Sahand Communications in Mathematical Analysis (SCMA) Vol. 9 No. 1 (2018), 33-43
http://scma.maragheh.ac.ir

Density near zero

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ABSTRACT. Let S be a dense subsemigroup of $(0,+\infty)$. In this paper, we state definition of thick near zero, and also we will introduce a definition that is equivalent to the definition of piecewise syndetic near zero which presented by Hindman and Leader in [6]. We define density near zero for subsets of S by a collection of nonempty finite subsets of S and we investigate the conditions under these concepts.

1. Introduction

Let $(S,+)$ be a discrete semigroup. The collection of all ultrafilters on S is denoted by βS. For $A \subseteq S$, define $\overline{A} = \{p \in \beta S : A \in p\}$, then $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets(also for the closed sets) of βS that is called the Stone-Čech compactification of S. There is a unique extension of the operation to βS making $(\beta S,+)$ a right topological semigroup (i.e., for each $p \in \beta S$, the right translation ρ_p is continuous where $\rho_p(q) = q+p$) and also for each $x \in S$, the left translation λ_x is continuous where $\lambda_x(q) = x+q$. The principal ultrafilters being identified with the points of S and S is a dense subset of βS. Given $p,q \in \beta S$ and $A \subseteq S$, $A \in p+q$ if and only if $\{x \in S : -x+A \in q\} \in p$, where $-x+A = \{y \in S : x+y \in A\}$.

Any compact Hausdorff right topological semigroup (S, +) has a smallest two sided ideal, denoted by K(S), which is the union of all minimal left ideals, and also the union of all minimal right ideals, as well. For more detail see [7].

2010 *Mathematics Subject Classification.* Primary: 54D80, 22A15; Secondary: 22A20.

Key words and phrases. The Stone-Cech compactification, Density, Piecewise syndetic set near zero.

Received: 24 September 2016, Accepted: 6 December 2016.

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Definition 1.1. Let S be a dense subset of $(0,\infty)$. Then

$$0^*(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\ (0,\epsilon) \cap S \in p\}.$$

The set $0^*(S)$ of all nonprincipal ultrafilters on $S = ((0,\infty),+)$ that converge to zero is a semigroup under the restriction of the usual operation $'+'$ on βS. Let X be a topological space. Recall that an ultrafilter p on X is said to converge to a point x of X if and only if p contains the filter of neighborhoods of x. In [6] Hindman and Imer Leader characterized the smallest ideal of $(0^*,+)$, its closure, and those sets central set in $(0^*,+)$, that is, those sets which are members of minimal idempotents in $(0^*,+)$. It is proved in [6] that $0^*(S)$ is a compact right topological semigroup of $(\beta S_d,+)$ which is disjoint from $K(\beta S_d)$. In [1] the authors applied the algebraic structure of $0^*(S)$ on their investigation of image partition regularity near 0 of finite and infinite matrices. In [2] used algebraic structure of $0^*(\mathbb{R})$ to investigate image partition regularity of matrices with real entries from $\mathbb{R}$.

Definition 1.2. Let $(S,+)$ be a semigroup, let $A \subseteq S$, and $p \in \beta S$. Then $A^*(p) = \{s \in A : -s + A \in p\}$.

Lemma 1.3. *Let $(S,+)$ be a semigroup, let $p+p = p \in \beta S$, and let $A \in p$. For each $s \in A^*(p)$, $-s + A^*(p) \in p$.*

Proof. See Lemma 4.14, in [7]. □

We have been considering semigroups which are dense in $((0,\infty),+)$ with natural topology. When passing to the Stone-Čech compactification of such a semigroup S, we deal with S_d, which is the set S with the discrete topology. In this paper, the collection of all nonempty finite subsets of S is denoted by $P_f(S)$ and $P(S)$ is the set of all subsets of S.

2. Additive properties near zero

Recall that $A \subseteq S$ is thick if and only if for every $F \in P_f(S)$ there exists $x \in S$ such that $F + x \subseteq A$, A is syndetic if and only if there exists $H \in P_f(S)$ such that $S = \bigcup_{t\in H} -t + A$ and A is piecewise syndetic if and only if there exists $H \in P_f(S)$ such that $\bigcup_{t\in H} -t + A$ is thick.

Now we define thick near zero, and piecewise syndetic near zero.

