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A GENERALIZATION OF KANNAN AND CHATTERJEA FIXED POINT THEOREMS ON COMPLETE b-METRIC SPACES

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Abstract. In this paper, we give some results on the common fixed point of self-mappings defined on complete b-metric spaces. Our results generalize Kannan and Chatterjea fixed point theorems on complete b-metric spaces. In particular, we show that two self-mappings satisfying a contraction type inequality have a unique common fixed point. We also give some examples to illustrate the given results.

1. Introduction

The noation of a b-metric space was introduced by Bakhtin [3]. Since then, b-metric fixed point theory grew up in the classical metric fixed point theory to obtain a generalization of some known metric version of fixed point results. For quantitive information on b-metric fixed point theory, we refer the readers to [1, 2, 4, 5, 7, 8, 10, 12, 13] and some references therein.

The following two theorems are due to Kannan [9] and Chattreja [6], respectively.

Theorem 1.1. *Let (X, d) be a complete metric space. If a map $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)), \tag{1.1}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$, then T has a unique fixed point.

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Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a map satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)). \tag{1.2}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point.

In this paper we give a generalization of two theorems above in the setting of b-metric spaces.

2. Main Results

We recall that a function $d : X \times X \to [0, \infty)$ on a nonempty set X is a b-metric with parameter $s \geq 1$ if the triangle inequality in the definition of a metric is replaced with the (b-triangular) inequality

$$d(x, y) \leq s[d(x, z) + d(z, y)],$$

for all $x, y, z \in X$. Then (X, d) is called a b-metric space.

The following definition will be needed for our main results.

Definition 2.1 ([11])**.** A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if

(i): ψ is continuous and strictly increasing,
(ii): $\psi(t) = 0$ if and only if $t = 0$.

The main idea of the following theorem is borrowed from Theorem 1 in [14].

Theorem 2.2. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and T, f be self-mappings on X which satisfy*

$$\begin{aligned} d(Sx, Ty) \leq\ & a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ & + a_4 d(y, Sx) + a_5 d(x, y), \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers satisfying

(i): $s^2 a_1 + s^2 a_2 + s^3 a_3 + s^3 a_4 + s^2 a_5 < 1$,
(ii): $a_1 = a_2$ *or* $a_3 = a_4$.

Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ and consider the sequence $\{x_n\}$ in which

$$x_{2n+1} = Sx_{2n}, \qquad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, 3, \ldots.$$

By (2.1), we have

$$\begin{aligned} d(x_1, x_2) &= d(Sx_0, Tx_1) \\ &\leq a_1 d(x_0, Sx_0) + a_2 d(x_1, Tx_1) \\ &\quad + a_3 d(x_0, Tx_1) + a_4 d(x_1, Sx_0) + a_5 d(x_0, x_1) \end{aligned}$$

$$\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + s a_3 d(x_0, x_1) + s a_3 d(x_1, x_2) + a_5 d(x_0, x_1).$$

Therefore

$$d(x_1, x_2) \leq \frac{a_1 + s a_3 + a_5}{1 - a_2 - s a_3} d(x_0, x_1).$$

So,

$$d(x_2, x_3) \leq \frac{a_2 + s a_4 + a_5}{1 - a_1 - s a_4} d(x_1, x_2).$$

By repeating this procedure, we get

$$d(x_{2n-1}, x_{2n}) \leq \big(r\big)^n \big(k\big)^{n-1} d(x_0, x_1), \quad n = 1, 2, 3, \ldots, \tag{2.2}$$

and

$$d(x_{2n}, x_{2n+1}) \leq \big(r\big)^n \big(k\big)^n d(x_0, x_1), \quad n = 1, 2, 3, \ldots, \tag{2.3}$$

where

$$r = \frac{a_1 + s a_3 + a_5}{1 - a_2 - s a_3}, \qquad k = \frac{a_2 + s a_4 + a_5}{1 - a_1 - s a_4}.$$

