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Frames in super Hilbert modules

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Abstract. In this paper, we define super Hilbert module and investigate frames in this space. Super Hilbert modules are generalization of super Hilbert spaces in Hilbert C*-module setting. Also, we define frames in a super Hilbert module and characterize them by using of the concept of g-frames in a Hilbert C*-module. Finally, disjoint frames in Hilbert C*-modules are introduced and investigated.

1. Introduction

Super Hilbert spaces arose naturally as the state space of a quantum field in the functional Schrodinger representation of spinor quantum field theory and it provided a means to bring super symmetric quantum field theories into a form resembling standard quantum mechanics, the super Hilbert space has certain advantages compared with the Hilbert space in quantum mechanics [7, 9, 23, 25]. Balan [4] introduced the concept of super frames and presented some density results for Weyl-Heisenberg super frames. In [18], Han and Larson derived necessary and sufficient conditions for the direct sum of two frames to be a super frame. And in [15], Gu and Han investigated the connection between decomposable Parseval wavelet frames and super wavelet frames and gave some necessary and sufficient conditions for extendable Parseval wavelet frames. In [1, 19, 30] frames, g-frames and g-Riesz frames in super Hilbert spaces has been studied.

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [8] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer

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[6], and popularized from then on. Today frames have many applications in different subjects, for example in image and signal processing [2, 3] and coherent states [12–14]

Let H be a Hilbert space, and J be a finitely or countably index set. A sequence $\{f_j\}_{j\in J} \subseteq H$ is called a frame for H if there exist the constants $C, D > 0$ such that

$$C\|f\|^2 \leq \sum_{j\in J} \mid \langle f, f_j\rangle \mid^2 \leq D\|f\|^2, \tag{1.1}$$

for all $f \in H$. The constants C and D are called frame bounds. If $C = D$ we call this frame a tight frame and if C = D =1 it is called a Parseval frame.

In [29], Sun introduced a generalization of frames and showed the other concepts of generalizations of frames can be presented by g-frames. Also, Sun proved that generalized frames have many properties of frames.

Let U and V be two Hilbert spaces and $\{V_j : j \in J\}$ be a sequence of subspaces of V, where J is a subset of $\mathbb{Z}$. $L(U, V_j)$ is the collection of all bounded linear operators from U into V_j . The sequence $\{\Lambda_j \in L(U, V_j) : j \in J\}$ is called to be a generalized frame, or simply a g-frame, for U with respect to $\{V_j : j \in J\}$ if there exist two positive constants C and D such that

$$C\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq D\|f\|^2, \tag{1.2}$$

for all $f \in U$. The constants C and D are called g-frame bounds. If $C = D$ we call this g-frame a tight g-frame and if C = D =1 it is called a Parseval g-frame.

The notion of frames in Hilbert C^*-modules has been introduced and has been investigated in [11]. Frank and Larson [10, 11] defined the standard frames in Hilbert C^*-modules in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. The extended results of this more general framework are not a routine generalization, because there are essential differences between Hilbert C^*-modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^*-module. Also, there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^*-modules.

We refer the readers for more details on Hilbert C^*-modules and a discussion of basic properties of frames in Hilbert spaces and Hilbert C^*-modules and their generalizations to [5, 10, 24, 26–28].

Let $\mathcal{H}$ be a Hilbert C^*-module, and J a set which is finite or countable. A sequence $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $C, D > 0$ such that

$$C\langle f, f\rangle \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq D\langle f, f\rangle, \tag{1.3}$$

for all $f \in \mathcal{H}$. The constants C and D are called the frame bounds. Basic properties of frames in Hilbert C^*-modules are studied in [10, 16, 17, 22].

A. Khosravi and B. Khosravi [21] defined g-frames in Hilbert C^*-modules. Let U and V be two Hilbert C^*-modules over the same C^*-algebra A and $\{V_j : j \in J\}$ be a sequence of subspaces of V, where J is a subset of $\mathbb{Z}$. Let $End_A^*(U, V_j)$ be the collection of all adjointable A-linear maps from U into V_j, i.e. let T be an A-linear map from U into V_j , if $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f \in U$ and $g \in V_j$ then, it implies that $T \in End_A^*(U, V_j)$. The sequence $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ is said to be a generalized frame (or simply a g-frame) for Hilbert C^*-module U with respect to $\{V_j : j \in J\}$ if there are two positive constants C and D such that

$$C\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle, \tag{1.4}$$

for all $f \in U$. The constants C and D are called g-frame bounds.

