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G-frames for operators in Hilbert spaces

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Abstract. K-frames as a generalization of frames were introduced by L. Găvruţa to study atomic systems on Hilbert spaces which allows, in a stable way, to reconstruct elements from the range of the bounded linear operator K in a Hilbert space. Recently some generalizations of this concept are introduced and some of its difference with ordinary frames are studied. In this paper, we give a new generalization of K-frames. After proving some characterizations of generalized K-frames, new results are investigated and some new perturbation results are established. Finally, we give several characterizations of K-duals.

1. Introduction and Basic Definitions

Frames in Hilbert spaces were introduced by J. Duffin and A.C. Schaffer [12] in 1952. In 1986, frames were brought to life by Daubechies, Grossmann and Meyer [9]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [16], sampling [13, 14], coding and communications [22], filter bank theory [5], system modeling [11], and so on.

Frames are generalizations of orthonormal basis in Hilbert spaces. The elements of an orthonormal basis are linearly independent which allows every vector to be uniquely represented as a linear combination of the basis elements. This is very restrictive for practical problems. A frame also allows each element in the space to be written as a linear

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combination of the elements in the frame, but linear independence between the frame elements is not required. For more information about frames see [6].

G-frames in a complex Hilbert space were introduced by W. Sun in [23] to deal with all existing frames as a united object and discussed some properties of them. Next many authors give new results, for instance see [24, 25].

On the other hand, a family of local atoms for a subspace H_0 of a separable Hilbert space H as a family of analysis and synthesis systems with frame-like properties H_0 was studied in [15] and generalized by L. Găvruţa in [17] and called atomic systems. Let H be a Hilbert space and $K \in B(H)$, the space of all bounded linear operators on H. K-frames as a generalization of frames, i. e. frames are a special case of K-frame when K is the identity operator. It is proved that an atomic system for K is a K-frame and vice versa (see [17]). We refer to [17] and [26] for more information on these concepts. Recently, generalized K-frames and generalized atomic systems for operators introduced in [4] and [27]. Some characterizations and perturbation results for generalized K-frames are investigated in these papers. Also, some results about these concepts in Hilbert modules and Banach spaces can be found in [7, 8], respectively.

The paper is organized as follows. We continue this introductory section with a review of the basic definitions and results. In Section 2, some characterizations of g-K-frames are studied. In section 3, some more properties of generalized K-frames will be discussed. Many of our results are generalizations of results in [4, 26, 27]. In the next section some other perturbation results for g-K-frames are established. In the last section of this paper K-duals are studied.

Let us first recall the concepts of the atomic systems for K and K-frames. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in the Hilbert space H is called an atomic system for the bounded linear operator K on H if the following statements hold

(i) the series $\sum_{n\in\mathbb{N}} c_n x_n$ converges for all $c = \{c_n\}_{n\in\mathbb{N}} \in l^2$;
(ii) for any $x \in H$, there exists $a_x = \{a_n\}_{n\in\mathbb{N}} \in l^2$ such that $Kx = \sum_{n\in\mathbb{N}} a_n x_n$ where $\|a_x\|_{l^2} \leq C\|x\|$, C is a positive constant.

Note that in (ii), the sequence a_x corresponding to $x \in H$ is not unique, in general.

It is well-known that every bounded linear operator K on a separable Hilbert space admits an atomic system (see [17, Theorem 2]).

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be a K-frame for H if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad (x \in H).$$

In the sequel, it is assumed that $\mathbb{J}$ is a finitely or countably index set. For two Hilbert spaces H and F we denote by $B(H, F)$ the collection of all bounded linear operators between H and F, and we abbreviate $B(H, H)$ by $B(H)$. Also we denote the range of $L \in B(H, F)$ by $R(L)$.

Recall that if H and H_j, $j \in \mathbb{J}$, are Hilbert spaces then a sequence $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is said to be a generalized frame (or simply g-frame) for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if there are two positive constants A and B such that

$$A\|x\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j(x)\|^2 \leq B\|x\|^2, \quad (x \in H).$$

The constants A and B are the lower and upper frame bounds, respectively. If the right-hand side of this inequality holds, then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be a g-Bessel sequence.
For a sequence of Hilbert spaces $\{H_j\}_{j\in\mathbb{J}}$, the space

$$\bigoplus_{j\in\mathbb{J}} H_j := \left\{ \{x_j\}_{j\in\mathbb{J}} : x_j \in H_j \text{ and } \sum_{j\in\mathbb{J}} \|x_j\|^2 < \infty \right\},$$

with the inner product $\langle\{x_j\}, \{y_j\}\rangle = \sum_{j\in\mathbb{J}}\langle x_j, y_j\rangle$ is a Hilbert space.

Now for a g-Bessel sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$, we denote by $T_\Lambda : \bigoplus_{j\in\mathbb{J}} H_j \to H$, T_Λ^* and S the the synthesis operator, analysis operator and frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$, respectively, which are defined as follows

$$T_\Lambda(x_j)_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} \Lambda_j^* x_j, \qquad Sx = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j x.$$

The following proposition is a characterization of g-Bessel sequences.

