Frames in super Hilbert modules

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Abstract. In this paper, we define super Hilbert module and investigate frames in this space. Super Hilbert modules are generalization of super Hilbert spaces in Hilbert C*-module setting. Also, we define frames in a super Hilbert module and characterize them by using of the concept of g-frames in a Hilbert C*-module. Finally, disjoint frames in Hilbert C*-modules are introduced and investigated.

1. Introduction

Super Hilbert spaces arose naturally as the state space of a quantum field in the functional Schrödinger representation of spinor quantum field theory and it provided a means to bring super symmetric quantum field theories into a form resembling standard quantum mechanics, the super Hilbert space has certain advantages compared with the Hilbert space in quantum mechanics [7, 9, 23, 25]. Balan [4] introduced the concept of super frames and presented some density results for Weyl-Heisenberg super frames. In [18], Han and Larson derived necessary and sufficient conditions for the direct sum of two frames to be a super frame. And in [13], Gu and Han investigated the connection between decomposable Parseval wavelet frames and super wavelet frames and gave some necessary and sufficient conditions for extendable Parseval wavelet frames. In [1, 19, 30] frames, g-frames and g-Riesz frames in super Hilbert spaces has been studied.

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [8] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer.

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and popularized from then on. Today frames have many applications in different subjects, for example in image and signal processing [2, 6] and coherent states [12-13].

Let $H$ be a Hilbert space, and $J$ be a finitely or countably index set. A sequence $\{f_j\}_{j \in J} \subseteq H$ is called a frame for $H$ if there exist the constants $C, D > 0$ such that

$$C \|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq D \|f\|^2;$$

for all $f \in H$. The constants $C$ and $D$ are called frame bounds. If $C = D$ we call this frame a tight frame and if $C = D = 1$ it is called a Parseval frame.

In [29], Sun introduced a generalization of frames and showed the other concepts of generalizations of frames can be presented by g-frames. Also, Sun proved that generalized frames have many properties of frames.

Let $U$ and $V$ be two Hilbert spaces and $\{V_j : j \in J\}$ be a sequence of subspaces of $V$, where $J$ is a subset of $\mathbb{Z}$. $L(U,V_j)$ is the collection of all bounded linear operators from $U$ into $V_j$. The sequence $\{\Lambda_j \in L(U,V_j) : j \in J\}$ is called to be a generalized frame, or simply a g-frame, for $U$ with respect to $\{V_j : j \in J\}$ if there exist two positive constants $C$ and $D$ such that

$$C \|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq D \|f\|^2;$$

for all $f \in U$. The constants $C$ and $D$ are called g-frame bounds. If $C = D$ we call this g-frame a tight g-frame and if $C = D = 1$ it is called a Parseval g-frame.

The notion of frames in Hilbert $C^*$-modules has been introduced and has been investigated in [11]. Frank and Larson [11, 13] defined the standard frames in Hilbert $C^*$-modules in finitely or countably generated Hilbert $C^*$-modules over unital $C^*$-algebras. The extended results of this more general framework are not a routine generalization, because there are essential differences between Hilbert $C^*$-modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert $C^*$-module. Also, there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert $C^*$-modules.

We refer the readers for more details on Hilbert $C^*$-modules and a discussion of basic properties of frames in Hilbert spaces and Hilbert $C^*$-modules and their generalizations to [3, 10, 24, 26-28].
Let $\mathcal{H}$ be a Hilbert $C^*$-module, and $J$ a set which is finite or countable. A sequence $\{f_j\}_{j \in J} \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $C, D > 0$ such that

\[
C \langle f, f \rangle \leq \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle \leq D \langle f, f \rangle,
\]

for all $f \in \mathcal{H}$. The constants $C$ and $D$ are called the frame bounds. Basic properties of frames in Hilbert $C^*$-modules are studied in [10, 16, 17, 22].

