f\|^2 \right)^{\frac{1}{2}} = \|\{\Lambda_j f\}_{j \in J}\|
\leq \|(\Lambda_j - \Theta_j)f\|_{j \in J} + \|\Theta_j f\|_{j \in J}\|
\leq (\sqrt{M} + 1) \|\Theta_j f\|_{j \in J},$$

which implies that

$$\sum_{j \in J} \|\Lambda_j f\|^2 \leq (\sqrt{M} + 1)^2 \sum_{j \in J} \|\Theta_j f\|^2.$$

So for any $f \in H$

$$\sum_{j \in J} \|\Theta_j f\|^2 \geq \frac{1}{(\sqrt{M} + 1)^2} \sum_{j \in J} \|\Lambda_j f\|^2
\geq \frac{A}{(\sqrt{M} + 1)^2} \|K^* f\|^2.$$  \hfill (4.2)

On the other hand

$$\left( \sum_{j \in J} \|\Theta_j f\|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j \in J} \|\Lambda_j f - \Theta_j f\|^2 \right)^{\frac{1}{2}} \leq \sqrt{M} \left( \sum_{j \in J} \|\Lambda_j f\|^2 \right)^{\frac{1}{2}}
\leq (\sqrt{M} + 1) \left( \sum_{j \in J} \|\Lambda_j f\|^2 \right)^{\frac{1}{2}}
\leq \sqrt{B}(1 + \sqrt{M}) \|f\|.$$

Combining (4.2) and (4.3), we conclude that $\{\Theta_j\}_{j \in J}$ is a $g$-$L$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with $g$-$K$-frame bounds $A/E(\sqrt{M} + 1)^2$, $B(1 + \sqrt{M})^2$, respectively. For the converse, suppose that $\{\Theta_j\}_{j \in J}$ is a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with frame bounds $C, D$, respectively. Closeness of the range of $K \in B(H)$ implies that its pseudo-inverse $K^+$
exists, so \( I_{R(K)} = KK^+|_{R(K)} f \), where \( R(K) \) is the range of \( K \). Hence \( I_{R(K)}^* = (K^+|_{R(K)})^* K^* \). Thus for any \( f \in R(K) \),

\[
\left( \sum_{j \in J} \| (A_j - \Theta_j) f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| f \| \\
= \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| (K^+|_{R(K)})^* K^* f \| \\
\leq \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| K^+|_{R(K)} \| \| K^* f \| \\
\leq \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| K^+|_{R(K)} \| \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} \\
= \left( 1 + \frac{\sqrt{B} \| K^+|_{R(K)} \|}{\sqrt{A}} \right) \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}}.
\]

On the other hand for any \( f \in R(K) \) we have

\[
\left( \sum_{j \in J} \| (A_j - \Theta_j) f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j \in J} \| A_j f \|^2 \right)^{\frac{1}{2}} + \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| f \| \\
= \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| (K^+|_{R(K)})^* K^* f \| \\
\leq \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| K^+|_{R(K)} \| \| K^* f \|.
\]
Thus for any $f \in R(K)$,
\[
\left( \sum_{j \in J} \| (\Lambda_j - \Theta_j) f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} + \sqrt{B} \| K^+ |_{R(K)} \| \| K^* f \|
\]
\[
\leq \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}} + \frac{\sqrt{B} \| K^+ |_{R(K)} \|}{\sqrt{C}} \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}}
\]
\[
= \left( 1 + \frac{\sqrt{B} \| K^+ |_{R(K)} \|}{\sqrt{C}} \right) \left( \sum_{j \in J} \| \Theta_j f \|^2 \right)^{\frac{1}{2}}.
\]

Letting
\[
M = \min \left\{ \left( 1 + \frac{\sqrt{D} \| K^+ |_{R(K)} \|}{\sqrt{A}} \right)^2, \left( 1 + \frac{\sqrt{B} \| K^+ |_{R(K)} \|}{\sqrt{C}} \right)^2 \right\},
\]
one can see that (4.1) holds for any $f \in R(K)$. \qed

For $K = I$ we have the following result which is proved in [25] for Hilbert $C^*$-modules.

**Corollary 4.2.** Let $\{\Lambda_j\}_{j \in J}$ be a $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$, with $g$-frame bounds $A, B > 0$. Let $\{\Theta_j \in B(H, H_j) : j \in J\}$. Then the following statements are equivalent:

(i) $\{\Theta_j\}_{j \in J}$ is a $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$;

(ii) There exists a constant $M > 0$, such that for any $f \in H$, we have
\[
\sum_{j \in J} \| (\Lambda_j - \Theta_j) f \|^2 \leq M \min \left( \sum_{j \in J} \| \Lambda_j f \|^2, \sum_{j \in J} \| \Theta_j f \|^2 \right).
\]

5. **G-\(K\)-DUALS**

In this section the duals of $g$-\(K\)-frames are studied. First, we give a new definition of generalized dual frame that generalize some results of previous work (see [20]).

**Definition 5.1.** Let $K \in B(H)$ and $\Xi : H \to F$ be a bounded operator. A bounded operator $\Upsilon \in B(H, F)$ is called a generalized $K$-dual (or $g$-$K$-dual) of $\Xi$ if $K = \Xi^\ast \Upsilon$.