Let $m, n \in \mathbb{N}$ and $m > n$. Then by (2.2) and (2.3), we have

$$\begin{aligned} d(x_{2n}, x_{2m}) &\leq s d(x_{2n}, x_{2n+1}) + \cdots + s^{2m-2n-1} d(x_{2m-2}, x_{2m-1}) \\ &\quad + s^{2m-2n} d(x_{2m-1}, x_{2m}) \\ &\leq s r^n k^n \lambda + \cdots + s^{2m-2n-1} r^{m-1} k^{m-1} \lambda + s^{2m-2n} r^m k^{m-1} \lambda \\ &= s \alpha^n \lambda + \cdots + s^{2m-2n-1} \alpha^{m-1} \lambda + s^{2m-2n} r \alpha^{m-1} \lambda \\ &= s \alpha^n \lambda (1 + sr) + \cdots + s^{2m-2n-1} \alpha^{m-1} \lambda (1 + sr) \\ &= s(1 + sr) \lambda \alpha^n \big(1 + s^2 \alpha + (s^2 \alpha)^2 + \cdots + (s^2 \alpha)^{m-n-1}\big), \end{aligned}$$

where $\alpha = rk$ and $\lambda = d(x_0, x_1)$. Since $s^2 \alpha < 1$, we get

$$d(x_{2n}, x_{2m}) \leq s(1 + sr) \lambda \frac{\alpha^n}{1 - s^2 \alpha}.$$

Therefore $\{x_{2n}\}$ is a Cauchy sequence. Let $x_{2n} \to x$. Using (2.3), we have

$$\begin{aligned} d(x, x_{2n+1}) &\leq s d(x, x_{2n}) + s d(x_{2n}, x_{2n+1}) \\ &\leq s d(x, x_{2n}) + \lambda \alpha^n \quad n = 0, 1, 2, 3, \ldots. \end{aligned}$$

So $\lim_{n \to \infty} x_{2n+1} = x$ and therefore $\lim_{n \to \infty} x_n = x$. Now, we show that x is the unique fixed point of T and S. Using (2.1), we have

$$\begin{aligned} d(x, Sx) &\leq s\big(d(x, x_{2n}) + d(x_{2n}, Sx)\big) \\ &= s d(x, x_{2n}) + s d(T x_{2n-1}, Sx) \\ &\leq s d(x, x_{2n}) + s a_1 d(x, Sx) + s a_2 d(x_{2n-1}, T x_{2n-1}) \end{aligned}$$

$$
\begin{aligned}
&+ sa_3 d(x, Tx_{2n-1}) + sa_4 d(x_{2n-1}, Sx) + sa_5 d(x, x_{2n-1})\\
\le\ & sd(x, x_{2n}) + sa_1 d(x, Sx) + sa_2 d(x_{2n-1}, Tx_{2n-1})\\
&+ sa_3 d(x, Tx_{2n-1}) + s^2 a_4 \big(d(x_{2n-1}, x) + d(x, Sx)\big)\\
&+ sa_5 d(x, x_{2n-1}),
\end{aligned}
$$

and so

$$d(x, Sx) \le sa_1 d(x, Sx) + sa_4 d(x, Sx).$$

This implies that $Sx = x$. Similarly, $Tx = x$. To see the uniqueness of the common fixed point of T and S, assume on the contrary that $Tx = Sx = x$ and $Ty = Sy = y$ but $x \neq y$. By (2.1), we have

$$
\begin{aligned}
d(x, y) = d(Sx, Ty) &\le a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty)\\
&\quad + a_4 d(y, Sx) + a_5 d(x, y)\\
&= \big(a_3 + a_4 + a_5\big) d(x, y) < d(x, y),
\end{aligned}
$$

which is a contradiction. □

Putting $T = S, a_1 = a_2, a_3 = a_4 = a_5 = 0$ and $s = 1$, Theorem 2.2 reduces to Theorem 1.1.