A. Khosravi and B. Khosravi [21] defined g-frames in Hilbert $C^*$-modules. Let $U$ and $V$ be two Hilbert $C^*$-modules over the same $C^*$-algebra $A$ and $\{V_j : j \in J\}$ be a sequence of subspaces of $V$, where $J$ is a subset of $\mathbb{Z}$. Let $\text{End}_A(U, V_j)$ be the collection of all adjointable $A$-linear maps from $U$ into $V_j$, i.e. let $T$ be an $A$-linear map from $U$ into $V_j$, if $\langle Tf, g \rangle = \langle f, T^*g \rangle$ for all $f \in U$ and $g \in V_j$ then, it implies that $T \in \text{End}_A(U, V_j)$.

The sequence $\{\Lambda_j \in \text{End}_A(U, V_j) : j \in J\}$ is said to be a generalized frame (or simply a g-frame) for Hilbert $C^*$-module $U$ with respect to $\{V_j : j \in J\}$ if there are two positive constants $C$ and $D$ such that

\[
C \langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D \langle f, f \rangle,
\]

for all $f \in U$. The constants $C$ and $D$ are called g-frame bounds.

In this paper, we define super Hilbert modules that they are generalization of super Hilbert spaces in Hilbert $C^*$-module setting. Also we define frames in a super Hilbert module and characterize them by using g-frames in a Hilbert $C^*$-module. Moreover disjoint frames and complementary frames in Hilbert $C^*$-modules are introduced and are investigated.

2. Preliminaries

Hilbert $C^*$-modules form a wide category between Hilbert spaces and Banach spaces.

**Definition 2.1.** Let $A$ be a $C^*$-algebra with involution $\ast$. An inner product $A$-module (or pre Hilbert $A$-module) is a complex linear space $\mathcal{H}$ which is a left $A$-module with an $A$-valued inner product map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to A$ which satisfies the following properties:

(i) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$;

(ii) $\langle af, g \rangle = a \langle f, g \rangle$ for all $f, g \in \mathcal{H}$ and $a \in A$;

(iii) $\langle f, g \rangle = \langle g, f \rangle^\ast$ for all $f, g \in \mathcal{H}$;

(iv) $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$ and $\langle f, f \rangle = 0$ if and only if $f = 0$. 

For $f \in \mathcal{H}$, we define a norm on $\mathcal{H}$ by $\|f\|_\mathcal{H} = \|\langle f, f \rangle\|_A^{1/2}$. If $\mathcal{H}$ is complete with this norm, it is called a (left) Hilbert $C^*$-module over $A$ or a (left) Hilbert $A$-module.

An element $a$ of a $C^*$-algebra $A$ is positive if $a^* = a$ and its spectrum is a subset of positive real numbers. In this case, we write $a \geq 0$. By the property (4) of the above formula, $\langle f, f \rangle \geq 0$ for every $f \in \mathcal{H}$, hence we define the absolute value of $f$ by $|f| = \langle f, f \rangle^{1/2}$.

For the frame $\{f_j : j \in J\}$ in a Hilbert $A$-module $\mathcal{H}$, the operator $S$ defined by

$$Sf = \sum_{j \in J} \langle f, f_j \rangle f_j, \quad (f \in \mathcal{H}),$$

is called the frame operator. The frame operator $S$ is invertible, positive, adjointable and self-adjoint. Since

$$\langle Sf, f \rangle = \left\langle \sum_{j \in J} \langle f, f_j \rangle f_j, f \right\rangle = \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle, \quad (f \in \mathcal{H}),$$

it follows that

$$C \langle f, f \rangle \leq \langle Sf, f \rangle \leq D \langle f, f \rangle, \quad (f \in \mathcal{H}),$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j \in J} \langle S^{-1}f, f_j \rangle f_j = \sum_{j \in J} \langle f, S^{-1}f_j \rangle f_j,$$

for all $f \in \mathcal{H}$.

Let $\tilde{f}_j = S^{-1}f_j$, then

$$f = \sum_{j \in J} \langle f, \tilde{f}_j \rangle \tilde{f}_j = \sum_{j \in J} \langle f, f_j \rangle \tilde{f}_j,$$

for any $f \in \mathcal{H}$. The sequence $\{\tilde{f}_j : j \in J\}$ is also a frame for $\mathcal{H}$ which is called the canonical dual frame of $\{f_j : j \in J\}$.

