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On Some Results in the Light of Generalized Relative Ritt Order of Entire Functions Represented by Vector Valued Dirichlet Series

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Abstract. In this paper, we study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of generalized relative Ritt order and generalized relative Ritt lower order.

1. Introduction, Definitions and Notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \tag{1.1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty,$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty.$$

If σ_a and σ_c denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

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The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1.1) is written as follows:

$$M_f(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma+it)|.$$

In the sequel the following two notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), \quad k = 1, 2, 3, \ldots;$$
$$\log^{[0]} x = x,$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right), \quad k = 1, 2, 3, \ldots;$$
$$\exp^{[0]} x = x.$$

Taking this into account, the Ritt order (See [1]) of $f(s)$, denoted by ρ_f, which is generally used in computational purpose, is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as follows:

$$\rho_f = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Similarly, one can define the Ritt lower order of $f(s)$, denoted by λ_f in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Further, an entire function $f(s)$ defined by (1.1) is said to be of regular Ritt growth if its Ritt order coincides with its Ritt lower order. Otherwise $f(s)$ is said to be of irregular Ritt-growth.

During the past decades, several authors {e.g., cf., [1, 2, 3, 5, 7], have made intensive investigations on the properties of entire Dirichlet series related to Ritt order. Further, Srivastava [6] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order.

Srivastava [4] introduced the relative Ritt order between two entire functions represented by vector valued Dirichlet series to avoid comparing growth just with $\exp \exp z$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf \{\mu > 0 : M_f(\sigma) < M_g(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\} \\ &= \limsup_{\sigma \to \infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}. \end{aligned}$$

Definition 1.1. The generalized relative Ritt order denoted by $\rho_g^{[k]}(f)$ of an entire function f with respect to another entire function g both represented by vector valued Dirichlet series is defined as follows:

$$\begin{aligned} \rho_g^{[k]}(f) &= \inf \left\{\mu > 0 : M_f(\sigma) < M_g\left(\exp^{[k]}(\sigma\mu)\right) \text{ for all } \sigma > \sigma_0(\mu)\right\} \\ &= \limsup_{\sigma \to \infty} \frac{\log^{[k]} M_g^{-1} M_f(\sigma)}{\sigma}, \end{aligned}$$

where $k = 0, 1, 2, \ldots$.

Similarly, one can define the generalized relative Ritt lower order of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g^{[k]}(f)$ in the following manner:

$$\lambda_g^{[k]}(f) = \liminf_{\sigma \to \infty} \frac{\log^{[k]} M_g^{-1} M_f(\sigma)}{\sigma},$$

where $k = 0, 1, 2, \ldots$.

In this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their generalized relative Ritt order and generalized relative Ritt lower order.

2. Theorems

In this section we present the main results of the paper.

Theorem 2.1. *If f, g,h and k be any four entire functions represented by vector valued Dirichlet series such that $0 < \lambda_h^{[m]}(f) \leq \rho_h^{[m]}(f) < \infty$ and $0 < \lambda_k^{[n]}(g) \leq \rho_k^{[n]}(g) < \infty$ where $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$,* *then*

$$\begin{aligned}
\frac{\lambda_h^{[m]}(f)}{\rho_k^{[n]}(g)} &\le \liminf_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \\
&\le \frac{\lambda_h^{[m]}(f)}{\lambda_k^{[n]}(g)} \\
&\le \limsup_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \\
&\le \frac{\rho_h^{[m]}(f)}{\lambda_k^{[n]}(g)}.
\end{aligned}$$

Proof. From the definitions of $\rho_k^{[n]}(g)$ and $\lambda_h^{[m]}(f)$, we have for arbitrary positive ε and for all sufficiently large values of σ that

$$\log^{[m]} M_h^{-1} M_f(\sigma) \ge \left(\lambda_h^{[m]}(f) - \varepsilon\right)\sigma, \tag{2.1}$$

and

$$\log^{[n]} M_k^{-1} M_g(\sigma) \le \left(\rho_k^{[n]}(g) + \varepsilon\right)\sigma. \tag{2.2}$$

Now from (2.1) and (2.2), it follows for all sufficiently large values of σ that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \ge \frac{\left(\lambda_h^{[m]}(f) - \varepsilon\right)\sigma}{\left(\rho_k^{[m]}(g) + \varepsilon\right)\sigma}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \ge \frac{\lambda_h^{[m]}(f)}{\rho_k^{[n]}(g)}. \tag{2.3}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M_h^{-1} M_f(\sigma) \le \left(\lambda_h^{[m]}(f) + \varepsilon\right)\sigma, \tag{2.4}$$

and for all sufficiently large values of σ,

$$\log^{[n]} M_k^{-1} M_g(\sigma) \ge \left(\lambda_k^{[n]}(g) - \varepsilon\right)\sigma. \tag{2.5}$$

Combining (2.4) and (2.5), we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \le \frac{\left(\lambda_h^{[m]}(f) + \varepsilon\right)\sigma}{\left(\lambda_k^{[n]}(g) - \varepsilon\right)\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\le\frac{\lambda_h^{[m]}(f)}{\lambda_k^{[n]}(g)}. \tag{2.6}$$

