

Frames in super Hilbert modules

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ABSTRACT. In this paper, we define super Hilbert module and investigate frames in this space. Super Hilbert modules are generalization of super Hilbert spaces in Hilbert C^* -module setting. Also, we define frames in a super Hilbert module and characterize them by using of the concept of g -frames in a Hilbert C^* -module. Finally, disjoint frames in Hilbert C^* -modules are introduced and investigated.

1. INTRODUCTION

Super Hilbert spaces arose naturally as the state space of a quantum field in the functional Schrodinger representation of spinor quantum field theory and it provided a means to bring super symmetric quantum field theories into a form resembling standard quantum mechanics, the super Hilbert space has certain advantages compared with the Hilbert space in quantum mechanics [7, 9, 23, 25]. Balan [4] introduced the concept of super frames and presented some density results for Weyl-Heisenberg super frames. In [18], Han and Larson derived necessary and sufficient conditions for the direct sum of two frames to be a super frame. And in [15], Gu and Han investigated the connection between decomposable Parseval wavelet frames and super wavelet frames and gave some necessary and sufficient conditions for extendable Parseval wavelet frames. In [1, 19, 30] frames, g -frames and g -Riesz frames in super Hilbert spaces has been studied.

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [8] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer

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[6], and popularized from then on. Today frames have many applications in different subjects, for example in image and signal processing [2, 3] and coherent states [12–14]

Let H be a Hilbert space, and J be a finitely or countably index set. A sequence $\{f_j\}_{j \in J} \subseteq H$ is called a frame for H if there exist the constants $C, D > 0$ such that

$$(1.1) \quad C\|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq D\|f\|^2,$$

for all $f \in H$. The constants C and D are called frame bounds. If $C = D$ we call this frame a tight frame and if $C = D = 1$ it is called a Parseval frame.

In [29], Sun introduced a generalization of frames and showed the other concepts of generalizations of frames can be presented by g-frames. Also, Sun proved that generalized frames have many properties of frames.

Let U and V be two Hilbert spaces and $\{V_j : j \in J\}$ be a sequence of subspaces of V , where J is a subset of \mathbb{Z} . $L(U, V_j)$ is the collection of all bounded linear operators from U into V_j . The sequence $\{\Lambda_j \in L(U, V_j) : j \in J\}$ is called to be a generalized frame, or simply a g-frame, for U with respect to $\{V_j : j \in J\}$ if there exist two positive constants C and D such that

$$(1.2) \quad C\|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq D\|f\|^2,$$

for all $f \in U$. The constants C and D are called g-frame bounds. If $C = D$ we call this g-frame a tight g-frame and if $C = D = 1$ it is called a Parseval g-frame.

The notion of frames in Hilbert C^* -modules has been introduced and has been investigated in [11]. Frank and Larson [10, 11] defined the standard frames in Hilbert C^* -modules in finitely or countably generated Hilbert C^* -modules over unital C^* -algebras. The extended results of this more general framework are not a routine generalization, because there are essential differences between Hilbert C^* -modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^* -module. Also, there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^* -modules.

We refer the readers for more details on Hilbert C^* -modules and a discussion of basic properties of frames in Hilbert spaces and Hilbert C^* -modules and their generalizations to [5, 10, 24, 26–28].

Let \mathcal{H} be a Hilbert C^* -module, and J a set which is finite or countable. A sequence $\{f_j\}_{j \in J} \subseteq \mathcal{H}$ is called a frame for \mathcal{H} if there exist constants $C, D > 0$ such that

$$(1.3) \quad C\langle f, f \rangle \leq \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle \leq D\langle f, f \rangle,$$

for all $f \in \mathcal{H}$. The constants C and D are called the frame bounds. Basic properties of frames in Hilbert C^* -modules are studied in [10, 16, 17, 22].