Proposition 1.1 ([19]). *For a sequence $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$, the following statements are equivalent:*

(i) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in\mathbb{J}}$;*
(ii) *The operator*

$$T : (\{x_j\}_{j\in\mathbb{J}}) \mapsto \sum_{j\in\mathbb{J}} \Lambda_j^* x_j,$$

is well-defined and bounded from $\bigoplus_{j\in\mathbb{J}} H_j$ to H.

Also it is well known that if $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with bounds A and B, then the g-frame operator S is an invertible bounded self-adjoint operator which satisfies

$$A\|x\|^2 \leq \langle Sx, x\rangle \leq B\|f\|^2,$$

and

$$B^{-1}\|x\|^2 \leq \langle S^{-1}x, x\rangle \leq A^{-1}\|x\|^2,$$

for all $x \in H$.

Definition 1.2 ([4, 27])**.** A sequence $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is said to be a generalized atomic system (or simply g-atomic system) for the operator $K \in B(H)$ if

(i) for every $\{x_j\}_{j\in\mathbb{J}} \in \bigoplus_{j\in\mathbb{J}} H_j$, $\sum_{j\in\mathbb{J}} \Lambda_j^* x_j$ converges in H,

(ii) there exists $C > 0$ such that for all $x \in H$, there is a sequence $a_x = \{a_j\}_{j\in\mathbb{J}} \in \bigoplus_{j\in\mathbb{J}} H_j$, for which

$$Kx = \sum_{j\in\mathbb{J}} \Lambda_j^* a_j \quad \text{and} \quad \|a_x\| = \left(\sum_{j\in\mathbb{J}} \|a_j\|^2\right)^{\frac{1}{2}} \leq C\|x\|.$$

Definition 1.3 ([4, 27])**.** Suppose that $K \in B(H)$. A sequence $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is said to be a generalized K-frame (or g-K-frame) for H with respect to $\{H_j : j \in \mathbb{J}\}$ if there are two positive constants A, B such that

$$A\|K^*x\| \leq \sum_{j\in\mathbb{J}} \|\Lambda_j x\|^2 \leq B\|x\|^2, \quad (x \in H).$$

The following result in operator theory has an important role in our discussion.

Theorem 1.4 ([10])**.** *Let H, H_1 and H_2 be Hilbert spaces and $L_1 \in B(H_1, H)$, $L_2 \in B(H_2, H)$. The following statements are equivalent:*

(i) *$R(L_1) \subseteq R(L_2)$;*

(ii) *$L_1L_1^* \leq \lambda L_2L_2^*$, for some $\lambda \geq 0$;*

(iii) *there exists a bounded operator $T \in B(H_1, H_2)$ such that $L_1 = L_2T$.*

Throughout the paper H, F and H_j for any $j \in \mathbb{J}$ are separable Hilbert spaces.

2. G-K-frames and g-atomic systems

First we give the following definition that a generalized atomic system in [4, 27] is the special case of g-atomic systems.

Definition 2.1. Let $K \in B(H)$. A bounded linear operator $\Xi : H \to F$ is called a generalized atomic system (or g-atomic system) for $K \in B(H)$ with respect to F if there exists $C > 0$ such that for all $x \in H$, there is $a_x \in F$ such that

$$Kx = \Xi^* a_x \quad \text{and} \quad \|a_x\| \le C\|x\|.$$

Theorem 2.2. *Let $K \in B(H)$. Then there exists a g-atomic system for $K \in B(H)$ with respect to F.*

Proof. Define $\Xi : H \to F$ by $\Xi(e_j) = \delta_j$ for all $j \in \mathbb{J}$, where $\{e_j\}_{j\in\mathbb{J}}$ and $\{\delta_j\}_{j\in\mathbb{J}}$ are orthonormal bases for H and F, respectively. Then we have $\Xi \in B(H, F)$ and $\Xi^*\Xi K x = Kx$. Therefore Ξ is a g-atomic system for $K \in B(H)$ with respect to F. Note that

$$\|a_x\| := \|\Xi K x\| \le \|\Xi\|\|K\|\|x\|.$$

□

Now we give a new definition of generalized K-frames.

Definition 2.3. A bounded linear operator Ξ from H to F is called a generalized K-frame (or g-K-frame) for H with respect to F if there are two positive constants A, B such that

$$A\|K^*x\|^2 \le \|\Xi x\|^2 \le B\|x\|^2, \quad (x \in H).$$

Now we have the following characterization of g-atomic systems.

Theorem 2.4. *Let $K \in B(H)$ and Ξ be a bounded linear operator from H to F. Then the following statements are equivalent:*

(i) *$\Xi : H \to F$ is a g-atomic system for $K \in B(H)$ with respect to F;*
(ii) *$\Xi : H \to F$ is a g-K-frame for H with respect to F;*
(iii) *An operator $\Xi : H \to F$ is bounded and there exists another bounded operator $Q : H \to F$ such that $K = \Xi^* Q$;*
(iv) *ΞL is a g-atomic system for L^*K, where L is surjective on H;*
(v) *An operator $\Xi : H \to F$ is bounded, linear and there exists another bounded operator $Q : H \to F$ such that $K^* = Q^*\Xi$;*
(vi) *An operator $\Xi : H \to F$ is bounded, linear and $R(K) \subseteq R(\Xi^*)$.*

Proof. Assume that $K \in B(H)$ and Ξ is the bounded linear operator from H to F.