Definition 2.1. Let S be a dense subsemigroup of $((0,\infty),+)$. A subset B of S is syndetic near zero if and only if for every $\epsilon > 0$, there exist $F \in P_f(0,\epsilon)$ and some $\delta > 0$ such that $S \cap (0,\delta) \subseteq \bigcup_{t\in F}(-t + B)$.

Definition 2.2. Let S be a dense subsemigroup of $((0,\infty),+)$. A subset A of S is piecewise syndetic near zero if and only if there exist sequences $\{F_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ such that

(1) for every $n \in \mathbb{N}$, $F_n \in P_f\left(\left(0, \frac{1}{n}\right) \cap S\right)$ and $\delta_n \in \left(0, \frac{1}{n}\right)$,

(2) for each $G \in P_f(S)$ and each $\mu > 0$, there exists $x \in (0, \mu) \cap S$ such that for each $n \in \mathbb{N}$, $(G \cap (0, \delta_n)) + x \subseteq \bigcup_{t \in F_n}(-t + A)$.

Theorem 2.3. *Let $A \subseteq S$, then $K(0^*) \cap \overline{A} \neq \emptyset$ if and only if A is piecewise syndetic near* 0*.*

Proof. See Theorem 13.35, in [7]. □

Definition 2.4. Let S be a dense subsemigroup of $((0, \infty), +)$ and let $A \subseteq S$.

(a) A is thick near zero if and only if

$$(\exists \epsilon > 0)(\forall F \in (P_f(0, \epsilon) \cap S))(\forall \delta > 0)(\exists y \in (0, \delta) \cap S)(F + y \subseteq A).$$

(b) A has the PS property if and only if for every $\delta > 0$ there exists $F \in P_f((0, \delta) \cap S)$ such that $\bigcup_{t \in F} -t + A$ is thick near zero.

Remark 2.5. (a) It is obvious that A is thick near zero if and only if for some $\epsilon > 0$

$$\{-t + A : t \in (0, \epsilon) \cap S\}$$

has the finite intersection property in $(0, \epsilon) \cap S$, i.e. for each $t_1, \ldots, t_n \in (0, \epsilon) \cap S$, there exists $x \in (0, \epsilon) \cap S$ such that $x \in \bigcap_{i=1}^{n} -t_i + A$.

(b) Let A be not thick near zero, so for each $\epsilon > 0$, for some $G \in P_f((0, \epsilon) \cap S)$, there exists $\eta > 0$, such that for each $y \in ((0, \eta) \cap S)$, we have $G + y \nsubseteq A$.

Hence $y \notin \bigcap_{t \in G}(-t + A)$ for each $y \in (0, \eta) \cap S$. This implies that

$$(0, \eta) \cap S \subseteq \bigcup_{t \in G}(-t + (S \backslash A)).$$

Thus $S \backslash A$ is syndetic near zero and also has the PS property.

In this paper, the minimal ideal in $0^*(S)$ is denoted by K.

Theorem 2.6. *Let S be a dense subsemigroup of $(0, +\infty)$ and let $p \in 0^*(S)$. The following statements are equivalent:*

(a) *$p \in K$,*

(b) *For all $A \in p$, $\{x \in S : -x + A \in p\}$ is syndetic near* 0*,*

(c) *For all $r \in 0^*(S)$, $p \in 0^*(S) + r + p$.*

Proof. See [6] or see Theorem 3.4 in [9]. □

Theorem 2.7. *Let $A \subseteq S$. Then $K \cap \overline{A} \neq \emptyset$ if and only if A has the PS property.*

Proof. Necessity. Pick $p \in K \cap \overline{A}$ and let $B = \{x \in S : -x + A \in p\}$. By Theorem 2.6, B is syndetic near zero. So for every $\varepsilon > 0$, there exist $F \in P_f((0,\varepsilon) \cap S)$ and $\delta > 0$ such that $(0,\delta) \cap S \subseteq \bigcup_{t \in F} -t + B$. So for each $x \in (0,\delta) \cap S$, there exists $t \in F$ such that $x \in -t + B$, and so $-(x+t)+A \in p$. Thus $-x+(\bigcup_{t\in F} -t+A) \in p$ and since $(0,\delta)\cap S \in p$ for each $\delta > 0$, hence $\{-x + (\bigcup_{t\in F} -t + A) : x \in (0,\delta) \cap S\}$ has the finite intersection property in $(0,\delta) \cap S$. By Remark 2.5 (a), A has the PS property.