Like ordinary frames in Hilbert spaces, the notion of analysis and synthesis operators can be defined as follows:

**Definition 2.2.** Let $\{f_j\}_{j \in J}$ be a frame in Hilbert $A$-module $\mathcal{H}$ over a unital $C^*$-algebra $A$, then the related analysis operator $U : \mathcal{H} \to \ell^2(A)$ is defined by

$$Uf = \{\langle f, f_j \rangle : j \in J\},$$

for all $f \in \mathcal{H}$. We define the synthesis operator $T : \ell^2(A) \to \mathcal{H}$ by

$$T(\{a_j\}) = \sum_{j \in J} a_j f_j,$$

for all $\{a_j\}_{j \in J} \in \ell^2(A)$. 
For any \( g = \{g_j\}_{j \in J} \in \ell^2(A) \) and \( f \in \mathcal{H} \),
\[
\langle Uf, g \rangle = \langle \{\langle f, f_j \rangle\}, \{g_j\} \rangle \\
= \sum_{j \in J} \langle f, f_j \rangle g_j^* \\
= \sum_{j \in J} \langle f, g_j f_j \rangle \\
= \left\langle f, \sum_{j \in J} g_j f_j \right\rangle \\
= \langle f, Tg \rangle,
\]
it follows that \( U \) is adjointable and \( U^* = T \). Also
\[
U^*Uf = U^*(\langle f, f_j \rangle) = \sum_{j \in J} \langle f, f_j \rangle f_j = Sf,
\]
for all \( f \in \mathcal{H} \).

The following theorems characterize frames and Bessel sequences and frames in Hilbert \( A \)-modules.

**Theorem 2.3** ([16]). A sequence \( \{f_j\}_{j \in J} \) in Hilbert \( A \)-module \( \mathcal{H} \) over an unital \( C^* \)-algebra \( A \) is a frame for \( \mathcal{H} \) if and only if the synthesis operator \( T \) is well defined and surjective.

**Corollary 2.4** ([16]). A sequence \( \{f_j\}_{j \in J} \) in Hilbert \( A \)-module \( \mathcal{H} \) over an unital \( C^* \)-algebra \( A \) is a Bessel for \( \mathcal{H} \) if and only if the synthesis operator \( T \) is well defined and \( \|T\| \leq \sqrt{D} \).

The definition of super Hilbert space is given in [9] as following:

**Definition 2.5.** Super Hilbert space \( \mathcal{H} \) is a direct sum \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) of two complex Hilbert spaces \( (\mathcal{H}_0, \langle \cdot, \cdot \rangle_0), (\mathcal{H}_1, \langle \cdot, \cdot \rangle_1) \) equipped with the inner product \( \langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_1 \).

It is easy to see that every super Hilbert space is a Hilbert space.

### 3. Main result

In this section, at first we define super Hilbert module that is a generalization of super Hilbert space in a Hilbert \( C^* \)-module setting. Then we investigate and characterize frames in super Hilbert modules.

**Definition 3.1.** Let \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be two Hilbert \( A \)-modules with inner products \( \langle \cdot, \cdot \rangle_0 \) and \( \langle \cdot, \cdot \rangle_1 \) respectively. Super Hilbert module space \( \mathcal{H} \) is a direct sum \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) equipped with the inner product
\[
\langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_1.
\]
Every direct sum of Hilbert $C^*$-modules is a Hilbert $C^*$-module [24]. Hence every super Hilbert module is a Hilbert $C^*$-module.

**Definition 3.2.** Let $\mathcal{H}$ be a Hilbert $A$-module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a sequence of elements of super Hilbert module $\mathcal{H} \oplus \mathcal{H}$ and $\Lambda_j$ be defined by

$$\Lambda_j f = (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle), \quad \forall f \in \mathcal{H}.$$ 

If $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert $A$-module $\mathcal{H}$ with respect to $A^2$, then we call it a g-frame associated with $\{(\varphi_j, \psi_j) : j \in J\}$.