(i) ⇒ (ii) For any $x \in H$ we have

$$\begin{aligned}\|K^*(x)\| &= \sup_{\|y\|=1} |\langle K^*x, y\rangle| \\ &= \sup_{\|y\|=1} |\langle x, Ky\rangle|,\end{aligned}$$

by definition of g-atomic system for $K \in B(H)$ with respect to F, there exists $C > 0$ such that for any $y \in H$, there is $a_y \in F$, for which $\|a_y\| \leq C\|y\|$ and $Ky = T^* a_y$. Thus

$$\begin{aligned} \|K^*x\| &= \sup_{\|y\|=1} |\langle x, \Xi^* a_y\rangle| \\ &= \sup_{\|y\|=1} |\langle \Xi x, a_y\rangle| \\ &\leq \sup_{\|y\|=1} \|\Xi x\| \, \|a_y\| \\ &\leq \sup_{\|y\|=1} C\|\Xi x\|. \end{aligned}$$

Hence

$$\frac{1}{C^2}\|K^*x\|^2 \leq \|\Xi x\|^2,$$

which implies that (ii) is holds.

(ii) $\Rightarrow$ (iii) By Theorem 1.4 there exists a bounded operator $Q : H \to F$ such that $K = \Xi^* Q$.

(iii) $\Rightarrow$ (i) Take $a_x := Qx$, for every $x \in H$.

(iv)$\Rightarrow$ (i) Let $\Xi L : H \to F$ be a g-atomic system for L^*K, then for all $x \in H$, there exists $b_x \in F$ such that $L^*K = (\Xi L)^* b_x = L^*\Xi^* b_x$ so $L^*(Kx - \Xi^* b_x) = 0$. Due to injectivity of L^*, we have $Kx = \Xi^* b_x$. Therefore Ξ is a g-atomic system for $K \in B(H)$ with respect to F. [(i)$\Rightarrow$ (iv)] Let Ξ be a g-atomic system for $K \in B(H)$ with respect to F as in Definition 2.1, then $Kx = \Xi a_x$ and $L^*K = L^*\Xi^* a_x = (\Xi L)^* a_x$, it means that ΞL is a g-atomic system for L^*K.

(iii) $\Leftrightarrow$ (v) It is obvious.

(v) $\Leftrightarrow$ (vi) It follows from Theorem 1.4.

□

In view of Theorem 2.4, suppose that $F = \bigoplus_{j\in\mathbb{J}} H_j$ and consider an operator $\Lambda_j : H \to H_j$ by $x \mapsto \Lambda_j x$, for all $j \in \mathbb{J}$. Now, define $\Xi : H \to \bigoplus_{j\in\mathbb{J}} H_j$ by $\Xi(x) = \{\Lambda_j x\}_{j\in\mathbb{J}}$. Then we have the following corollary generalized [4, Theorem 2.5] and [27, Theorem 3.8].

Corollary 2.5. *For $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$, the following statements are equivalent:*

(i) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-atomic system for $K \in B(H)$ with respect to $\{H_j\}_{j\in\mathbb{J}}$;*

(ii) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$;*

(iii) $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a g-Bessel sequence and there exists another g-Bessel sequence* $\{\Delta_j\}_{j\in\mathbb{J}}$ *such that for all* $x \in H$*,*

$$K(x) = \sum_{j\in\mathbb{J}} \Lambda_j^* \Delta_j(x); \tag{2.1}$$

(iv) $\{\Lambda_j L\}_{j\in\mathbb{J}}$ *is a g-atomic system for* L^*K*, where* L *is surjective on* H*;*

(v) $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a g-Bessel sequence and there exists another Bessel sequence* $\{\Delta_j\}_{j\in\mathbb{J}}$ *such that*

$$K^*(x) = \sum_{j\in\mathbb{J}} \Delta_j^* \Lambda_j(x);$$

(vi) $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a g-Bessel sequence and* $R(K) \subseteq R(T)$*, where* T *is the synthesis operator of* $\{\Lambda_j\}_{j\in\mathbb{J}}$*.*

Also with the notations of the previous corollary, the following characterization of g-atomic systems is valid.

Theorem 2.6. $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a g-K-frame for* H *with respect to* $\{H_j\}_{j\in\mathbb{J}}$ *if and only if there exist* $A, B > 0$ *such that*

$$AKK^* \leq \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j \leq BI,$$

where I *is the identity operator on* H*. Moreover, in this case*

$$\|K\| \leq \sqrt{\frac{B}{A}}.$$

Proof. Since $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ then for any $x \in H$ we have

$$\begin{aligned} A\|K^*x\|^2 &\leq \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x\rangle \\ &= \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Lambda_j x, x \right\rangle \\ &\leq B\|x\|^2, \end{aligned}$$

which is equivalent to

$$\langle AKK^*x, x\rangle \leq \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Lambda_j x, x \right\rangle \leq \langle Bx, x\rangle, \quad (x \in H).$$

This proves the first part.
Furthermore $AKK^* \leq BI$ implies that $A\|K\|^2 \leq B$ and so

$$\|K\| \leq \sqrt{\frac{B}{A}}.$$

□

The following equivalent conditions for the sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ to be a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ may be useful which can be proved by Theorem 1.4 and the Theorem 2.6.