Also for a sequence of values of σ tending to infinity we have

$$\log^{[n]} M_k^{-1}M_g(\sigma)\le\left(\lambda_k^{[n]}(g)+\varepsilon\right)\sigma. \tag{2.7}$$

Now from (2.1) and (2.7), we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\ge\frac{\left(\lambda_h^{[m]}(f)-\varepsilon\right)\sigma}{\left(\lambda_k^{[n]}(g)+\varepsilon\right)\sigma}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\ge\frac{\lambda_h^{[m]}(f)}{\lambda_k^{[n]}(g)}. \tag{2.8}$$

Also for all sufficiently large values of σ,

$$\log^{[m]} M_h^{-1}M_f(\sigma)\le\left(\rho_h^{[m]}(f)+\varepsilon\right)\sigma. \tag{2.9}$$

Now it follows from (2.5) and (2.9), for all sufficiently large values of σ we have

$$\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\le\frac{\left(\rho_h^{[m]}(f)+\varepsilon\right)\sigma}{\left(\lambda_k^{[n]}(g)-\varepsilon\right)\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\le\frac{\rho_h^{[m]}(f)}{\lambda_k^{[n]}(g)}. \tag{2.10}$$

Thus the theorem follows from (2.3), (2.6), (2.8) and (2.10). □

Theorem 2.2. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0<\rho_h^{[m]}(f)<\infty$ and $0<\rho_k^{[n]}(g)<\infty$ where $m=0,1,2,\dots$ and $n=0,1,2,\dots$, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}\le\frac{\rho_h^{[m]}(f)}{\rho_k^{[n]}(g)}\le\limsup_{\sigma\to\infty}\frac{\log^{[m]} M_h^{-1}M_f(\sigma)}{\log^{[n]} M_k^{-1}M_g(\sigma)}.$$

Proof. From the definition of $\rho_k^{[n]}(g)$, we get for a sequence of values of σ tending to infinity that

$$\log^{[n]} M_k^{-1}M_g(\sigma)\ge\left(\rho_k^{[n]}(g)-\varepsilon\right)\sigma. \tag{2.11}$$

Now from (2.9) and (2.11) , it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\left(\rho_h^{[m]}(f)+\varepsilon\right)\sigma}{\left(\rho_k^{[n]}(g)-\varepsilon\right)\sigma}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{[m]}(f)}{\rho_k^{[n]}(g)}. \tag{2.12}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M_h^{-1} M_f(\sigma) \geq \left(\rho_h^{[m]}(f)-\varepsilon\right)\sigma. \tag{2.13}$$

So combining (2.2) and (2.13) , we get for a sequence of values of σ tending to infinity:

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geq \frac{\left(\rho_h^{[m]}(f)-\varepsilon\right)\sigma}{\left(\rho_k^{[n]}(g)+\varepsilon\right)\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geq \frac{\rho_h^{[m]}(f)}{\rho_k^{[n]}(g)}. \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14) . □

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0<\lambda_h^{[m]}(f)\leq\rho_h^{[m]}(f)<\infty$ and $0<\lambda_k^{[n]}(g)\leq\rho_k^{[n]}(g)<\infty$ where $m=0,1,2,\ldots$ and $n=0,1,2,\ldots$, then*

$$\begin{aligned}\liminf_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} &\leq \min\left\{\frac{\lambda_h^{[m]}(f)}{\lambda_k^{[n]}(g)}, \frac{\rho_h^{[m]}(f)}{\rho_k^{[n]}(g)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{[m]}(f)}{\lambda_k^{[n]}(g)}, \frac{\rho_h^{[m]}(f)}{\rho_k^{[n]}(g)}\right\}\\ &\leq \limsup_{\sigma\to\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)}.\end{aligned}$$

The proof is omitted.

Theorem 2.4. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series such that $\rho_k^{[m]}(g) < \infty$. If $\lambda_h^{[n]}(f) = \infty$ where $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$, then*

$$\lim_{r \to \sigma} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of σ tending to infinity,

$$\log^{[m]} M_h^{-1} M_f(\sigma) \leq \beta \log^{[n]} M_k^{-1} M_g(\sigma). \tag{2.15}$$

Again from the definition of $\rho_k^{[m]}(g)$, it follows that for all sufficiently large values of σ that

$$\log^{[n]} M_k^{-1} M_g(\sigma) \leq \left(\rho_k^{[n]}(g) + \varepsilon\right)\sigma. \tag{2.16}$$

Thus from (2.15) and (2.16), we have for a sequence of values of σ tending to infinity that

$$\log^{[m]} M_h^{-1} M_f(\sigma) \leq \beta\left(\rho_k^{[n]}(g) + \varepsilon\right)\sigma,$$

i.e.,

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\sigma} \leq \frac{\beta\left(\rho_k^{[n]}(g) + \varepsilon\right)\sigma}{\sigma},$$

i.e.,

$$\liminf_{r \to \sigma} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\sigma} = \lambda_h^{[m]}(f) < \infty.$$

This is a contradiction.

This proves the theorem. □

Remark 2.5. Theorem 2.4 is also valid with "limit superior" instead of "limit" if $\lambda_h^{[m]}(f) = \infty$ is replaced by $\rho_h^{[m]}(f) = \infty$ and the other conditions remaining the same.

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