The following lemma gives a necessary condition for $\{(\varphi_j, \psi_j) : j \in J\}$ to be a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$.

**Lemma 3.3.** Suppose that $\mathcal{H}$ is a Hilbert $C^*$-module and that $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$. Then both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert $C^*$-module $\mathcal{H}$.

**Proof.** Since $\{\Phi_j = (\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$, there exist constants $C, D > 0$ such that

$$C \langle \langle f, f \rangle \rangle \leq \sum_{j \in J} \langle \langle f, \Phi_j \rangle \rangle \langle \langle \Phi_j, f \rangle \rangle \leq D \langle \langle f, f \rangle \rangle,$$

for all $f = (f_1, f_2) \in \mathcal{H} \oplus \mathcal{H}$.

By the definition of the inner product in super Hilbert module, we would have

$$C \langle \langle f_1, f_1 \rangle \rangle + \langle \langle f_2, f_2 \rangle \rangle \leq \sum_{j \in J} \langle \langle f_1, \varphi_j \rangle \rangle + \langle \langle f_2, \psi_j \rangle \rangle \langle \langle \varphi_j, f_1 \rangle \rangle + \langle \langle \psi_j, f_2 \rangle \rangle \leq D \langle \langle f_1, f_1 \rangle \rangle + \langle \langle f_2, f_2 \rangle \rangle,$$

for all $f = (f_1, f_2) \in \mathcal{H} \oplus \mathcal{H}$. Substituting $f_2 = 0$ into the above inequality we obtain

$$C \langle \langle f_1, f_1 \rangle \rangle \leq \sum_{j \in J} \langle \langle f_1, \varphi_j \rangle \rangle \langle \langle \varphi_j, f_1 \rangle \rangle \leq D \langle \langle f_1, f_1 \rangle \rangle,$$

for all $f_1 \in \mathcal{H}$. This means $\{\varphi_j : j \in J\}$ is a frame for Hilbert $C^*$-module $\mathcal{H}$.

The same conclusion can be driven for $\{\psi_j : j \in J\}$ by letting $f_1 = 0$. $\square$

The following theorem gives a necessary condition for a frame in super Hilbert module $\mathcal{H} \oplus \mathcal{H}$ by using g-frames in Hilbert $C^*$-modules.

**Theorem 3.4.** Let $\mathcal{H}$ be a Hilbert $C^*$-module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$. Then $\{\Lambda_j : j \in J\}$ is a g-frame.
for Hilbert A-module with respect to $A^2$ associated with $\{(\varphi_j, \psi_j) : j \in J\}$.

Proof. By Lemma $6.3$, both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert $C^*$-module $\mathcal{H}$. Now by regard to this fact that

$$\langle \Lambda_j f, \Lambda_j f \rangle = \langle \langle f, \varphi_j \rangle, \langle f, \varphi_j \rangle \rangle + \langle \langle f, \psi_j \rangle, \langle f, \psi_j \rangle \rangle = \langle f, \varphi_j \rangle \langle f, \varphi_j \rangle + \langle f, \psi_j \rangle \langle f, \psi_j \rangle,$$

the operator sequence $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to $A^2$ associated with $\{(\varphi_j, \psi_j) : j \in J\}$. $\square$

The following proposition is a generalization of a similar proposition in [30] for super Hilbert modules.

**Proposition 3.5.** Let $\{(\varphi_j, \psi_j) : j \in J\} \subseteq \mathcal{H} \bigoplus \mathcal{H}$ and

$$\Lambda_j f = ((f, \varphi_j), (f, \psi_j))^T,$$

for any $f \in \mathcal{H}$ and $j \in J$. Then $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to $A^2$ associated with $\{(\varphi_j, \psi_j) : j \in J\}$ if and only if $\{\varphi_j : j \in J\} \bigcup \{\psi_j : j \in J\}$ is a frame for Hilbert A-module $\mathcal{H}$.