Proposition 2.7. *For a sequence $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$, the following statements are equivalent:*

(i) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$;*

(ii) *The operator*

$$T : (\{x_j\}_{j\in\mathbb{J}}) \mapsto \sum_{j\in\mathbb{J}} \Lambda_j^* x_j,$$

is well-defined and bounded from $\bigoplus_{j\in\mathbb{J}} H_j$ to H and $R(K) \subseteq R(T)$;

(iii) *The operator*

$$S : x \mapsto \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j x,$$

is well-defined and bounded from H to H and there exists $A > 0$ such that $AKK^* \leq S$.

Let $\{e_{j,k} : k \in \mathbb{K}_j$ be an orthonormal basis for H_j. For $j \in \mathbb{J}$ and $k \in \mathbb{K}_j$, define $F_{j,k} := (0, 0, \ldots, \underbrace{e_{j,k}}_{j-th}, 0, \ldots)$. Then $\{F_{j,k} : j \in \mathbb{J}, \quad k \in \mathbb{K}_j\}$ is an orthonormal basis for $\bigoplus_{j\in I} H_j$.
With these notations we state now another characterization of g-K-frames as follows.

Theorem 2.8. *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if and only if there exists a bounded linear operator $\Theta : \bigoplus_{j\in\mathbb{J}} H_j \to H$ such that for any $i \in \mathbb{J}$ and $k \in \mathbb{K}_i$,*

$$\langle x, \Theta(F_{i,k})\rangle = \langle \{\Lambda_j x\}_{j\in\mathbb{J}}, F_{i,k}\rangle,$$

and $R(K) \subset R(\Theta)$.

Proof. $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, so for some $A, B > 0$,

$$A\|K^* x\|^2 \leq \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x\rangle \leq B\|x\|^2, \quad (x \in H). \tag{2.2}$$

Let $T : H \to \bigoplus_{j\in\mathbb{J}} H_j$ be defined by $T(x) = \{\Lambda_j x\}_{j\in\mathbb{J}}$. Trivially T is a bounded linear operator and

$$\langle x, T^*F_{i,k}\rangle = \langle Tx, F_{i,k}\rangle = \langle \{\Lambda_j x\}_{j\in\mathbb{J}}, F_{i,k}\rangle, \quad i \in \mathbb{J}, k \in \mathbb{K}_i.$$

Now (2.2) implies that

$$A\|K^*x\|^2 \leq \|T(x)\|^2.$$

Hence $AKK^* \leq \Theta\Theta^*$, where $\Theta = T^*$. Therefore by Theorem 1.4, $R(K) \subset R(\Theta)$.

Conversely, let $\langle x, \Theta(F_{i,k})\rangle = \langle \{\Lambda_j x\}_{j\in\mathbb{J}}, F_{i,k}\rangle$, $i \in \mathbb{J}$ and $k \in \mathbb{K}_i$, where $\Theta \in B(\bigoplus_{j\in\mathbb{J}} H_j, H)$ and $R(K) \subset R(\Theta)$. We have $\Theta^*(x) = \{\Lambda_j x\}_{j\in\mathbb{J}}$. Indeed,

$$\langle \{\Lambda_j x\}_{j\in\mathbb{J}}, F_{i,k}\rangle = \langle x, \Theta(F_{i,k})\rangle = \langle \Theta^*x, F_{i,k}\rangle, \quad i \in \mathbb{J}, k \in \mathbb{K}_i.$$

Obviously $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence, since

$$\sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x\rangle = \|\Theta^*x\|^2 \leq \|\Theta^*\|^2\|x\|^2, \quad (x \in H).$$

Also $R(K) \subset R(\Theta)$ and Theorem 1.4 imply that there exists $B > 0$ such that $BKK^* \leq \Theta\Theta^*$. Therefore

$$B\|K^*x\|^2 \leq \|\Theta^*x\|^2 = \sum_{j=1}^{\infty} \langle \Lambda_j x, \Lambda_j x\rangle, \quad (x \in H).$$

□

Recall that if X and Y are two Banach spaces and $Q \in B(X, Y)$, then Q^+ is said to be the pseudo-inverse of Q if $QQ^+Q = Q$. In particular, for any $y \in R(Q)$, $QQ^+y = y$. For more details one can see [26]. An important difference of ordinary g-frames with g-K-frames is the fact that with the notations of Corollary 2.5, $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Delta_j\}_{j\in\mathbb{J}}$ are not interchangeable for any $x \in R(K)$, in general. One may see Example 3.2. of [26] for details.
We are going to show that there exists another type such that $\{\Lambda_j\}_{j\in\mathbb{J}}$ and a sequence derived by $\{\Delta_j\}_{j\in\mathbb{J}}$ are interchangeable in $R(K)$.