Sufficiency. Let A has the PS property. So for each $n \in \mathbb{N}$ there exists $F_n \in P_f((0, \frac{1}{n}) \cap S)$ such that for some $\epsilon_n > 0$ and each $G_n \in P_f((0,\epsilon_n) \cap S)$, for every δ_n there exists $y_n \in P_f((0,\delta_n) \cap S)$ such that $G_n + y_n \subseteq \bigcup_{t\in F_n}(-t + A)$. Pick $\delta_n = \min\left\{\frac{1}{n}, \epsilon_n\right\}$ for $G \in P_f(S)$ and $\mu > 0$, let

$$C(G,\mu) = \left\{x \in (0,\mu) : \forall n \in \mathbb{N}, (G \cap (0,\delta_n)) + x \subseteq \bigcup_{t\in F_n}(-t + A)\right\}.$$

It is obvious that $C(G,\mu) \neq \emptyset$ for each $\mu > 0$ and every $G \in P_f(S)$. Also,

$$C(G_1 \cup G_2, \min\{\mu_1,\mu_2\}) \subseteq C(G_1,\mu_1) \cap C(G_2,\mu_2),$$

for each $G_1, G_2 \in P_f(S)$ and for every $\mu_1, \mu_2 > 0$. Therefore

$$\{C(G,\mu) : G \in P_f(S) \text{ and } \mu > 0\}$$

has the finite intersection property, so pick $p \in \beta S_d$ with

$$\{C(G,\mu) : G \in P_f(S) \text{ and } \mu > 0\} \subseteq p.$$

Since $C(G,\mu) \subseteq (0,\mu) \cap S$, so $p \in 0^*(S)$.

Now we claim that for each $n \in \mathbb{N}$, $0^*(S) + p \subseteq cl_{\beta S_d} \bigcup_{t\in F_n}(-t + A)$. Pick $q \in 0^*(S)$. To show that $\bigcup_{t\in F_n}(-t + A) \in q + p$, we show that

$$(0,\delta_n) \cap S \subseteq \left\{y > 0 : -y + \bigcup_{t\in F_n}(-t + A) \in p\right\}.$$

So let $y \in (0,\delta_n) \cap S$. Then $C(\{y\},\delta_n) \in p$ and $C(\{y\},\delta_n) \subseteq -y + \bigcup_{t\in F_n}(-t + A)$. Now pick $r \in (0^*(S) + p) \cap K$(since $0^*(S) + p$ is a left ideal of $0^*(S)$). Given $n \in \mathbb{N}$, $\bigcup_{t\in F_n}(-t + A) \in r$ so pick $t_n \in F_n$ such that $-t_n + A \in r$. Since for each $n \in \mathbb{N}$, $t_n \in F_n \subseteq (0, \frac{1}{n}) \cap S$ then $lim_{n\to\infty} t_n = 0$. Now pick $q \in 0^*(S) \cap cl_{\beta S_d}\{t_n : n \in \mathbb{N}\}$. Then $q + r \in K$ and $\{t_n : n \in \mathbb{N}\} \subseteq \{t \in S : -t + A \in r\}$ so $A \in q + r$. □

By Theorem 2.7, $A \subseteq S$ is piecewise syndetic near zero if and only if A has the PS property.

3. Density determined by nets of finite sets

Let S be a dense subsemigroup of $((0,\infty),+)$. Since $((0,\infty),+)$ has an uncountable subsemigroup, we define densities by nets [8]. Given any net in $P_f(S)$, there are corresponding natural notions of density for subsets of $(0,+\infty)$.

Definition 3.1. Let $\mathcal{F}=\{F_n\}_{n\in D}$ be a net in $P_f(S)$, and let $A\subseteq S$.

(a) $\underline{d}_{\mathcal{F}}(A)=\sup\{\alpha:(\exists m\in D)(\forall n\geq m)(|A\cap F_n|\geq \alpha\cdot|F_n|\}$.
(b) $\overline{d}_{\mathcal{F}}(A)=\sup\{\alpha:(\forall m\in D)(\exists n\geq m)(|A\cap F_n|\geq \alpha\cdot|F_n|)\}$.
(c) $d^*_{\mathcal{F}}(A)=\sup\{\alpha:(\forall m\in D)(\exists n\geq m)(\exists x\in S\cup\{0\}\})(|A\cap(F_n+x)|\geq\alpha\cdot|F_n|)\}$.