A. Khosravi and B. Khosravi [21] defined g-frames in Hilbert C^* -modules. Let U and V be two Hilbert C^* -modules over the same C^* -algebra A and $\{V_j : j \in J\}$ be a sequence of subspaces of V , where J is a subset of \mathbb{Z} . Let $End_A^*(U, V_j)$ be the collection of all adjointable A -linear maps from U into V_j , i.e. let T be an A -linear map from U into V_j , if $\langle Tf, g \rangle = \langle f, T^*g \rangle$ for all $f \in U$ and $g \in V_j$ then, it implies that $T \in End_A^*(U, V_j)$. The sequence $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ is said to be a generalized frame (or simply a g-frame) for Hilbert C^* -module U with respect to $\{V_j : j \in J\}$ if there are two positive constants C and D such that

$$(1.4) \quad C\langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D\langle f, f \rangle,$$

for all $f \in U$. The constants C and D are called g-frame bounds.

In this paper, we define super Hilbert modules that they are generalization of super Hilbert spaces in Hilbert C^* -module setting. Also we define frames in a super Hilbert module and characterize them by using g-frames in a Hilbert C^* -module. Moreover disjoint frames and complementary frames in Hilbert C^* -modules are introduced and are investigated.

2. PRELIMINARIES

Hilbert C^* -modules form a wide category between Hilbert spaces and Banach spaces.

Definition 2.1. Let A be a C^* -algebra with involution $*$. An inner product A -module (or pre Hilbert A -module) is a complex linear space \mathcal{H} which is a left A -module with an A -valued inner product map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow A$ which satisfies the following properties:

- (i) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$;
- (ii) $\langle af, g \rangle = a \langle f, g \rangle$ for all $f, g \in \mathcal{H}$ and $a \in A$;
- (iii) $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in \mathcal{H}$;
- (iv) $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$ and $\langle f, f \rangle = 0$ if and only if $f = 0$.

For $f \in \mathcal{H}$, we define a norm on \mathcal{H} by $\|f\|_{\mathcal{H}} = \|\langle f, f \rangle\|_A^{1/2}$. If \mathcal{H} is complete with this norm, it is called a (left) Hilbert C^* -module over A or a (left) Hilbert A -module.

An element a of a C^* -algebra A is positive if $a^* = a$ and its spectrum is a subset of positive real numbers. In this case, we write $a \geq 0$. By the property (4) of the above formula, $\langle f, f \rangle \geq 0$ for every $f \in \mathcal{H}$, hence we define the absolute value of f by $|f| = \langle f, f \rangle^{1/2}$.

For the frame $\{f_j : j \in J\}$ in a Hilbert A -module \mathcal{H} , the operator S defined by

$$Sf = \sum_{j \in J} \langle f, f_j \rangle f_j, \quad (f \in \mathcal{H}),$$

is called the frame operator. The frame operator S is invertible, positive, adjointable and self-adjoint. Since

$$\langle Sf, f \rangle = \left\langle \sum_{j \in J} \langle f, f_j \rangle f_j, f \right\rangle = \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle, \quad (f \in \mathcal{H}),$$

it follows that

$$C\langle f, f \rangle \leq \langle Sf, f \rangle \leq D\langle f, f \rangle, \quad (f \in \mathcal{H}),$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j \in J} \langle S^{-1}f, f_j \rangle f_j = \sum_{j \in J} \langle f, S^{-1}f_j \rangle f_j,$$

for all $f \in \mathcal{H}$.

Let $\tilde{f}_j = S^{-1}f_j$, then

$$f = \sum_{j \in J} \langle f, \tilde{f}_j \rangle f_j = \sum_{j \in J} \langle f, f_j \rangle \tilde{f}_j,$$

for any $f \in \mathcal{H}$. The sequence $\{\tilde{f}_j : j \in J\}$ is also a frame for \mathcal{H} which is called the canonical dual frame of $\{f_j : j \in J\}$.

Like ordinary frames in Hilbert spaces, the notion of analysis and synthesis operators can be defined as follows:

Definition 2.2. Let $\{f_j\}_{j \in J}$ be a frame in Hilbert A -module \mathcal{H} over a unital C^* -algebra A , then the related analysis operator $U : \mathcal{H} \rightarrow \ell^2(A)$ is defined by

$$Uf = \{\langle f, f_j \rangle : j \in J\},$$

for all $f \in \mathcal{H}$. We define the synthesis operator $T : \ell^2(A) \rightarrow \mathcal{H}$ by

$$T(\{a_j\}) = \sum_{j \in J} a_j f_j,$$

for all $\{a_j\}_{j \in J} \in \ell^2(A)$.