Theorem 2.9. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Delta_j\}_{j\in\mathbb{J}}$ are as in Corollary 2.5. If the range of $K \in B(H)$ is closed and K^+ is the pseudo-inverse of K, then there exists a sequence of operators $\{\Omega_j\}_{j\in\mathbb{J}} = \{\Delta_j K^+|_{R(K)}\}_{j\in\mathbb{J}}$ derived by $\{\Delta_j\}_{j\in\mathbb{J}}$ such that*

$$x = \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x, \quad (x \in R(K)), \tag{2.3}$$

and

$$x = \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x, \quad (x \in R(K)) . \tag{2.4}$$

Proof. Existence of K^+ is guaranteed by the fact that the range of K is closed. By (2.1), for any $x \in R(K)$ we have

$$\begin{aligned}
\langle x, x \rangle &= \langle KK^+ x, x \rangle \\
&= \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Delta_j K^+|_{R(K)} x, x \right\rangle \\
&= \langle x, \sum_{j=1}^{\infty} (K^+|_{R(K)})^* \Delta_j^* \Lambda_j x \rangle \\
&= \langle x, \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x \rangle .
\end{aligned}$$

So

$$x = \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j x,$$

for all $x \in R(K)$. Now define an operator $L : H \to H$ by

$$Lx = \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x.$$

Let the upper bounds of $\{\Omega_j\}_{j\in\mathbb{J}}$, $\{\Lambda_j\}_{j\in\mathbb{J}}$ are B and C, respectively, then

$$\begin{aligned}
\|L\| &= \sup_{\|x\|=1} |\langle Lx, x \rangle| \\
&\leq \sup_{\|x\|=1} \left(\sum_{j\in\mathbb{J}} \|\Omega_j x\|^2 \right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{J}} \|\Lambda_j x\|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{BC}.
\end{aligned}$$

So $L \in B(H)$. For $x, y \in H$, we have

$$\begin{aligned}
\langle Lx, y \rangle &= \left\langle \sum_{j=1}^{\infty} \Lambda_j^* \Omega_j x, y \right\rangle \\
&= \sum_{j=1}^{\infty} \langle \Omega_j x, \Lambda_j y \rangle
\end{aligned}$$

and

$$\begin{aligned}\langle x, y\rangle &= \langle x, \sum_{j=1}^{\infty} \Omega_j^* \Lambda_j y\rangle \\ &= \sum_{j=1}^{\infty} \langle \Omega_j x, \Lambda_j y\rangle.\end{aligned}$$

So $\langle Lx, y\rangle = \langle x, y\rangle$, for all $x, y \in H$, which implies that $L = I_{R(K)}$. This completes the proof. □

Let $\{\Omega_j\}_{j\in\mathbb{J}}$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ be as in Theorem 2.9. Then it is obvious that (2.3) or (2.4) hold if and only if $T_\Omega^* T_\Lambda = I_{R(K)}$, where T_Λ^* and T_Ω^* are the analysis operators for $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Omega_j\}_{j\in\mathbb{J}}$, respectively.

Corollary 2.10. *If $\{\Omega_j\}_{j\in\mathbb{J}}$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ are g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ and T_Λ^* and T_Ω^* are analysis operator for $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Omega_j\}_{j\in\mathbb{J}}$, respectively, then $R(\Omega_j) \perp R(\Lambda_j)$ for all $j \in \mathbb{J}$ if and only if $R(T_\Omega) \perp R(T_\Lambda)$.*

3. Some more properties of g-K-frames

In this section, first using a g-atomic system and some elements of $B(H)$, we are going to construct new g-atomic systems.

Proposition 3.1. *Let $K, L \in B(H)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the frame bounds A, B.*

- (i) *If $T : H \to H$ is an isometry such that $K^*T = TK^*$, then $\{\Lambda_j T^*\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the same frame bounds.*
- (ii) *$\{\Lambda_j L^*\}_{j\in\mathbb{J}}$ is a g-LK-frame with the frame bounds A and $B\|L\|^2$, respectively.*
- (iii) *For any $n \in \mathbb{N}$, $\{\Lambda_j (L^*)^n\}_{j\in\mathbb{J}}$ is a g-L^nK-frame.*
- (iv) *If $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ and $R(L) \subseteq R(K)$, then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is also a g-L-frame.*

Proof. Since $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the frame bounds $A, B > 0$, so

$$A\|K^*x\|^2 \le \sum_{j\in\mathbb{J}} \langle \Lambda_j x, \Lambda_j x\rangle \le B\|x\|^2, \quad (x \in H).$$

Hence for any $x \in H$

$$\begin{aligned}\sum_{\in\mathbb{J}} \|\Lambda_j T x\|^2 &\le B\|Tx\|^2 \\ &= B\|x\|^2.\end{aligned}$$

On the other hand for all $x \in H$ we have

$$\begin{aligned}\sum_{j=1}^{\infty}\|\Lambda_j Tx\|^2 &\geq A\|K^*Tx\|^2\\ &= A\langle K^*Tx, K^*Tx\rangle\\ &= A\langle TK^*x, TK^*x\rangle\\ &= A\langle K^*x, K^*x\rangle\\ &= A\|K^*x\|^2,\end{aligned}$$

which proves (i).
For proving (ii), one may see that for any $x \in H$,

$$\begin{aligned}A\|(LK)^*x\|^2 &= A\|K^*L^*x\|^2\\ &\leq \sum_{j=1}^{\infty}\|\Lambda_j L^*x\|^2\\ &\leq B\|L^*x\|^2\\ &\leq B\|L\|^2\|x\|^2.\end{aligned}$$