For discrete case, you can see Definition 3.1 in [8]. In this paper, our main purpose is about density of subsets that infinitely closed to zero.

Example 3.2. (a) Let $\{F_n\}_{n\in D}$ be a net in $P_f(S)$ such that

$$\inf\{\min F_n:n\in D\}=\delta>0.$$

Now let $A=(\delta,+\infty)$, then $\overline{d}_{\mathcal{F}}(A)=1$.

(b) Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence in $P_f(S)$ and let $\lim_{n\to\infty}\max F_n=0$. Then $\overline{d}_{\mathcal{F}}([\delta,+\infty)\cap S)=0$ and $\overline{d}_{\mathcal{F}}((0,\delta)\cap S)=1$.

Definition 3.3. Let $\mathcal{F}=\{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D}\max F_n=0$.

(a) $D_{\mathcal{F}}=\{p\in\beta S_d:(\forall A\in p)(\overline{d}_{\mathcal{F}}(A)>0)\}$.
(b) $D^*_{\mathcal{F}}=\{p\in\beta S_d:(\forall A\in p)(d^*_{\mathcal{F}}(A)>0)\}$.
(c) $N_{\mathcal{F}}=\{p\in 0^*(S):(\forall A\in p)(\overline{d}_{\mathcal{F}}(A)>0)\}$.
(d) $N^*_{\mathcal{F}}=\{p\in 0^*(S):(\forall A\in p)(d^*_{\mathcal{F}}(A)>0)\}$.

If $\overline{d}_{\mathcal{F}}(A)>0$, then $0\in\overline{A}$. Now pick $p\in D_{\mathcal{F}}$, so $\overline{d}_{\mathcal{F}}(A)>0$ for each $A\in p$ and hence $p\in 0^*(S)$. This implies that $N_{\mathcal{F}}=D_{\mathcal{F}}$. It is obvious that $N^*_{\mathcal{F}}=0^*(S)\cap D^*_{\mathcal{F}}$.

Lemma 3.4. *Let $\mathcal{F}=\{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D}\max F_n=0$. If A and B are subsets of S, then $\overline{d}_{\mathcal{F}}(A\cup B)\leq\overline{d}_{\mathcal{F}}(A)+\overline{d}_{\mathcal{F}}(B)$ and*

$$d^*_{\mathcal{F}}(A\cup B)\leq d^*_{\mathcal{F}}(A)+d^*_{\mathcal{F}}(B).$$

Consequently,

(a) *if $A\subseteq S$ and $\overline{d}_{\mathcal{F}}(A)>0$, then $\overline{A}\cap D_{\mathcal{F}}\neq\emptyset$ and if $A\subseteq S$ and $d^*_{\mathcal{F}}(A)>0$, then $\overline{A}\cap D^*_{\mathcal{F}}\neq\emptyset$.*
(b) *If $A\subseteq S$ and $d^*_{\mathcal{F}}(A)>0$, then $\overline{A}\cap N^*_{\mathcal{F}}\neq\emptyset$.*
(c) *For each $\delta>0$, $\overline{d}_{\mathcal{F}}(A)=\overline{d}_{\mathcal{F}}(A\cap(0,\delta))$.*

Proof. By Lemma 2.3 in [8], (a) and (b) are obvious. Since $\overline{d}_{\mathcal{F}}(A\cap(\delta,\infty))=0$, (c) is hold. □

Lemma 3.5. *Let* $\mathcal{F} = \{F_n\}_{n\in D}$ *be a net in* $P_f(S)$ *and let* $\lim_{n\in D} \max F_n = 0$. *Then* $N^*_{\mathcal{F}}$ *is a right ideal of* $(0^*(S), +)$.

Proof. By Theorem 2.4 in [8], $D^*_{\mathcal{F}}$ is a right ideal in βS_d. This implies that $N^*_{\mathcal{F}}$ is a right ideal of $(0^*(S), +)$. □

Remark 3.6. Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D} \max F_n = 0$. If A is a thick near zero subset of S, then $d^*_{\mathcal{F}}(A) = 1$.