In this paper, we define super Hilbert modules that they are generalization of super Hilbert spaces in Hilbert C*-module setting. Also we define frames in a super Hilbert module and characterize them by using g-frames in a Hilbert C*-module. Moreover disjoint frames and complementary frames in Hilbert C*-modules are introduced and are investigated.

2. Preliminaries

Hilbert C^*-modules form a wide category between Hilbert spaces and Banach spaces.

Definition 2.1. Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space $\mathcal{H}$ which is a left A-module with an A-valued inner product map $\langle ., .\rangle :$ $\mathcal{H} \times \mathcal{H} \to A$ which satisfies the following properties:

(i) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$;
(ii) $\langle af, g\rangle = a\langle f, g\rangle$ for all $f, g \in \mathcal{H}$ and $a \in A$;
(iii) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g \in \mathcal{H}$;
(iv) $\langle f, f\rangle \geq 0$ for all $f \in \mathcal{H}$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.

For $f \in \mathcal{H}$, we define a norm on $\mathcal{H}$ by $\|f\|_{\mathcal{H}} = \|\langle f, f\rangle\|_A^{1/2}$. If $\mathcal{H}$ is complete with this norm, it is called a (left) Hilbert C^*-module over A or a (left) Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^* = a$ and its spectrum is a subset of positive real numbers. In this case, we write $a \geq 0$. By the property (4) of the above formula, $\langle f, f\rangle \geq 0$ for every $f \in \mathcal{H}$, hence we define the absolute value of f by $|f| = \langle f, f\rangle^{1/2}$.

For the frame $\{f_j : j \in J\}$ in a Hilbert A-module $\mathcal{H}$, the operator S defined by

$$Sf = \sum_{j\in J}\langle f, f_j\rangle f_j, \quad (f \in \mathcal{H}),$$

is called the frame operator. The frame operator S is invertible, positive, adjointable and self-adjoint. Since

$$\langle Sf, f\rangle = \left\langle \sum_{j\in J}\langle f, f_j\rangle f_j, f \right\rangle = \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle, \quad (f \in \mathcal{H}),$$

it follows that

$$C\langle f, f\rangle \leq \langle Sf, f\rangle \leq D\langle f, f\rangle, \quad (f \in \mathcal{H}),$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j\in J}\langle S^{-1}f, f_j\rangle f_j = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j,$$

for all $f \in \mathcal{H}$.
Let $\tilde{f}_j = S^{-1}f_j$, then

$$f = \sum_{j\in J}\langle f, \tilde{f}_j\rangle f_j = \sum_{j\in J}\langle f, f_j\rangle \tilde{f}_j,$$

for any $f \in \mathcal{H}$. The sequence $\{\tilde{f}_j : j \in J\}$ is also a frame for $\mathcal{H}$ which is called the canonical dual frame of $\{f_j : j \in J\}$.

Like ordinary frames in Hilbert spaces, the notion of analysis and synthesis operators can be defined as follows:

Definition 2.2. Let $\{f_j\}_{j\in J}$ be a frame in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A, then the related analysis operator $U : \mathcal{H} \to \ell^2(A)$ is defined by

$$Uf = \{\langle f, f_j\rangle : j \in J\},$$

for all $f \in \mathcal{H}$. We define the synthesis operator $T : \ell^2(A) \to \mathcal{H}$ by

$$T(\{a_j\}) = \sum_{j\in J} a_j f_j,$$

for all $\{a_j\}_{j\in J} \in \ell^2(A)$.

For any $g = \{g_j\}_{j\in J} \in \ell^2(A)$ and $f \in \mathcal{H}$,

$$\begin{aligned}\langle Uf, g\rangle &= \langle \{\langle f, f_j\rangle\}, \{g_j\}\rangle \\ &= \sum_{j\in J} \langle f, f_j\rangle g_j^* \\ &= \sum_{j\in J} \langle f, g_j f_j\rangle \\ &= \left\langle f, \sum_{j\in J} g_j f_j \right\rangle \\ &= \langle f, Tg\rangle,\end{aligned}$$

it follows that U is adjointable and $U^* = T$. Also

$$U^*Uf = U^*(\langle f, f_j\rangle) = \sum_{j\in J} \langle f, f_j\rangle f_j = Sf,$$

for all $f \in \mathcal{H}$.