Example 2.3. Let $X = \{1, 2, 3\}$ and $d : X \times X \to [0, \infty)$ be defined as follows: $d(1, 2) = d(2, 1) = 1, d(3, 2) = d(2, 3) = \frac{6}{9}, d(1, 3) = d(3, 1) = \frac{1}{9}, d(0, 0) = d(1, 1) = d(2, 2) = 0$. It is easy to check that (X, d) is a b-metric space with parameter $s = \frac{3}{2}$. Define the mappings $T, S : X \to X$ by $T1 = T3 = 1, T2 = 3$ and $S1 = S2 = S3 = 1$. Let $a_1 = a_2 = a_3 = a_5 = 0, a_4 = \frac{2}{9}$. Then the conditions of Theorem 2.2 are satisfied.

Consider the following notation:

$$
\begin{aligned}
\Phi = \Big\{ &\varphi : [0, \infty) \times [0, \infty) \to [0, \infty) | \varphi(0, 0) \ge 0, \varphi(x, y) > 0 \text{ if } (x, y) \neq (0, 0)\\
&\text{and } \varphi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty} \varphi(a_n, b_n) \Big\}.
\end{aligned}
$$

Theorem 2.4. *Let (X, d) be a complete b-metric space with the parameter $s \ge 1$ and T, f be self-mappings on X which satisfy*

$$\psi(sd(Tx, fy)) \le \frac{\psi\left(\dfrac{d(x, fy) + \frac{d(y, Tx)}{s^3}}{s + 1}\right)}{1 + \varphi\big(d(x, fy), d(y, Tx)\big)}, \tag{2.4}$$

for all $x, y \in X$, where ψ is an altering distance function, $\varphi \in \Phi$ and T is continuous. Then T and f have a unique common fixed point.

Proof. Let $x_0 \in X, x_1 = Tx_0$ and $x_2 = fx_1$. Define the sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. By the inequality (2.4), we have

$$\begin{aligned}
\psi(sd(x_{2n+1}, x_{2n+2})) &= \psi(sd(Tx_{2n}, fx_{2n+1})) \\
&\leq \frac{\psi\left(\dfrac{d(x_{2n}, fx_{2n+1}) + \dfrac{d(x_{2n+1}, Tx_{2n})}{s^3}}{s+1}\right)}{1 + \varphi(d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, Tx_{2n}))} \\
&\leq \frac{\psi\left(\dfrac{d(x_{2n}, x_{2n+2})}{s+1}\right)}{1 + \varphi(d(x_{2n}, x_{2n+2}), 0)},
\end{aligned} \tag{2.5}$$

for each $n \geq 0$. Since φ is nonnegative,

$$\psi\big(sd(x_{2n+1}, x_{2n+2})\big) \leq \psi\left(\frac{d(x_{2n}, x_{2n+2})}{s+1}\right), \quad n = 0, 1, 2, \ldots.$$

This implies that

$$\begin{aligned}
sd(x_{2n+1}, x_{2n+2}) &\leq \frac{d(x_{2n}, x_{2n+2})}{s+1} \\
&\leq \frac{s}{s+1}\big(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big),
\end{aligned} \tag{2.6}$$

for each $n \geq 0$. So

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}), \quad n = 0, 1, 2, \ldots. \tag{2.7}$$

Similarly, we deduce that

$$d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2}), \quad n = 0, 1, 2, \ldots. \tag{2.8}$$

Using (2.7) and (2.8), by induction we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad n = 0, 1, 2, \ldots.$$

Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Passing to the limit as $n \to \infty$ in (2.6), we have

$$sr \leq \frac{1}{s+1} \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) \leq \frac{s}{2}(r + r) = sr.$$