Proof. Since

$$\langle \Lambda_j f, \Lambda_j f \rangle = \langle f, \varphi_j \rangle \langle f, \varphi_j \rangle + \langle f, \psi_j \rangle \langle f, \psi_j \rangle,$$

we conclude $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert A-module $\mathcal{H}$ with respect to $A^2$ associated with $\{(\varphi_j, \psi_j) : j \in J\}$ if and only if $\{\varphi_j : j \in J\} \bigcup \{\psi_j : j \in J\}$ is a frame for Hilbert A-module $\mathcal{H}$. $\square$

By the previous propositions, we get

**Proposition 3.6.** Suppose $\mathcal{H}$ is a Hilbert A-module. Let $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be two frames for Hilbert A-module $\mathcal{H}$ and $\{\Lambda_j : j \in J\}$ be a g-frames for Hilbert A-module $\mathcal{H}$ with respect to $A^2$ where

$$\Lambda_j f = ((f, \varphi_j), (f, \psi_j))$$

for all $f \in \mathcal{H}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator

$$T : \ell^2(A^2) \rightarrow \mathcal{H},$$

defined by

$$T(\{(a_j, b_j)\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* (a_j, b_j) = \sum_{j \in J} (a_j \varphi_j + b_j \psi_j),$$

for all $\{(a_j, b_j)\}_{j \in J} \in \ell^2(A^2)$.
The analysis operator for \( \{ \Lambda_j : j \in J \} \) is the operator 
\[
T^* : \mathcal{H} \rightarrow l^2(A^2),
\]
defined by 
\[
T^* f = \{ \Lambda_j f \}_{j \in J} = \{ \langle (f, \varphi_j), (f, \psi_j) \rangle \}_{j \in J},
\]
for all \( f \in \mathcal{H} \).

Also the g-frame operator for \( \{ \Lambda_j : j \in J \} \) is the operator 
\[
S_{\Lambda} : \mathcal{H} \rightarrow \mathcal{H},
\]
defined by 
\[
S_{\Lambda} f = T T^* f = \sum_{j \in J} (\langle f, \varphi_j \rangle \varphi_j + \langle f, \psi_j \rangle \psi_j) = S_{\varphi} f + S_{\psi} f,
\]
for all \( f \in \mathcal{H} \).

In the following, we check super Hilbert modules by different space and different C*-algebra.

**Proposition 3.7.** Let \( A \) and \( B \) be unital C*-algebras, \( \{ \varphi_j : j \in J \} \) and \( \{ \psi_j : j \in J \} \) be sequences in Hilbert A-module \( \mathcal{H} \) and Hilbert B-module \( \mathcal{K} \), respectively. Then \( \{ \varphi_j : j \in J \} \) and \( \{ \psi_j : j \in J \} \) are frames in Hilbert A-module \( \mathcal{H} \) and Hilbert B-module \( \mathcal{K} \) respectively if and only if \( \{ \Lambda_j : j \in J \} \) is a g-frame in super Hilbert C*-module \( \mathcal{H} \oplus \mathcal{K} \) with respect to \( A \oplus B \) where \( \Lambda_j(f, g) = \langle (f, \varphi_j), (g, \psi_j) \rangle \) for all \( (f, g) \in \mathcal{H} \oplus \mathcal{K} \).

**Proof.** Since 
\[
\langle \Lambda_j(f, g), \Lambda_j(f, g) \rangle = \langle \langle f, \varphi_j \rangle, \langle g, \psi_j \rangle \rangle = \langle (f, \varphi_j), (g, \psi_j) \rangle \]
\[
= \langle \langle f, \varphi_j \rangle, \langle f, \varphi_j \rangle \rangle + \langle \langle g, \psi_j \rangle, \langle g, \psi_j \rangle \rangle = \langle f, \varphi_j \rangle \langle \varphi_j, f \rangle + \langle g, \psi_j \rangle \langle \psi_j, g \rangle,
\]
then \( \{ \varphi_j : j \in J \} \) and \( \{ \psi_j : j \in J \} \) are frames in Hilbert A-module \( \mathcal{H} \) and Hilbert B-module \( \mathcal{K} \) respectively if and only if \( \{ \Lambda_j : j \in J \} \) is a g-frame in super Hilbert C*-module \( \mathcal{H} \oplus \mathcal{K} \) with respect to \( A \oplus B \). □

In this case, we have the following proposition.