The following theorems characterize frames and Bessel sequences and frames in Hilbert A-modules.

Theorem 2.3 ([16]). *A sequence $\{f_j\}_{j\in J}$ in Hilbert A-module $\mathcal{H}$ over an unital C^*-algebra A is a frame for $\mathcal{H}$ if and only if the synthesis operator T is well defined and surjective.*

Corollary 2.4 ([16]). *A sequence $\{f_j\}_{j\in J}$ in Hilbert A-module $\mathcal{H}$ over an unital C^*-algebra A is a Bessel for $\mathcal{H}$ if and only if the synthesis operator T is well defined and $\|T\| \leq \sqrt{D}$.*

The definition of super Hilbert space is given in [9] as following:

Definition 2.5. Super Hilbert space $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_0 \bigoplus \mathcal{H}_1$ of two complex Hilbert spaces $(\mathcal{H}_0, \langle ., .\rangle_0), (\mathcal{H}_1, \langle ., .\rangle_1)$ equipped with the inner product $\langle\langle ., .\rangle\rangle = \langle ., .\rangle_0 + \langle ., .\rangle_1$.

It is easy to see that every super Hilbert space is a Hilbert space.

3. Main result

In this section, at first we define super Hilbert module that is a generalization of super Hilbert space in a Hilbert C*-module setting. Then we investigate and characterize frames in super Hilbert modules.

Definition 3.1. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be two Hilbert A-modules with inner products $\langle ., .\rangle_0$ and $\langle ., .\rangle_1$ respectively. Super Hilbert module space $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_0 \bigoplus \mathcal{H}_1$ equipped with the inner product

$$\langle\langle ., .\rangle\rangle = \langle ., .\rangle_0 + \langle ., .\rangle_1.$$

Every direct sum of Hilbert C^*-modules is a Hilbert C^*-modules [24]. Hence every super Hilbert module is a Hilbert C^*-module.

Definition 3.2. Let $\mathcal{H}$ be a Hilbert A-module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a sequence of elements of super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$ and Λ_j be defined by

$$\Lambda_j f = (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle), \quad \forall f \in \mathcal{H}.$$

If $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to A^2, then we call it a g-frame associated with $\{(\varphi_j, \psi_j) : j \in J\}$.

The following lemma gives a necessary condition for $\{(\varphi_j, \psi_j) : j \in J\}$ to be a frame for super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$.

Lemma 3.3. *Suppose that $\mathcal{H}$ is a Hilbert C^*-module and that $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$. Then both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert C^*-module $\mathcal{H}$.*

Proof. Since $\{\Phi_j = (\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$, there exist constants $C, D > 0$ such that

$$C\langle\langle f, f\rangle\rangle \leq \sum_{j\in J} \langle\langle f, \Phi_j\rangle\rangle\langle\langle \Phi_j, f\rangle\rangle \leq D\langle\langle f, f\rangle\rangle,$$

for all $f = (f_1, f_2) \in \mathcal{H} \bigoplus \mathcal{H}$.

By the definition of the inner product in super Hilbert module, we would have

$$\begin{aligned} C\left(\langle f_1, f_1\rangle + \langle f_2, f_2\rangle\right) &\leq \sum_{j\in J} \left(\langle f_1, \varphi_j\rangle + \langle f_2, \psi_j\rangle\right)\left(\langle \varphi_j, f_1\rangle + \langle \psi_j, f_2\rangle\right) \\ &\leq D\left(\langle f_1, f_1\rangle + \langle f_2, f_2\rangle\right), \end{aligned}$$

for all $f = (f_1, f_2) \in \mathcal{H} \bigoplus \mathcal{H}$. Substituting $f_2 = 0$ into the above inequality we obtain

$$C\langle f_1, f_1\rangle \leq \sum_{j\in J} \langle f_1, \varphi_j\rangle\langle \varphi_j, f_1\rangle \leq D\langle f_1, f_1\rangle,$$

for all $f_1 \in \mathcal{H}$. This means $\{\varphi_j : j \in J\}$ is a frame for Hilbert C^*-module $\mathcal{H}$.