For any $g = \{g_j\}_{j \in J} \in \ell^2(A)$ and $f \in \mathcal{H}$,

$$\begin{aligned} \langle Uf, g \rangle &= \langle \{\langle f, f_j \rangle\}, \{g_j\} \rangle \\ &= \sum_{j \in J} \langle f, f_j \rangle g_j^* \\ &= \sum_{j \in J} \langle f, g_j f_j \rangle \\ &= \left\langle f, \sum_{j \in J} g_j f_j \right\rangle \\ &= \langle f, Tg \rangle, \end{aligned}$$

it follows that U is adjointable and $U^* = T$. Also

$$U^*Uf = U^*(\langle f, f_j \rangle) = \sum_{j \in J} \langle f, f_j \rangle f_j = Sf,$$

for all $f \in \mathcal{H}$.

The following theorems characterize frames and Bessel sequences and frames in Hilbert A -modules.

Theorem 2.3 ([16]). *A sequence $\{f_j\}_{j \in J}$ in Hilbert A -module \mathcal{H} over an unital C^* -algebra A is a frame for \mathcal{H} if and only if the synthesis operator T is well defined and surjective.*

Corollary 2.4 ([16]). *A sequence $\{f_j\}_{j \in J}$ in Hilbert A -module \mathcal{H} over an unital C^* -algebra A is a Bessel for \mathcal{H} if and only if the synthesis operator T is well defined and $\|T\| \leq \sqrt{D}$.*

The definition of super Hilbert space is given in [9] as following:

Definition 2.5. Super Hilbert space \mathcal{H} is a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of two complex Hilbert spaces $(\mathcal{H}_0, \langle \cdot, \cdot \rangle_0)$, $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$ equipped with the inner product $\langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_1$.

It is easy to see that every super Hilbert space is a Hilbert space.

3. MAIN RESULT

In this section, at first we define super Hilbert module that is a generalization of super Hilbert space in a Hilbert C^* -module setting. Then we investigate and characterize frames in super Hilbert modules.

Definition 3.1. Let \mathcal{H}_0 and \mathcal{H}_1 be two Hilbert A -modules with inner products $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ respectively. Super Hilbert module space \mathcal{H} is a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ equipped with the inner product

$$\langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle_0 + \langle \cdot, \cdot \rangle_1.$$

Every direct sum of Hilbert C^* -modules is a Hilbert C^* -modules [24]. Hence every super Hilbert module is a Hilbert C^* -module.

Definition 3.2. Let \mathcal{H} be a Hilbert A -module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a sequence of elements of super Hilbert module $\mathcal{H} \oplus \mathcal{H}$ and Λ_j be defined by

$$\Lambda_j f = (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle), \quad \forall f \in \mathcal{H}.$$

If $\{\Lambda_j : j \in J\}$ is a g -frame for Hilbert A -module \mathcal{H} with respect to A^2 , then we call it a g -frame associated with $\{(\varphi_j, \psi_j) : j \in J\}$.

The following lemma gives a necessary condition for $\{(\varphi_j, \psi_j) : j \in J\}$ to be a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$.

Lemma 3.3. *Suppose that \mathcal{H} is a Hilbert C^* -module and that $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$. Then both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert C^* -module \mathcal{H} .*

Proof. Since $\{\Phi_j = (\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$, there exist constants $C, D > 0$ such that

$$C \langle \langle f, f \rangle \rangle \leq \sum_{j \in J} \langle \langle f, \Phi_j \rangle \rangle \langle \langle \Phi_j, f \rangle \rangle \leq D \langle \langle f, f \rangle \rangle,$$

for all $f = (f_1, f_2) \in \mathcal{H} \oplus \mathcal{H}$.