**Proposition 3.8.** Let \( A \) and \( B \) be unital C*-algebras, \( \{ \varphi_j : j \in J \} \) and \( \{ \psi_j : j \in J \} \) be frames in Hilbert A-module \( \mathcal{H} \) and Hilbert B-module \( \mathcal{K} \), respectively and \( \{ \Lambda_j : j \in J \} \) be a g-frame in super Hilbert C*-module \( \mathcal{H} \oplus \mathcal{K} \) with respect to \( A \oplus B \) where \( \Lambda_j(f, g) = \langle (f, \varphi_j), (g, \psi_j) \rangle \) for all \( (f, g) \in \mathcal{H} \oplus \mathcal{K} \). Then the synthesis operator for \( \{ \Lambda_j : j \in J \} \) is the operator 
\[
T : l^2(A \oplus B) \rightarrow \mathcal{H} \oplus \mathcal{K},
\]
defined by
\[
T((a_j, b_j)_{j \in J}) = \sum_{j \in J} \Lambda_j^*(a_j, b_j)
\]
\[
= \left( \sum_{j \in J} a_j \varphi_j, \sum_{j \in J} b_j \psi_j \right),
\]
for all \((a_j, b_j) : j \in J \subseteq A \oplus B.\)

The analysis operator for \(\{\Lambda_j : j \in J\}\) is the operator
\[
T^* : \mathcal{H} \oplus \mathcal{K} \to \ell^2(A \oplus B),
\]
defined by
\[
T^*(f, g) = \{\Lambda_j(f, g)\}_{j \in J}
\]
\[
= \{\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle\}_{j \in J},
\]
for all \((f, g) \in \mathcal{H} \oplus \mathcal{K}.\)

Also the g-frame operator for \(\{\Lambda_j : j \in J\}\) is the operator
\[
S_\Lambda : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K},
\]
defined by
\[
S_\Lambda(f, g) = TT^*(f, g)
\]
\[
= \left( \sum_{j \in J} \langle f, \varphi_j \rangle \varphi_j, \sum_{j \in J} \langle g, \psi_j \rangle \psi_j \right)
\]
\[
= (S_\varphi f, S_\psi g),
\]
for all \((f, g) \in \mathcal{H} \oplus \mathcal{K}.\)

Now we state the following propositions for the case of super Hilbert modules by different spaces and same C*-algebra.

**Proposition 3.9.** Let \(A\) be an unital C*-algebras, \(\{\varphi_j : j \in J\}\) and \(\{\psi_j : j \in J\}\) be sequences in Hilbert \(A\)-modules \(\mathcal{H}\) and \(\mathcal{K}\) respectively. Then \(\{\varphi_j : j \in J\}\) and \(\{\psi_j : j \in J\}\) are frames in Hilbert \(A\)-modules \(\mathcal{H}\) and \(\mathcal{K}\) respectively if and only if \(\{(\varphi_j, \psi_j) : j \in J\}\) is a frame in super Hilbert C*-module \(\mathcal{H} \oplus \mathcal{K}\).