The same conclusion can be driven for $\{\psi_j : j \in J\}$ by letting $f_1 = 0$. □

The following theorem gives a necessary condition for a frame in super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$ by using g-frames in Hilbert C*-modules.

Theorem 3.4. *Let $\mathcal{H}$ be a Hilbert C^*-module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a frame for super Hilbert module $\mathcal{H} \bigoplus \mathcal{H}$. Then $\{\Lambda_j : j \in J\}$ is a g-frame* *for Hilbert A-module with respect to* A^2 *associated with* $\{(\varphi_j, \psi_j) : j \in J\}$.

Proof. By Lemma 3.3, both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert C^*-module $\mathcal{H}$. Now by regard to this fact that

$$\begin{aligned}\langle \Lambda_j f, \Lambda_j f\rangle &= \langle (\langle f, \varphi_j\rangle, \langle f, \psi_j\rangle), (\langle f, \varphi_j\rangle, \langle f, \psi_j\rangle)\rangle \\ &= \langle \langle f, \varphi_j\rangle, \langle f, \varphi_j\rangle\rangle + \langle \langle f, \psi_j\rangle, \langle f, \psi_j\rangle\rangle \\ &= \langle f, \varphi_j\rangle\langle \varphi_j, f\rangle + \langle f, \psi_j\rangle\langle \psi_j, f\rangle,\end{aligned}$$

the operator sequence $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$. □

The following proposition is a generalization of a similar proposition in [30] for super Hilbert modules.

Proposition 3.5. *Let* $\{(\varphi_j, \psi_j) : j \in J\} \subseteq \mathcal{H} \bigoplus \mathcal{H}$ *and*

$$\Lambda_j f = (\langle f, \varphi_j\rangle, \langle f, \psi_j\rangle)^T,$$

for any $f \in \mathcal{H}$ *and* $j \in J$. *Then* $\{\Lambda_j : j \in J\}$ *is a g-frame for Hilbert A-module* $\mathcal{H}$ *with respect to* A^2 *associated with* $\{(\varphi_j, \psi_j) : j \in J\}$ *if and only if* $\{\varphi_j : j \in J\} \bigcup \{\psi_j : j \in J\}$ *is a frame for Hilbert A-module* $\mathcal{H}$.

Proof. Since

$$\langle \Lambda_j f, \Lambda_j f\rangle = \langle f, \varphi_j\rangle\langle \varphi_j, f\rangle + \langle f, \psi_j\rangle\langle \psi_j, f\rangle,$$

we conclude $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$ if and only if $\{\varphi_j : j \in J\} \bigcup \{\psi_j : j \in J\}$ is a frame for Hilbert A-module $\mathcal{H}$. □

By the previous propositions, we get

Proposition 3.6. *Suppose* $\mathcal{H}$ *is a Hilbert A-module. Let* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *be two frames for Hilbert A-module* $\mathcal{H}$ *and* $\{\Lambda_j : j \in J\}$ *be a g-frames for Hilbert A-module* $\mathcal{H}$ *with respect to* A^2 *where* $\Lambda_j f = (\langle f, \varphi_j\rangle, \langle f, \psi_j\rangle)$ *for all* $f \in \mathcal{H}$. *Then the synthesis operator for* $\{\Lambda_j : j \in J\}$ *is the operator*

$$T : \ell^2(A^2) \to \mathcal{H},$$

defined by

$$\begin{aligned}T(\{(a_j, b_j)\}_{j\in J}) &= \sum_{j\in J} \Lambda_j^*(a_j, b_j) \\ &= \sum_{j\in J} (a_j\varphi_j + b_j\psi_j),\end{aligned}$$

for all $\{(a_j, b_j)\}_{j\in J} \in l^2(A^2)$.

The analysis operator for $\{\Lambda_j : j \in J\}$ *is the operator*

$$T^* : \mathcal{H} \to \ell^2(A^2),$$

defined by

$$T^* f = \{\Lambda_j f\}_{j\in J} = \{(\langle f, \varphi_j\rangle, \langle f, \psi_j\rangle)\}_{j\in J},$$

for all $f \in \mathcal{H}$.