Therefore

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = sr(s+1). \tag{2.9}$$

From (2.5) and (2.9), we get

$$\psi(\limsup_{n\to\infty} sd(x_{2n+1}, x_{2n+2})) \leq \frac{\limsup_{n\to\infty} \psi\left(\dfrac{d(x_{2n}, x_{2n+2})}{s+1}\right)}{1 + \liminf_{n\to\infty} \varphi(d(x_{2n}, x_{2n+2}), 0)}$$

$$\leq \frac{\psi\left(\dfrac{\limsup_{n\to\infty} d(x_{2n}, x_{2n+2})}{s+1}\right)}{1+\varphi(\liminf_{n\to\infty} d(x_{2n}, x_{2n+2}), 0)}.$$

Therefore

$$\psi(sr) \leq \frac{\psi\left(\dfrac{sr(s+1)}{s+1}\right)}{1+\varphi(sr(s+1), 0)},$$

and so $1+\varphi\big(sr(s+1), 0\big) \leq 1$. Since $\varphi \in \Phi$, we get $r = 0$. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.10}$$

Now we show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and$\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon, \tag{2.11}$$

and

$$d(x_{2m(k)}, x_{2n(k)-2}) < \varepsilon. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\begin{aligned}
\varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\
&\leq s\big(d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)})\big) \\
&\leq s\varepsilon + s^2\big(d(x_{2n(k)-2}, x_{2n(k)-1}) \\
&\quad + d(x_{2n(k)-1}, x_{2n(k)})\big),
\end{aligned}$$

for all $k \geq 1$. Passing to the limit as $k \to \infty$ in the above inequality and using (2.10) we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) \leq s\varepsilon. \tag{2.13}$$

Moreover, from (2.11) we get

$$\begin{aligned}
\varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\
&\leq s\big(d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)})\big),
\end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$, we have

$$\varepsilon \leq s \lim_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}). \tag{2.14}$$

On the other hand, we have

$$\begin{aligned}
d(x_{2n(k)-1}, x_{2m(k)+1}) &\leq s(d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)+1})) \\
&\leq sd(x_{2n(k)-1}, x_{2n(k)}) \\
&\quad + s^2(d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1})),
\end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$, we get

$$\limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) \leq s^3\varepsilon. \tag{2.15}$$

Also from (2.11) one can show that

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}). \tag{2.16}$$

Using (2.4),(2.13),(2.14) and (2.15), we have

$$\begin{aligned}
\psi(\varepsilon) &\leq \psi\big(s \limsup_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)})\big) \\
&= \psi\big(s \limsup_{k\to\infty} d(Tx_{2m(k)}, fx_{2n(k)-1})\big) \\
&\leq \limsup_{k\to\infty} \frac{\psi\left(\dfrac{d(x_{2m(k)}, fx_{2n(k)-1}) + \frac{1}{s^3} d(x_{2n(k)-1}, Tx_{2m(k)})}{s+1}\right)}{1+\varphi\big(d(x_{2m(k)}, fx_{2n(k)-1}), d(x_{2n(k)-1}, Tx_{2m(k)})\big)} \\
&\leq \frac{\psi\left(\limsup_{k\to\infty} \dfrac{d(x_{2m(k)}, x_{2n(k)}) + \frac{1}{s^3} d(x_{2n(k)-1}, x_{2m(k)+1})}{s+1}\right)}{1+\liminf_{k\to\infty}\varphi\big(d(x_{2m(k)}, x_{2n(k)}), d(x_{2n(k)-1}, Tx_{2m(k)})\big)} \\
&\leq \frac{\psi\left(\limsup_{k\to\infty} \dfrac{d(x_{2m(k)}, x_{2n(k)}) + \frac{1}{s^3} d(x_{2n(k)-1}, x_{2m(k)+1})}{s+1}\right)}{1+\varphi\big(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\big)} \\
&\leq \frac{\psi\left(\dfrac{s\varepsilon+\varepsilon}{s+1}\right)}{1+\varphi\big(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{n\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\big)} \\
&= \frac{\psi(\varepsilon)}{1+\varphi\big(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\big)}.
\end{aligned}$$