(iii) is trivial by applying (ii).
For proving (iv), if A and B are the g-K-frame bounds of $\{\Lambda_j\}_{j\in\mathbb{J}}$ then by the fact that $R(L) \subseteq R(K)$ and Theorem 1.4, there exists $\lambda > 0$ such that for any $x \in H$, $\|L^*x\|^2 \leq \lambda\|K^*x\|^2$ and

$$\begin{aligned}\frac{A}{\lambda}\|L^*x\|^2 &\leq A\|K^*x\|^2\\ &\leq \sum_{j\in\mathbb{J}}\|\Lambda_j x\|^2\\ &\leq B\|x\|^2.\end{aligned}$$

□

As a corollary of (iv) one can easily see that every g-frame is indeed a g-K-frame, for any $K \in B(H)$.

Proposition 3.2. *Let $K \in B(H)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the frame bounds A, B, then $\{\Lambda_j K^*\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the frame bounds $A, B\|K\|^2$. The frame operator of $\{\Lambda_j K^*\}_{j\in\mathbb{J}}$ is $S' = KSK^*$, where S is the frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$.*

Proof. Since $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, for any $x \in H$ we have

$$\begin{aligned} A\|K^*x\|^2 &\leq \sum_{j\in\mathbb{J}} \|\Lambda_j K^*x\|^2 \\ &\leq B\|K^*x\|^2 \\ &\leq B\|K\|^2\|x\|^2. \end{aligned}$$

But by definition of S

$$SK^*x = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j K^*x.$$

Thus

$$\begin{aligned} KSK^*x &= \sum_{j\in\mathbb{J}} K\Lambda_j^*\Lambda_j K^*x \\ &= \sum_{j\in\mathbb{J}} (\Lambda_j K^*)^*(\Lambda_j K^*)x. \end{aligned} \tag{3.1}$$

Hence $S' = KSK^*$. □

Corollary 3.3. *Let $K \in B(H)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-orthonormal basis, then $\{\Lambda_j K^*\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$.*

Proposition 3.4. *Suppose that $K \in B(H)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, then $\{\Lambda_j S^{-1}K\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, with the frame operator $S' = K^*S^{-1}K$, where S is the frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$.*

Proof. By Corollary 2.5, it is enough to show that $\{\Lambda_j S^{-1}K\}_{j\in\mathbb{J}}$ is a g-atomic system. If S is the frame operator of $\{\Lambda_j S^{-1}K\}_{j\in\mathbb{J}}$, then it is well-known that

$$x = \sum_{j\in\mathbb{J}}^{\infty} \Lambda_j^*\Lambda_j S^{-1}x,$$

for all $x \in H$. Thus

$$Kx = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j S^{-1}Kx, \quad (x \in H).$$

Trivially $\{\Lambda_j S^{-1}K\}_{j\in\mathbb{J}}$ is a g-Bessel sequence, since for $x \in H$,

$$\begin{aligned} \sum_{j\in\mathbb{J}} \|\Lambda_j S^{-1}Kx\|^2 &\leq B\|S^{-1}Kx\|^2 \\ &\leq B\|S^{-1}\|^2\|K\|^2\|x\|^2. \end{aligned}$$

Also

$$SS^{-1}Kx = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j S^{-1}Kx, \quad (x \in H),$$

and so

$$\begin{aligned} K^*S^{-1}SS^{-1}Kx &= \sum_{j\in\mathbb{J}} KS^{-1}\Lambda_j^*\Lambda_j S^{-1}Kx \\ &= \sum_{j\in\mathbb{J}} (\Lambda_j S^{-1}K)^*(\Lambda_j S^{-1}K)x, \end{aligned}$$

which implies that $S^{'} = K^*S^{-1}K$. □

Corollary 3.5. *If $K \in B(H)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-orthonormal basis, then $\{\Lambda_j S^{-1}K\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$.*

Proposition 3.6. *If $L \in B(H)$, $R(K) \subset R(L^*)$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, then $\{\Lambda_j L\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with the frame operator $S^{'} = L^*SL$, where S is the frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$.*

Proof. By the facts that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ and $R(K) \subset R(L^*)$, we may find positive real numbers A, B such that

$$A\|K^*x\|^2 \leq A\|Lx\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j Lx\|^2 \leq B\|Lx\|^2 \leq B\|L\|^2\|x\|^2, \quad (x \in H).$$

The proof of $S^{'} = L^*SL$ is similar to the proof of (3.1). □

If $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-K_1-frame, $\{\Omega_j\}_{j\in\mathbb{J}}$ is a g-K_2-frame and $R(K_1) \subset R(K_2)$, then applying Theorem 1.4 one may easily see that $\{\Omega_j\}_{j\in\mathbb{J}}$ is a g-K_1-frame.

4. Some perturbation results

In this section, a perturbation result for generalized atomic systems is investigated. A version of the following theorem for Hilbert C^*-modules can be seen in [25].