Also the g-frame operator for $\{\Lambda_j : j \in J\}$ *is the operator*

$$S_\Lambda : \mathcal{H} \to \mathcal{H},$$

defined by

$$\begin{aligned} S_\Lambda f &= TT^* f \\ &= \sum_{j\in J} (\langle f, \varphi_j\rangle \varphi_j + \langle f, \psi_j\rangle \psi_j) \\ &= S_\varphi f + S_\psi f, \end{aligned}$$

for all $f \in \mathcal{H}$.

In the following, we check super Hilbert modules by different space and different C*-algebra.

Proposition 3.7. *Let* A *and* B *be unital* C^**-algebras,* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *be sequences in Hilbert* A*-module* $\mathcal{H}$ *and Hilbert* B*-module* $\mathcal{K}$*, respectively. Then* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *are frames in Hilbert* A*-module* $\mathcal{H}$ *and Hilbert* B*-module* $\mathcal{K}$ *respectively if and only if* $\{\Lambda_j : j \in J\}$ *is a g-frame in super Hilbert* C^**-module* $\mathcal{H} \bigoplus \mathcal{K}$ *with respect to* $A \oplus B$ *where* $\Lambda_j(f, g) = (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle)$ *for all* $(f, g) \in \mathcal{H} \bigoplus \mathcal{K}$.

Proof. Since

$$\begin{aligned} \langle \Lambda_j(f, g), \Lambda_j(f, g)\rangle &= \langle (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle), (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle)\rangle \\ &= \langle \langle f, \varphi_j\rangle, \langle f, \varphi_j\rangle\rangle + \langle \langle g, \psi_j\rangle, \langle g, \psi_j\rangle\rangle \\ &= \langle f, \varphi_j\rangle\langle \varphi_j, f\rangle + \langle g, \psi_j\rangle\langle \psi_j, g\rangle, \end{aligned}$$

then $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames in Hilbert A-module $\mathcal{H}$ and Hilbert B-module $\mathcal{K}$ respectively if and only if $\{\Lambda_j : j \in J\}$ is a g-frame in super Hilbert C^*-module $\mathcal{H} \bigoplus \mathcal{K}$ with respect to $A \oplus B$. □

In this case, we have the following proposition.

Proposition 3.8. *Let* A *and* B *be unital* C^**-algebras,* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *be frames in Hilbert* A*-module* $\mathcal{H}$ *and Hilbert* B*-module* $\mathcal{K}$ *respectively and* $\{\Lambda_j : j \in J\}$ *be a g-frame in super Hilbert* C^**-module* $\mathcal{H} \bigoplus \mathcal{K}$ *with respect to* $A \oplus B$ *where* $\Lambda_j(f, g) = (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle)$ *for all* $(f, g) \in \mathcal{H} \bigoplus \mathcal{K}$*. Then the synthesis operator for* $\{\Lambda_j : j \in J\}$ *is the operator*

$$T : \ell^2(A \oplus B) \to \mathcal{H} \bigoplus \mathcal{K},$$

defined by

$$T(\{(a_j, b_j)\}_{j\in J}) = \sum_{j\in J} \Lambda_j^*(a_j, b_j)$$
$$= \left(\sum_{j\in J} a_j\varphi_j, \sum_{j\in J} b_j\psi_j \right),$$

for all $\{(a_j, b_j) : j \in J\} \subseteq A \oplus B$.

The analysis operator for $\{\Lambda_j : j \in J\}$ *is the operator*

$$T^* : \mathcal{H} \bigoplus \mathcal{K} \to \ell^2(A \oplus B),$$

defined by

$$T^*(f, g) = \{\Lambda_j(f, g)\}_{j\in J}$$
$$= \{\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle\}_{j\in J},$$

for all $(f, g) \in \mathcal{H} \bigoplus \mathcal{K}$.

Also the g-frame operator for $\{\Lambda_j : j \in J\}$ *is the operator*

$$S_\Lambda : \mathcal{H} \bigoplus \mathcal{K} \to \mathcal{H} \bigoplus \mathcal{K},$$

defined by

$$S_\Lambda(f, g) = TT^*(f, g)$$
$$= \left(\sum_{j\in J} \langle f, \varphi_j\rangle\varphi_j, \sum_{j\in J} \langle g, \psi_j\rangle\psi_j \right)$$
$$= (S_\varphi f, S_\psi g),$$

for all $(f, g) \in \mathcal{H} \bigoplus \mathcal{K}$.