Consequently

$$\varphi(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})) = 0.$$

Because $\varphi \in \Phi$, we have

$$\liminf_{n\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \liminf_{n\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) = 0.$$

which contradicts (2.16). This implies that $\{x_{2n}\}$ is a Cauchy sequence and so is $\{x_n\}$. Hence, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Since T is continuous, we have

$$Tx^* = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = x^*,$$

i.e., x^* is a fixed point of T. Moreover, from (2.4) we have

$$\begin{aligned}\psi(sd(x^*, fx^*)) &= \psi(sd(Tx^*, fx^*))\\ &\le \frac{\psi\left(\dfrac{d(x^*, fx^*) + \dfrac{d(x^*, Tx^*)}{s^3}}{s+1}\right)}{1+\varphi(d(x^*, fx^*), d(x^*, Tx^*))}\\ &= \frac{\psi\left(\dfrac{d(x^*, fx^*)}{s+1}\right)}{1+\varphi(d(x^*, fx^*), 0)}\\ &\le \psi\left(\frac{d(x^*, fx^*)}{s+1}\right).\end{aligned}$$

Since ψ is a strictly increasing function, we have

$$sd(x^*, fx^*) \le \frac{d(x^*, fx^*)}{s+1}.$$

Therefore $fx^* = x^*$. Hence x^* is a common fixed point of T and f. To see the uniqueness of the common fixed point of T and f, assume on the contrary that $Tu = fu = u$ and $Tv = fv = v$ but $u \neq v$. We have

$$\begin{aligned}\psi(sd(u, v)) &= \psi(sd(Tu, fv))\\ &\le \frac{\psi\left(\dfrac{d(u, fv) + \dfrac{d(v, Tu)}{s^3}}{s+1}\right)}{1+\varphi(d(u, fv), d(v, Tu))}.\end{aligned}$$

Since $s \ge 1$, we get

$$\psi(sd(u, v)) \le \frac{\psi\left(\dfrac{d(u, v) + d(v, u)}{2}\right)}{1+\varphi(d(u, v), d(v, u))}.$$

Then

$$\psi(d(u, v)) \le \frac{\psi(d(u, v))}{1+\varphi(d(u, v), d(v, u))},$$

i.e, $\varphi(d(u, v), d(v, u)) = 0$. This implies that $u = v$. □

In Theorem 2.4, if $\psi(t) = t$ and $\varphi(u, v) = \dfrac{1}{s(s+1)\alpha} - 1$, where $\alpha \in \left[0, \frac{1}{s(s+1)}\right)$, we get the following corollary.

Corollary 2.5. *Let (X, d) be a complete b-metric space with the parameter $s \geq 1$ and T, f be self-mappings on X which satisfy*

$$d(Tx, fy) \leq \alpha \left(d(x, fy) + \frac{1}{s^3} d(y, Tx) \right),$$

for all $x, y \in X$, where $\alpha \in \left[0, \frac{1}{s(s+1)}\right)$ and T is continuous. Then T and f have a unique common fixed point.

Also, in the case that $s = 1$ and $T = f$, Corollary 2.5 would be an extension of Chatterjea Theorem [6].

Example 2.6. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be defined as follows: $d(0,1) = d(1,0) = 1, d(0,2) = d(2,0) = \frac{1}{5}, d(1,2) = d(2,1) = \frac{3}{5}, d(0,0) = d(1,1) = d(2,2) = 0$. It is easy to check that (X, d) is a b-metric space with parameter $s = \frac{5}{4}$. Define $T : X \to X$ by $T0 = 0, T1 = 2, T2 = 0$ and $f(x) = 0$ for all $x \in X$. Define $\psi : [0, \infty) \to [0, \infty)$ and $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\varphi(u, v) = \frac{1}{15}$ for all $u, v \in [0, \infty)$. Then, the inequality (2.4) holds for all $x, y \in X$.

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