**Proposition 3.10.** Let \(A\) be an unital C*-algebras, \(\{\varphi_j : j \in J\}\) and \(\{\psi_j : j \in J\}\) be sequences in Hilbert \(A\)-modules \(\mathcal{H}\) and \(\mathcal{K}\) respectively. Then \(\{\varphi_j : j \in J\}\) and \(\{\psi_j : j \in J\}\) are frames in Hilbert \(A\)-modules \(\mathcal{H}\) and \(\mathcal{K}\) respectively if and only if \(\{\Lambda_j : j \in J\}\) is a g-frame in super Hilbert C*-module \(\mathcal{H} \oplus \mathcal{K}\) with respect to \(A^2\) where \(\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)\) for all \((f, g) \in \mathcal{H} \oplus \mathcal{K}.\)
Proposition 3.11. Let $A$ be an unital $C^*$-algebra, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be frames in Hilbert $A$-module $\mathcal{H}$ and $\mathcal{K}$ respectively and $\{\Lambda_j : j \in J\}$ be a $g$-frame in super Hilbert $C^*$-module $\mathcal{H} \oplus \mathcal{K}$ with respect to $A$ where $\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)$ for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator
\[
T : \ell^2(A^2) \to \mathcal{H} \oplus \mathcal{K},
\]
defined by
\[
T\left(\{(a_j, b_j)\}_{j \in J}\right) = \sum_{j \in J} \Lambda_j^*(a_j, b_j) = \left(\sum_{j \in J} a_j f_j, \sum_{j \in J} b_j g_j\right),
\]
for all $\{(a_j, b_j) : j \in J\} \subseteq A^2$. The analysis operator for $\{\Lambda_j : j \in J\}$ is the operator
\[
T^* : \mathcal{H} \oplus \mathcal{K} \to \ell^2(A^2),
\]
defined by
\[
T^*(f, g) = \{\Lambda_j(f, g)\}_{j \in J} = \{(f, \varphi_j), (g, \psi_j)\}_{j \in J},
\]
for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Also the $g$-frame operator for $\{\Lambda_j : j \in J\}$ is the operator
\[
S_\Lambda : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K},
\]
defined by
\[
S_\Lambda(f, g) = TT^*(f, g) = \left(\sum_{j \in J} \langle f, \varphi_j \rangle \varphi_j, \sum_{j \in J} \langle g, \psi_j \rangle \psi_j\right) = (S_\varphi f, S_\psi g),
\]
for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert $C^*$-modules on an unital $C^*$-algebra $A$. We say that the frame pairs $\{(\varphi_j), (\psi_j)\} \subset \mathcal{H} \oplus \mathcal{K}$ and $\{(\mu_j), (\nu_j)\} \subset \mathcal{H} \oplus \mathcal{K}$ are similar if there are bounded invertible operators $T_1 \in L(\mathcal{H})$ and $T_2 \in L(\mathcal{K})$ such that $T_1 \varphi_j = \mu_j$ and $T_2 \psi_j = \nu_j$ for all $j \in J$. A pair of frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{K}$. A pair of Parseval frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called strongly disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a Parseval frame for $\mathcal{H} \oplus \mathcal{K}$, and a pair of general
frames \( \{ \varphi_j : j \in J \} \) and \( \{ \psi_j : j \in J \} \) is called strongly disjoint if it is similar to a strongly disjoint pair of parseval frames.

Han and Larson in [18] have proved that \( \{ \varphi_j : j \in J \} \) is a parseval frame in a Hilbert space \( \mathcal{H} \) if and only if there are a Hilbert space \( \mathcal{K} \) and a parseval frame \( \{ \psi_j : j \in J \} \) in \( \mathcal{K} \) such that \( \{ (\varphi_j, \psi_j) : j \in J \} \) is an orthonormal basis for \( \mathcal{H} \oplus \mathcal{K} \). We extend this result for frames in Hilbert \( C^* \)-modules.

The following proposition in Hilbert \( C^* \)-module setting may be proved in much the same way as Proposition 1.1 in [18]. Also, the following results are related to Theorem 4.1, Propositions 5.1 and 5.2 in reference [10].

**Proposition 3.12.** Let \( \mathcal{H} \) be Hilbert \( C^* \)-module on unital \( C^* \)-algebra \( A \). Suppose that \( \{ \varphi_j : j \in J \} \) is a Parseval frame for \( \mathcal{H} \). Then there exist a Hilbert \( C^* \)-module \( \mathcal{M} \) and an orthonormal basis \( \{ e_j : j \in J \} \) for \( \mathcal{M} \) such that \( \varphi_j = Pe_j \), where \( P \) is the orthogonal projection from \( \mathcal{K} \) to \( \mathcal{H} \).