Now we state the following propositions for the case of super Hilbert modules by different spaces and same C*-algebra.

Proposition 3.9. *Let* A *be an unital* C^**-algebras,* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *be sequences in Hilbert* A*-modules* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively. Then* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *are frames in Hilbert* A*-modules* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively if and only if* $\{(\varphi_j, \psi_j) : j \in J\}$ *is a frame in super Hilbert* C^**-module* $\mathcal{H} \bigoplus \mathcal{K}$.

Proposition 3.10. *Let* A *be an unital* C^**-algebras,* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *be sequences in Hilbert* A*-modules* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively. Then* $\{\varphi_j : j \in J\}$ *and* $\{\psi_j : j \in J\}$ *are frames in Hilbert* A*-modules* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively if and only if* $\{\Lambda_j : j \in J\}$ *is a g-frame in super Hilbert* C^**-module* $\mathcal{H} \bigoplus \mathcal{K}$ *with respect to* A^2 *where* $\Lambda_j(f, g) = (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle)$ *for all* $(f, g) \in \mathcal{H} \bigoplus \mathcal{K}$.

Proposition 3.11. *Let A be an unital C^*-algebra, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be frames in Hilbert A-module $\mathcal{H}$ and $\mathcal{K}$ respectively and $\{\Lambda_j : j \in J\}$ be a g-frame in super Hilbert C^*-module $\mathcal{H} \bigoplus \mathcal{K}$ with respect to A where $\Lambda_j(f,g) = (\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle)$ for all $(f,g) \in \mathcal{H} \bigoplus \mathcal{K}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator*

$$T : \ell^2(A^2) \to \mathcal{H} \bigoplus \mathcal{K},$$

defined by

$$\begin{aligned} T(\{(a_j, b_j)\}_{j\in J}) &= \sum_{j\in J} \Lambda_j^*(a_j, b_j) \\ &= \left(\sum_{j\in J} a_j \varphi_j, \sum_{j\in J} b_j \psi_j \right), \end{aligned}$$

for all $\{(a_j, b_j) : j \in J\} \subseteq A^2$. The analysis operator for $\{\Lambda_j : j \in J\}$ is the operator

$$T^* : \mathcal{H} \bigoplus \mathcal{K} \to \ell^2(A^2),$$

defined by

$$\begin{aligned} T^*(f,g) &= \{\Lambda_j(f,g)\}_{j\in J} \\ &= \{\langle f, \varphi_j\rangle, \langle g, \psi_j\rangle\}_{j\in J}, \end{aligned}$$

for all $(f,g) \in \mathcal{H} \bigoplus \mathcal{K}$. Also the g-frame operator for $\{\Lambda_j : j \in J\}$ is the operator

$$S_\Lambda : \mathcal{H} \bigoplus \mathcal{K} \to \mathcal{H} \bigoplus \mathcal{K},$$

defined by

$$\begin{aligned} S_\Lambda(f,g) &= TT^*(f,g) \\ &= \left(\sum_{j\in J} \langle f, \varphi_j\rangle \varphi_j, \sum_{j\in J} \langle g, \psi_j\rangle \psi_j \right) \\ &= (S_\varphi f, S_\psi g), \end{aligned}$$

for all $(f,g) \in \mathcal{H} \bigoplus \mathcal{K}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert C^*-modules on an unital C^*-algebra A. We say that the frame pairs $(\{\varphi_j\}, \{\psi_j\}) \subset \mathcal{H} \bigoplus \mathcal{K}$ and $(\{\mu_j\}, \{\nu_j\}) \subset \mathcal{H} \bigoplus \mathcal{K}$ are similar if there are bounded invertible operators $T_1 \in L(\mathcal{H})$ and $T_2 \in L(\mathcal{K})$ such that $T_1\varphi_j = \mu_j$ and $T_2\psi_j = \nu_j$ for all $j \in J$. A pair of frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \bigoplus \mathcal{K}$. A pair of parseval frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called strongly disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a parseval frame for $\mathcal{H} \bigoplus \mathcal{K}$, and a pair of general frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called strongly disjoint if it is similar to a strongly disjoint pair of parseval frames.