In view of Definition 5.1, one can see that $\Xi$ and $\Upsilon$ are $g$-atomic system for $K \in B(H)$ and $K^* \in B(H)$ with respect to $F$, respectively. It is obvious that $\Xi$ and $\Upsilon$ are not interchangeable in general unless $K$ is self adjoint. Let $\Xi$ be a $g$-$K$-frame for $H$ with respect to $F$. If
$K \in B(H)$ has closed range then it has a pseudo-inverse $K^+$ and by applying a similar process to [26], one may prove that

$$A\|K^+\|^2 \|x\| \leq \|\Xi^* \Xi x\| \leq B\|x\|, \quad (x \in H).$$

So $S : R(K) \to S(R(K))$ is a homeomorphism. Moreover

$$B^{-1}\|x\| \leq \|\Xi^* \Xi x\| \leq A^{-1}\|K^+\|^2 \|x\|, \quad (x \in S(R(K))).$$

Also $\Xi P(S^{-1})K$ is a $g$-$K$-dual of $\Xi$, where $P$ is the orthogonal projection of $H$ onto $S(R(K))$ and $S := \Xi^* \Xi$. Indeed,

$$\Xi^* \Xi P(S^{-1})K = SP(S^{-1})K = K.$$

This $g$-$K$-dual of $\Xi$, $\Xi P(S^{-1})K$, is denoted by $\Pi$.

**Theorem 5.2.** Let $\Xi$ be a $g$-$K$-frame for $H$ with respect to $F$ with $g$-$K$-frame bounds $A$ and $B$, respectively. Then there exists a one-to-one correspondence between $g$-$K$-duals of $\Xi$ and operator $\Psi \in B(H,F)$ such that $\Xi^* \Psi = 0$.

**Proof.** Suppose that $\Phi$ is a $g$-$K$-dual of $\Xi$ with the bounds $A_1$ and $B_1$, respectively. Define $\Psi : H \to F; x \mapsto \Psi x$ by

$$\Psi x = \Phi x - \Pi x.$$

Then $\Psi$ is bounded by (5.1). Indeed,

$$\|\Psi x\|^2 = \|\Phi x - \Pi x\|^2 \leq \|\Phi x\|^2 + \|\Xi P(S^{-1})K x\|^2 + 2\|\Phi x\|\|\Xi P(S^{-1})K x\| \leq \left(B_1 + A^{-1}\|K^+\|^2\|K\|^2 + 2\sqrt{B_1 A^{-1}\|K^+\|^2\|K\|^2}\right)\|x\|^2.$$

Moreover,

$$\Xi^* \Psi x = \Xi^* \Phi x = \Xi^* \Phi x - \Xi^* \Pi x = K x - \Xi^* \Xi P(S^{-1})K x = K x - K x = 0.$$

Conversely, Let $\psi \in B(H,F)$ such that $\Xi \Psi = 0$. Set

$$\Phi x = \Pi x + \Psi x, \quad (x \in H),$$

Then $\Phi$ is a bounded operator. Moreover,

$$\Xi^* \Phi x = \Xi^* \Pi x + \Xi^* \Psi x = K x.$$

Therefore $\Phi$ is a $g$-$K$-dual of $\Xi$. $\square$

The following corollary in $g$-$K$-frame is a generalization of [3, Theorem 3.4].

**Corollary 5.3.** Let $\{\Lambda_j\}_{j \in J}$ be a $g$-$K$-frame for $H$ with respect to $\{H_j\}_{j \in J}$. Then there exists a one-to-one correspondence between $K$-duals of $\{\Lambda_j\}_{j \in J}$ and operator $\psi \in B(H, \bigoplus_{j \in J} H_j)$ such that $T \psi = 0$, where $T$ is the synthesis operator of $\{\Lambda_j\}_{j \in J}$. 
Theorem 5.4. Let $K \in B(H)$. A bounded operator $\Xi : H \to F$ has a $K$-dual if and only if there exists a Bessel sequence $\{y_j\}_{j \in J}$ such that for every $x \in H$

$$Kx = \sum_{j \in J} \langle x, y_j \rangle x_j,$$

where $x_j := \Xi^*\delta_j$, $j \in J$, and $\{\delta_j\}_{j \in J}$ is an orthonormal basis for $F$.

Proof. It follows from Theorem 1.4, [17, Theorem 3] and the equality $\|\Xi x\|^2 = \sum_j |\langle x, x_j \rangle|^2$, for any $x \in H$. Indeed,

$$\|\Xi x\|^2 = \langle \Xi x, \Xi x \rangle = \left\langle \sum_j \langle \Xi x, \delta_j \rangle \delta_j, \Xi x \right\rangle = \sum_j \langle \Xi x, \delta_j \rangle \langle \delta_j, \Xi x \rangle = \sum_j \langle x, \Xi^*\delta_j \rangle \langle \Xi^*\delta_j, x \rangle = \sum_j |\langle x, x_j \rangle|^2.$$ 

□

References


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