**Proof.** Let \( \mathcal{K} = \ell^2(A) \) and let \( \theta : \mathcal{H} \to \mathcal{K} \) be defined by

\[
\theta(f) = \{ \langle f, \varphi_j \rangle : j \in J \},
\]

for all \( f \in \mathcal{H} \). Since \( \{ \varphi_j : j \in J \} \) is a Parseval frame for Hilbert \( C^* \)-module \( \mathcal{H} \), we have

\[
\| \theta(f) \|^2 = \sum_{j \in J} |\langle f, \varphi_j \rangle|^2 = \| f \|^2.
\]

Thus \( \theta \) is well defined and is an isometry. So we can embed \( \mathcal{H} \) into \( \mathcal{K} \) by identifying \( \mathcal{H} \) with \( \theta(\mathcal{H}) \). Let \( P \) be the orthogonal projection from \( \mathcal{K} \) onto \( \theta(\mathcal{H}) \). Denote the standard orthonormal basis for \( \mathcal{K} \) by \( \{ e_j : j \in J \} \). We claim that \( Pe_j = \theta(\varphi_j) \). For any \( m \in J \), we have

\[
\langle \theta(\varphi_m), Pe_j \rangle = \langle P\theta(\varphi_m), e_j \rangle = \langle \theta(\varphi_m), e_n \rangle = \langle \varphi_m, \varphi_j \rangle = \langle \theta(\varphi_m), \theta(\varphi_j) \rangle.
\]

Since the vectors \( \theta(\varphi_j) \) span \( \theta(\mathcal{H}) \), it follows that \( Pe_j - \theta(\varphi_j) \perp \theta(\mathcal{H}) \). But \( \text{ran}(P) = \theta(\mathcal{H}) \). Hence \( Pe_j - \theta(\varphi_j) = 0 \), as required. \( \square \)

**Corollary 3.13.** A sequence \( \{ \varphi_j : j \in J \} \) is a parseval frame for Hilbert \( C^* \)-module \( \mathcal{H} \) if and only if there exist a Hilbert \( C^* \)-module \( \mathcal{M} \) and a parseval frame \( \{ \psi_j : j \in J \} \) for \( \mathcal{M} \) such that \( \{ (\varphi_j, \psi_j) : j \in J \} \) is an orthonormal basis for \( \mathcal{H} \oplus \mathcal{M} \).

**Proof.** By Proposition 3.12 there is a Hilbert \( C^* \)-module \( \mathcal{K} \supseteq \mathcal{H} \) and an orthonormal basis \( \{ e_j : j \in J \} \) of \( \mathcal{K} \) such that \( \varphi_j = Pe_j \), where \( P \) is the
projection from $\mathcal{K}$ onto $\mathcal{H}$. Let $M = (I - P)K$ and $\psi_j = (I - P)e_j$, $j \in J$.

Since every orthonormal basis is Parseval frame, the pair of $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ in Corollary is strongly disjoint.

**Proposition 3.14.** The extension of a tight frame to an orthonormal basis described in the statement of Corollary is unique up to unitary equivalence. That is if $\mathcal{N}$ is another Hilbert $C^*$-module and $\{\phi_j : j \in J\}$ is a tight frame for $\mathcal{N}$ such that $\{\langle \varphi_j, \psi_j \rangle : j \in J\}$ is an orthonormal basis for $\mathcal{H} \oplus \mathcal{N}$, then there is an unitary transformation $U$ mapping $M$ onto $\mathcal{N}$ such that $U\psi_j = \phi_j$ for all $j \in J$. In particular, $\dim M = \dim \mathcal{N}$.

**Proof.** The proof is similar to Proposition 1.4. in [18] for Hilbert space.

If $\{\varphi_j : j \in J\}$ is a Parseval frame, we will call any normalized tight frame $\{\psi_j : j \in J\}$ such that $\{\langle \varphi_j, \psi_j \rangle : j \in J\}$ is an orthonormal basis for the direct sum space, as in Proposition, a strong complementary frame (or strong complement) to $\{\varphi_j : j \in J\}$. The above result says that every Parseval frame has a strong complement which is unique up to unitary equivalence.

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**References**


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