The following three requirements will guarantee certain properties of densities determined by nets of finite sets. This definition is stated in [8] for general case, and we use it for $(0, +\infty)$.

Definition 3.7. Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$. Followings are three properties that $\mathcal{F}$ might satisfy.

- $(*)$ $(\forall \epsilon > 0)(\forall t \in S)(\exists c \in D)(\exists m \in D)(\forall n \geq m)(\exists k \geq n)(\exists z \in S)(|(t + F_n) \setminus (F_k + z)| < \epsilon|F_n|)$ and $(|F_k| \leq c|F_n|)$.
- $(*')$ $(\forall \epsilon \in (0, 1))(\forall t \in S)(\exists c \in D)(\exists m \in D)(\forall n \geq m)(\exists k \geq n)(|(t + F_n) \setminus F_k| < \epsilon|F_n|)$ and $(|F_k| \leq c|F_n|)$.
- $(**)$ $(\forall H \in \mathcal{F}s(S))(\exists c \in D)(\exists m \in D)(\forall n \geq m)(|F_n| \leq c|\bigcap_{a\in H}(-a+F_n)|)$.

Theorem 3.8. *Let* $\mathcal{F} = \{F_n\}_{n\in D}$ *be a net in* $P_f(S)$ *and let* $\lim_{n\in D} \max F_n = 0$. *If* $\mathcal{F}$ *satisfies* $(*)$, $B \subseteq S$, $t \in S$, *and* $d^*_{\mathcal{F}}(-t+B) > 0$, *then* $d^*_{\mathcal{F}}(B) > 0$. *In particular,* $N^*_{\mathcal{F}}$ *is a left ideal of* $(0^*(S), +)$.

Proof. See Theorem 2.7 in [8]. □

Theorem 3.9. *Let* $\mathcal{F} = \{F_n\}_{n\in D}$ *be a net in* $P_f(S)$ *and let* $\inf\{maxF_n : n \in D\} = 0$. *If* $\mathcal{F}$ *satisfies* $(*')$, $B \subseteq S$, $t \in S$ *and* $\overline{d}_{\mathcal{F}}(-t + B) > 0$ *then* $\overline{d}_{\mathcal{F}}(B) > 0$. *In particular,* $N_{\mathcal{F}}$ *is a left ideal of* $(0^*(S), +)$.

Proof. Pick $\alpha > 0$ such that $\overline{d}_{\mathcal{F}}(-t+B) > \alpha$ and let $\epsilon = \overline{d}_{\mathcal{F}}(-t+B)-\alpha$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by $(*')$ for $\epsilon/2$ and t. We claim that $\overline{d}_{\mathcal{F}}(B) \geq \alpha/c$. To this end, let $r \in D$. Pick $n \in D$ such that $n \geq r$, $n \geq m$, and $|(-t + B) \cap F_n| \geq (\alpha + \epsilon/2) \cdot |F_n|$. Pick $k \geq n$ such that $|t + F_n \setminus F_k| < (\epsilon/2)|F_n|$ and $|F_k| \leq c|F_n|$. Since $B \cap (t + F_n) \subseteq (B \cap F_k) \cup (t + F_n \setminus F_k)$,

$$\begin{aligned}|B \cap (t + F_n)| &\leq |B \cap F_k| + |t + F_n \setminus F_k| \\ &= |B \cap F_k| + |t + F_n \setminus F_k|\end{aligned}$$

so

$$\begin{aligned}|B \cap F_k| &\geq |B \cap t + F_n| - |t + F_n \setminus F_k| \\ &\geq |-t + B \cap F_n| - (\epsilon/2)|F_n| \\ &\geq \alpha|F_n| \\ &\geq (\alpha/c)|F_k|.\end{aligned}$$

$N_{\mathcal{F}}$ is a left ideal, see Theorem 2.8 in [8]. □

Remark 3.10. Let S be a dense subsemigroup of $(0, +\infty)$ and let $A = S \cap (1, 2)$. Let $\{F_n\}_{n\in D} \subseteq P_f((0, 1) \cap S)$ such that $\inf\{\max F_n : n \in D\} = 0$. Since $\overline{d}_{\mathcal{F}}(A) = 0$ and $\overline{d}_{\mathcal{F}}(-1 + A) = 1$, this implies that Theorem 3.9 is not true for net $\{F_n\}_{n\in D}$. Therefore, if $\{F_n\}_{n\in D}$ is a net in $P_f(S)$ with $\sup\{\max F_n : n \in D\} < \infty$, then $\{F_n\}_{n\in D}$ does not has $(*')$ property.