Theorem 4.1. *Assume that $K \in B(H)$. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, with g-K-frame bounds $A, B > 0$. If there exists a constant $M > 0$, such that*

$$\sum_{j\in\mathbb{J}} \|(\Lambda_j - \Theta_j)f\|^2 \leq M \min\left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2, \sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right), \tag{4.1}$$

for any $f \in H$, then $\{\Theta_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$. The converse is valid for any $f \in R(K)$, when $R(K)$ is closed.

Proof. First suppose that (4.1) is valid. For any $f \in H$, we have

$$\begin{aligned}\left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} &= \|\{\Lambda_j f\}_{j\in\mathbb{J}}\| \\ &\le \|\{(\Lambda_j - \Theta_j)f\}_{j\in\mathbb{J}}\| + \|\{\Theta_j f\}_{j\in\mathbb{J}}\| \\ &\le (\sqrt{M}+1)\,\|\{\Theta_j f\}_{j\in\mathbb{J}}\|\,,\end{aligned}$$

which implies that

$$\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2 \le (\sqrt{M}+1)^2\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2.$$

So for any $f \in H$

$$\begin{aligned}\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2 &\ge \frac{1}{(\sqrt{M}+1)^2}\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2 \\ &\ge \frac{A}{(\sqrt{M}+1)^2}\|K^* f\|^2.\end{aligned} \tag{4.2}$$

On the other hand

$$\begin{aligned}\left(\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right)^{\frac{1}{2}} &\le \left(\sum_{j\in\mathbb{J}}\|\Lambda_j f - \Theta_j f\|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &\le \sqrt{M}\left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &= (\sqrt{M}+1)\left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &\le \sqrt{B}(1+\sqrt{M})\|f\|.\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we conclude that $\{\Theta_j\}_{j\in\mathbb{J}}$ is a g-L-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with g-K-frame bounds

$$\frac{A}{E(\sqrt{M}+1)^2}, \qquad B(1+\sqrt{M})^2,$$

respectively. For the converse, suppose that $\{\Theta_j\}_{j\in\mathbb{J}}$ is a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with frame bounds C, D, respectively. Closeness of the range of $K \in B(H)$ implies that its pseudo-inverse K^+

exists, so $I_{R(K)} = KK^{+}|_{R(K)}f$, where $R(K)$ is the range of K. Hence $I^{*}_{R(K)} = (K^{+}|_{R(K)})^{*}K^{*}$. Thus for any $f \in R(K)$,

$$\begin{aligned}
\left(\sum_{j\in\mathbb{J}} \|(\Lambda_j - \Theta_j)f\|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \sqrt{D}\|f\| \\
&= \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \sqrt{D}\|(K^{+}|_{R(K)})^{*}K^{*}f\| \\
&\leq \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \sqrt{D}\|K^{+}|_{R(K)}\|\|K^{*}f\| \\
&\leq \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \frac{\sqrt{D}\|K^{+}|_{R(K)}\|}{\sqrt{A}} \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\
&= \left(1 + \frac{\sqrt{D}\|K^{+}|_{R(K)}\|}{\sqrt{A}}\right) \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand for any $f \in R(K)$ we have

$$\begin{aligned}
\left(\sum_{j\in\mathbb{J}} \|(\Lambda_j - \Theta_j)f\|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right)^{\frac{1}{2}} + \sqrt{B}\|f\| \\
&= \left(\sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right)^{\frac{1}{2}} + \sqrt{B}\|(K^{+}|_{R(K)})^{*}K^{*}f\| \\
&\leq \left(\sum_{j\in\mathbb{J}} \|\Theta_j f\|^2\right)^{\frac{1}{2}} + \sqrt{B}\|K^{+}|_{R(K)}\|\|K^{*}f\|.
\end{aligned}$$

Thus for any $f \in R(K)$,

$$\begin{aligned}\left(\sum_{j\in\mathbb{J}}\|(\Lambda_j-\Theta_j)f\|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right)^{\frac{1}{2}} + \sqrt{B}\|K^+|_{R(K)}\|\|K^*f\| \\ &\leq \left(\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right)^{\frac{1}{2}} + \frac{\sqrt{B}\|K^+|_{R(K)}\|}{\sqrt{C}}\left(\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right)^{\frac{1}{2}} \\ &= \left(1+\frac{\sqrt{B}\|K^+|_{R(K)}\|}{\sqrt{C}}\right)\left(\sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Letting

$$M = \min\left\{\left(1+\frac{\sqrt{D}\|K^+|_{R(K)}\|}{\sqrt{A}}\right)^2, \left(1+\frac{\sqrt{B}\|K^+|_{R(K)}\|}{\sqrt{C}}\right)^2\right\},$$

one can see that (4.1) holds for any $f \in R(K)$. □

For $K = I$ we have the following result which is proved in [25] for Hilbert C^*-modules.

Corollary 4.2. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, with g-frame bounds $A, B > 0$. Let $\{\Theta_j \in B(H, H_j) : j \in \mathbb{J}\}$. Then the following statements are equivalent:*

(i) *$\{\Theta_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$;*
(ii) *There exists a constant $M > 0$, such that for any $f \in H$, we have*

$$\sum_{j\in\mathbb{J}}\|(\Lambda_j-\Theta_j)f\|^2 \leq M\min\left(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2, \sum_{j\in\mathbb{J}}\|\Theta_j f\|^2\right).$$

5. G-K-duals

In this section the duals of g-K-frames are studied. First, we give a new definition of generalized dual frame that generalize some results of previous work (see [20]).