Han and Larson in [18] have proved that $\{\varphi_j : j \in J\}$ is a parseval frame in a Hilbert space $\mathcal{H}$ if and only if there are a Hilbert space $\mathcal{K}$ and a parseval frame $\{\psi_j : j \in J\}$ in $\mathcal{K}$ such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \bigoplus \mathcal{K}$. We extend this result for frames in Hilbert C^*-modules.

The following proposition in Hilbert C^*-module setting may be proved in much the same way as Proposition 1.1 in [18]. Also, the following results are related to Theorem 4.1, Propositions 5.1 and 5.2 in reference [10].

Proposition 3.12. *Let $\mathcal{H}$ be Hilbert C^*-module on unital C^*-algebra A. Suppose that $\{\varphi_j : j \in J\}$ is a Parseval frame for $\mathcal{H}$. Then there exist a Hilbert C^*-module $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_j : j \in J\}$ for $\mathcal{K}$ such that $\varphi_j = Pe_j$, where P is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$.*

Proof. Let $\mathcal{K} = \ell^2(A)$ and let $\theta : \mathcal{H} \to \mathcal{K}$ be defined by

$$\theta(f) = \{\langle f, \varphi_j\rangle : j \in J\},$$

for all $f \in \mathcal{H}$. Since $\{\varphi_j : j \in J\}$ is a Parseval frame for Hilbert C*-module $\mathcal{H}$, we have

$$\|\theta(f)\|^2 = \sum_{j\in J} |\langle f, \varphi_j\rangle|^2 = \|f\|^2.$$

Thus θ is well defined and is an isometry. So we can embed $\mathcal{H}$ into $\mathcal{K}$ by identifying $\mathcal{H}$ with $\theta(\mathcal{H})$. Let P be the orthogonal projection from $\mathcal{K}$ onto $\theta(\mathcal{H})$. Denote the standard orthonormal basis for $\mathcal{K}$ by $\{e_j : j \in J\}$. We claim that $Pe_j = \theta(\varphi_j)$. For any $m \in J$, we have

$$\begin{aligned}\langle \theta(\varphi_m), Pe_j\rangle &= \langle P\theta(\varphi_m), e_j\rangle \\ &= \langle \theta(\varphi_m), e_n\rangle \\ &= \langle \varphi_m, \varphi_j\rangle \\ &= \langle \theta(\varphi_m), \theta(\varphi_j)\rangle.\end{aligned}$$

Since the vectors $\theta(\varphi_j)$ span $\theta(\mathcal{H})$, it follows that $Pe_j - \theta(\varphi_j) \perp \theta(\mathcal{H})$. But $ran(P) = \theta(\mathcal{H})$. Hence $Pe_j - \theta(\varphi_j) = 0$, as required. $\square$

Corollary 3.13. *A sequence $\{\varphi_j : j \in J\}$ is a parseval frame for Hilbert C^*-module $\mathcal{H}$ if and only if there exist a Hilbert C^*-module $\mathcal{M}$ and a parseval frame $\{\psi_j : j \in J\}$ for $\mathcal{M}$ such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \bigoplus \mathcal{M}$.*

Proof. By Proposition 3.12 there is a Hilbert C*-module $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_j : j \in J\}$ of $\mathcal{K}$ such that $\varphi_j = Pe_j$, where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$. Let $M = (I-P)K$ and $\psi_j = (I-P)e_j,\ j \in J$. □

Since every orthonormal basis is Parseval frame, the pair of $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ in Corollary 3.13 is strongly disjoint.

Proposition 3.14. *The extension of a tight frame to an orthonormal basis described in the statement of Corollary 3.13 is unique up to unitary equivalence. That is if $\mathcal{N}$ is another Hilbert C^*-module and $\{\phi_j : j \in J\}$ is a tight frame for $\mathcal{N}$ such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \bigoplus \mathcal{N}$, then there is an unitary transformation U mapping M onto N such that $U\psi_j = \phi_j$ for all $j \in J$. In particular, $dim\mathcal{M} = dim\mathcal{N}$.*

Proof. The proof is similar to Proposition 1.4. in [18] for Hilbert space. □

If $\{\varphi_j : j \in J\}$ is a parseval frame, we will call any normalized tight frame $\{\psi_j : j \in J\}$ such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for the direct sum space, as in Proposition 3.14, a strong complementary frame (or strong complement) to $\{\varphi_j : j \in J\}$. The above result says that every parseval frame has a strong complement which is unique up to unitary equivalence.

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