Theorem 3.11. *Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D} \max F_n = 0$. If $\mathcal{F}$ satisfies $(*)$ and $B \subset S$ is piecewise syndetic near zero then $d^*_{\mathcal{F}}(B) > 0$.*

Proof. Since B is piecewise syndetic near zero, so for each $\varepsilon > 0$, there exists $G \in P_f((0, \varepsilon) \cap S)$ such that $\bigcup_{t\in G} -t + B$ is thick near zero. So there exists $\gamma > 0$ such that for each $F \in P_f((0, \gamma) \cap S)$ and every $\delta > 0$, for some $x \in (0, \delta) \cap S$ with $F + x \subseteq \bigcup_{t\in G} -t + B$. Given $m \in D$ and $\gamma > 0$ with $\max F_m < \gamma$, pick $x \in (0, \delta) \cap S$. So $F_m + x \subseteq \bigcup_{t\in G}(-t + B)$, and

$$\left|\left(\bigcup_{t\in G}(-t + B) \cap (F_m + x)\right)\right| = |F_m|.$$

We conclude that $d^*_{\mathcal{F}}(\bigcup_{t\in G} -t + B) = 1$. Thus by Lemma 3.4, there is some $t \in G$ such that $d^*_{\mathcal{F}}(-t + B) > 0$, and so by Theorem 3.8, $d^*_{\mathcal{F}}(B) > 0$. □

Theorem 3.12. *Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net such that* $\inf\{\max F_n : n \in D\} = 0$. *If $\mathcal{F}$ satisfies $(**)$ and B is a syndetic near zero subset of S, then $\overline{d}_{\mathcal{F}}(B) > 0$.*

Proof. B is syndetic near zero, so for each $\varepsilon > 0$ there exist $H \in P_f((0, \varepsilon) \cap S)$ and $\delta > 0$ such that $(0, \delta) \cap S \subseteq \bigcup_{t\in H}(-t + B)$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by $(**)$ for H such that for some $n > m$ implies that $maxF_n < \delta$. Let $k = |H|$, we show that $|B \cap F_n| \geq (\frac{1}{ck})|F_n|$ and thus $\overline{d}_{\mathcal{F}}(B) \geq \frac{1}{ck}$. Let $G_1 = \bigcap_{a\in H}(-a + F_n)$, so that $|F_n| \leq c|G_1|$ and $G_1 \subseteq (0, \delta) \cap S$. For $s \in G_1$, pick $t \in H$ such that $t + s \in B$. Since $s \in G_1$, $t + s \in F_n$. So $G_1 \subseteq \bigcup_{t\in H} -t + (B \cap F_n)$. This implies $|G_1| \leq k|B \cap F_n|$ as required. □

Theorem 3.13. *Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D} \max F_n = 0$. If $\mathcal{F}$ satisfies $(**)$ and $d^*_{\mathcal{F}}(B) = 1$, then B is thick near zero.*

Proof. Suppose that B is not thick near zero, so that $G = S \setminus B$ is syndetic near zero, see Remark 2.5 (b). So for every $\varepsilon > 0$ there exist $H \in P_f(0,\varepsilon)$ and some $\delta > 0$ such that $(0,\delta) \cap S \subseteq \bigcup_{t\in H}(-t+G)$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by $(**)$ for H. Let $k = |H|$. Since $d^*_{\mathcal{F}}(B) > 1 - \frac{1}{ck}$, pick $x \in S$ and $n \geq m$ such that $|B \cap (F_n + x)| > (1 - \frac{1}{ck})|F_n|$ and $F_n \subseteq (0,\delta) \cap S$. Let $G_1 = \bigcap_{a\in H}(-a + F_n)$ and for $t \in H$, let

$$E_t = \{s \in G_1 : t + s + x \in G\}.$$

$G_1 = \bigcup_{t\in H} E_t$ and $|G_1| \geq \frac{1}{c}|F_n|$ so pick $t \in H$ such that $|E_t| \geq \frac{1}{ck}|F_n|$. Now $t + E_t \subseteq (G \setminus B) \cap (F_n + x)$, therefore

$$\begin{aligned} |B \cap (F_n + x)| &\leq |F_n + x| - |t + E_t + x| \\ &\leq |F_n| - |E_t| \\ &\leq \left(1 - \frac{1}{ck}\right)|F_n| \end{aligned}$$

which is a contradiction. □

Theorem 3.14. *Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ with $lim_{n\in D} max F_n = 0$. If $\mathcal{F}$ satisfies $(**)$ and B is piecewise syndetic near zero subset of S, then $d^*_{\mathcal{F}}(B) > 0$.*