Definition 5.1. Let $K \in B(H)$ and $\Xi : H \to F$ be a bounded operator. A bounded operator $\Upsilon \in B(H, F)$ is called a generalized K-dual (or g-K-dual) of Ξ if $K = \Xi^*\Upsilon$.

In view of Definition 5.1, one can see that Ξ and Υ are g-atomic system for $K \in B(H)$ and $K^* \in B(H)$ with respect to F, respectively. It is obvious that Ξ and Υ are not interchangeable in general unless K is self adjoint. Let Ξ be a g-K-frame for H with respect to F. If

$K \in B(H)$ has closed range then it has a pseudo-inverse K^+ and by applying a similar process to [26], one may prove that

$$A\|K^+\|^{-2}\|x\| \leq \|\Xi^*\Xi x\| \leq B\|x\|, \quad (x \in H).$$

So $S : R(K) \to S(R(K))$ is a homeomorphism. Moreover

$$B^{-1}\|x\| \leq \|\Xi^*\Xi x\| \leq A^{-1}\|K^+\|^2\|x\|, \quad (x \in S(R(K))) \,. \tag{5.1}$$

Also $\Xi P(S^{-1})^*K$ is a g-K-dual of Ξ, where P is the orthogonal projection of H onto $S(R(K))$ and $S := \Xi^*\Xi$. Indeed,

$$\Xi^*\Xi P(S^{-1})^*K = SP(S^{-1})^*K = K.$$

This g-K-dual of Ξ, $\Xi P(S^{-1})^*K$, is denoted by Π.

Theorem 5.2. *Let Ξ be a g-K-frame for H with respect to F with g-K-frame bounds A and B, respectively. Then there exists a one-to-one correspondence between g-K-duals of Ξ and operator $\Psi \in B(H, F)$ such that $\Xi^*\Psi = 0$.*

Proof. Suppose that Φ is a g-K-dual of Ξ with the bounds A_1 and B_1, respectively. Define $\Psi : H \to F; x \mapsto \Psi x$ by

$$\Psi x = \Phi x - \Pi x.$$

Then Ψ is bounded by (5.1). Indeed,

$$\begin{aligned}\|\Psi x\|^2 &= \|\Phi x - \Pi x\|^2 \\ &\leq \|\Phi x\|^2 + \|\Xi P(S^{-1})^*Kx\|^2 + 2\|\Phi x\|\|\Xi P(S^{-1})^*Kx\| \\ &\leq \left(B_1 + A^{-1}\|K^+\|^2\|K\|^2 + 2\sqrt{B_1A^{-1}}\|K^+\|\|K\|\right)\|x\|^2.\end{aligned}$$

Moreover,

$$\Xi^*\Psi x = \Xi^*\Psi x = \Xi^*\Phi x - \Xi^*\Pi x = Kx - \Xi^*\Xi P(S^{-1})^*Kx = Kx - Kx = 0.$$

Conversely, Let $\psi \in B(H, F)$ such that $\Xi\Psi = 0$. Set

$$\Phi x = \Pi x + \Psi x, \quad (x \in H),$$

Then Φ is a bounded operator. Moreover,

$$\Xi^*\Phi x = \Xi^*\Pi x + \Xi^*\Psi x = Kx.$$

Therefore Φ is a g-K-dual of Ξ. □

The following corollary in g-K-frame is a generalization of [3, Theorem 3.4].

Corollary 5.3. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-K-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$. Then there exists a one-to-one correspondence between K-duals of $\{\Lambda_j\}_{j\in\mathbb{J}}$ and operator $\psi \in B(H, \bigoplus_{j\in\mathbb{J}} H_j)$ such that $T\psi = 0$, where T is the synthesis operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$.*

Theorem 5.4. *Let $K \in B(H)$. A bounded operator $\Xi : H \to F$ has a K-dual if and only if there exists a Bessel sequence $\{y_j\}_{j\in\mathbb{J}}$ such that for every $x \in H$*

$$Kx = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j,$$

where $x_j := \Xi^*\delta_j$, $j \in \mathbb{J}$, and $\{\delta_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for F.

Proof. It follows from Theorem 1.4, [17, Theorem 3] and the equality $\|\Xi x\|^2 = \sum_j |\langle x, x_j\rangle|^2$, for any $x \in H$. Indeed,

$$\begin{aligned}\|\Xi x\|^2 &= \langle \Xi x, \Xi x\rangle \\ &= \left\langle \sum_j \langle \Xi x, \delta_j\rangle \delta_j, \Xi x \right\rangle \\ &= \sum_j \langle \Xi x, \delta_j\rangle \langle \delta_j, \Xi x\rangle \\ &= \sum_j \langle x, \Xi^*\delta_j\rangle \langle \Xi^*\delta_j, x\rangle \\ &= \sum_j |\langle x, x_j\rangle|^2.\end{aligned}$$

□

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