Proof. Since B is piecewise syndetic near zero, for all $\delta > 0$ there exists $F \in P_f((0,\delta) \cap S)$ such that $\bigcup_{t\in F} -t + B$ is thick near zero. So there exists $\varepsilon > 0$, such that for all $K \in P_f((0,\varepsilon)\cap S)$ there exists $x \in (0,\varepsilon)\cap S$ with $K + x \subseteq \bigcup_{t\in F} -t + B$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed for F by $(**)$. Let $k = |F|$. Let $n \geq m$ such that $\max F_n < \varepsilon$ and let $G_1 = \bigcap_{a\in F} -a + F_n$, so $G_1 \in P_f((0,\varepsilon) \cap S)$. Pick z such that $G_1 + z \subseteq \bigcup_{t\in F} -t+B$. Given $y \in G_1$, pick $t \in F$ such that $t+y+z \in B$. Therefore $G_1+z \subseteq \bigcup_{t\in F} -t+(B\cap(F_n+z))$ and so $|G_1| = |G_1+z| \leq k|B\cap(F_n+z)|$, Thus

$$|B \cap (F_n + z)| \geq \frac{1}{k}|G_1| \geq \frac{1}{ck}|F_n|.$$

So $d^*_{\mathcal{F}}(B) > 0$. □

Let $B \subseteq S$ and let there exists $\delta > 0$ such that $(0,\delta) \cap S \subseteq B$. Then $\overline{d}_{\mathcal{F}}(B) = 1$. □

Theorem 3.15. *Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$, and let B be a syndetic near zero subset of S. If*

(a) *$\mathcal{F}$ satisfies $(**)$ and $lim_{n\in D} \max F_n = 0$, or*
(b) *$\mathcal{F}$ satisfies $(*')$ and $inf\{\max F_n : n \in D\} = 0$, then $\overline{d}_{\mathcal{F}}(B) > 0$.*

Proof. If $\mathcal{F}$ satisfies $(**)$, the conclusion follows from Theorem 3.11 so assume $\mathcal{F}$ satisfies $(*')$. So for each $\varepsilon > 0$, there exist $H \in P_f((0,\varepsilon) \cap S)$ and $\delta > 0$ such that $(0,\delta) \cap S \subseteq \bigcup_{t \in H} -t+B$. Then $\overline{d}_{\mathcal{F}}(\bigcup_{t \in H} -t+B) = 1$ so by Lemma 3.4 there is $t \in H$ such that $\overline{d}_{\mathcal{F}}(-t+B) > 0$ and thus by Theorem 3.9 $\overline{d}_{\mathcal{F}}(B) > 0$. □

Definition 3.16. Let $S \subseteq (0,+\infty)$, and $\mathcal{F} = \{F_n\}_{n \in D}$ be a net in $P_f(S)$. Then $\mathcal{F}$ is a Følner net if for each $s \in S$, the net

$$\left\{ \frac{|(s+F_n)\Delta F_n|}{|F_n|} \right\}_{n \in D}$$

converges to 0.

Lemma 3.17. *Let S be a left cancellative semigroup and let $\mathcal{F} = \{F_n\}_{n \in D}$ be a Følner net in $P_f(S)$. Then for each $s \in S$, the set*

$$\left\{ \frac{|(-s+F_n)\Delta F_n|}{|F_n|} \right\}_{n \in D}$$

converges to 0.

Proof. See Lemma 4.3 in [8]. □

Theorem 3.18. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \{F_n\}_{n \in D}$ be a Følner net in $P_f(S)$. Then for all $A \subseteq S$ and all $s \in S$,*

(a) $\underline{d}_{\mathcal{F}}(A) = \underline{d}_{\mathcal{F}}(s+A) = \underline{d}_{\mathcal{F}}(-s+A)$.
(b) $\overline{d}_{\mathcal{F}}(A) = \overline{d}_{\mathcal{F}}(s+A) = \overline{d}_{\mathcal{F}}(-s+A)$.
(c) $d^*_{\mathcal{F}}(A) = d^*_{\mathcal{F}}(s+A) = d^*_{\mathcal{F}}(-s+A)$.

Proof. See Theorem 4.5 in [8]. □

Remark 3.19. Let S be a dense subsemigroup of $((0,\infty),+)$ and let $\mathcal{F}$ be a net in $P_f(S)$ with $\lim_{n \in D} \max F_n = 0$. For each $s \in S$, $\overline{d}_{\mathcal{F}}(S) = 1$ and $\overline{d}_{\mathcal{F}}(s+S) = 0$, so Theorem 3.18 is not true for $\mathcal{F}$. Therefore $\mathcal{F}$ is not a Følner net.

Theorem 3.20. *Let $\mathcal{F} = \{F_n\}_{n \in D}$ be a net in $P_f(S)$ such that $\lim_{n \in D} \max F_n = 0$, and let $A \subseteq S$. There is a countable additive regular measure μ on the set $\mathcal{B}$ of Borel subsets of βS_d such that*

(1) $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A)$,
(2) *for all* $B \subseteq S$, $\mu(\overline{B}) \leq \overline{d}_{\mathcal{F}}(B)$, *and*
(3) $\mu(0^*(S)) = 1$.

Proof. By the proof of Theorem 4.7 in [8], (1) and (2) are trivial. By Theorem 4.7 in [8], $\mu(\beta S_d) = 1$. If $\delta > 0$, then $\overline{d}_{\mathcal{F}}((\delta,+\infty)) = 0$ and by Lemma 3.4, $\overline{d}_{\mathcal{F}}(A) = \overline{d}_{\mathcal{F}}(A \cap (0,\delta))$. Now pick $\delta > 0$, and let $A_\delta = S \cap (0,\delta)$. Then

$$1 = \mu(\beta S_d) = \mu(\overline{A_\delta} \cup \overline{S - A_\delta}) = \mu(\overline{A_\delta}).$$

Therefore

$$1 = \lim_{\delta \to 0^+} \mu(\overline{A_\delta}) = \mu(0^*(S)).$$

□

Theorem 3.21. *Let S be a dense subsemigroup of $((0,\infty),+)$. Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ such that $\lim_{n\in D} \max F_n = 0$, and let $A \subseteq S$ be piecewise syndetic near zero. Then for each $\delta > 0$, there exists $F \in P_f(S \cap (0,\epsilon)))$ such that $d^*_{\mathcal{F}}(\bigcup_{t\in F}(-t + A)) = 1$.*

Proof. By Remark 3.6 and Definition 2.4 (c), it is obvious. □

Theorem 3.22. *Let S be a dense subsemigroup of $((0,\infty),+)$. Let $\mathcal{F} = \{F_n\}_{n\in D}$ be a net in $P_f(S)$ and let $\lim_{n\in D} \max F_n = 0$. If $\mathcal{F}$ satisfies $(**)$, let for each $\delta > 0$, there exists $F \in P_f(S \cap (0,\epsilon))$ such that $d^*_{\mathcal{F}}(\bigcup_{t\in F}(-t + A)) = 1$, then A is piecewise syndetic near zero.*

Proof. By Theorem 3.13 and $(**)$, $\bigcup_{t\in F}(-t + A)$ is thick near zero. □

Theorem 3.23. *Let S be a dense subsemigroup of $((0,\infty),+)$. Let $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ be a sequence in $P_f(S)$ and let $\lim_{n\to\infty} \max F_n = 0$. Let $A \subseteq S$ and $d^*_{\mathcal{F}}(A) = \alpha > 0$ then there exists a sequence $\mathcal{A} = \{G_n\}_{n=1}^{\infty}$ such that $\overline{d}_{\mathcal{A}}(A) = \alpha$.*

Proof. Let for each $n \in \mathbb{N}$, there exists $m \geq n$ and $x \in S \cup \{0\}$ such that $|A \cap (F_m + x)| \geq \left(d^*_{\mathcal{F}}(A) - \frac{1}{n}\right)|F_m|$ thus we have $\overline{d}_{\mathcal{A}}(A) = \alpha$. □

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