By the definition of the inner product in super Hilbert module, we would have

$$\begin{aligned} C (\langle f_1, f_1 \rangle + \langle f_2, f_2 \rangle) &\leq \sum_{j \in J} (\langle f_1, \varphi_j \rangle + \langle f_2, \psi_j \rangle) (\langle \varphi_j, f_1 \rangle + \langle \psi_j, f_2 \rangle) \\ &\leq D (\langle f_1, f_1 \rangle + \langle f_2, f_2 \rangle), \end{aligned}$$

for all $f = (f_1, f_2) \in \mathcal{H} \oplus \mathcal{H}$. Substituting $f_2 = 0$ into the above inequality we obtain

$$C \langle f_1, f_1 \rangle \leq \sum_{j \in J} \langle f_1, \varphi_j \rangle \langle \varphi_j, f_1 \rangle \leq D \langle f_1, f_1 \rangle,$$

for all $f_1 \in \mathcal{H}$. This means $\{\varphi_j : j \in J\}$ is a frame for Hilbert C^* -module \mathcal{H} .

The same conclusion can be driven for $\{\psi_j : j \in J\}$ by letting $f_1 = 0$. \square

The following theorem gives a necessary condition for a frame in super Hilbert module $\mathcal{H} \oplus \mathcal{H}$ by using g -frames in Hilbert C^* -modules.

Theorem 3.4. *Let \mathcal{H} be a Hilbert C^* -module and $\{(\varphi_j, \psi_j) : j \in J\}$ be a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{H}$. Then $\{\Lambda_j : j \in J\}$ is a g -frame*

for Hilbert A -module with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$.

Proof. By Lemma 3.3, both $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames for Hilbert C^* -module \mathcal{H} . Now by regard to this fact that

$$\begin{aligned} \langle \Lambda_j f, \Lambda_j f \rangle &= \langle (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle), (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle) \rangle \\ &= \langle \langle f, \varphi_j \rangle, \langle f, \varphi_j \rangle \rangle + \langle \langle f, \psi_j \rangle, \langle f, \psi_j \rangle \rangle \\ &= \langle f, \varphi_j \rangle \langle \varphi_j, f \rangle + \langle f, \psi_j \rangle \langle \psi_j, f \rangle, \end{aligned}$$

the operator sequence $\{\Lambda_j : j \in J\}$ is a g -frame for Hilbert A -module \mathcal{H} with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$. \square

The following proposition is a generalization of a similar proposition in [30] for super Hilbert modules.

Proposition 3.5. *Let $\{(\varphi_j, \psi_j) : j \in J\} \subseteq \mathcal{H} \oplus \mathcal{H}$ and*

$$\Lambda_j f = (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle)^T,$$

for any $f \in \mathcal{H}$ and $j \in J$. Then $\{\Lambda_j : j \in J\}$ is a g -frame for Hilbert A -module \mathcal{H} with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$ if and only if $\{\varphi_j : j \in J\} \cup \{\psi_j : j \in J\}$ is a frame for Hilbert A -module \mathcal{H} .

Proof. Since

$$\langle \Lambda_j f, \Lambda_j f \rangle = \langle f, \varphi_j \rangle \langle \varphi_j, f \rangle + \langle f, \psi_j \rangle \langle \psi_j, f \rangle,$$

we conclude $\{\Lambda_j : j \in J\}$ is a g -frame for Hilbert A -module \mathcal{H} with respect to A^2 associated with $\{(\varphi_j, \psi_j) : j \in J\}$ if and only if $\{\varphi_j : j \in J\} \cup \{\psi_j : j \in J\}$ is a frame for Hilbert A -module \mathcal{H} . \square

By the previous propositions, we get

Proposition 3.6. *Suppose \mathcal{H} is a Hilbert A -module. Let $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be two frames for Hilbert A -module \mathcal{H} and $\{\Lambda_j : j \in J\}$ be a g -frames for Hilbert A -module \mathcal{H} with respect to A^2 where $\Lambda_j f = (\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle)$ for all $f \in \mathcal{H}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator*

$$T : \ell^2(A^2) \rightarrow \mathcal{H},$$

defined by

$$\begin{aligned} T(\{(a_j, b_j)\}_{j \in J}) &= \sum_{j \in J} \Lambda_j^*(a_j, b_j) \\ &= \sum_{j \in J} (a_j \varphi_j + b_j \psi_j), \end{aligned}$$

for all $\{(a_j, b_j)\}_{j \in J} \in \ell^2(A^2)$.

The analysis operator for $\{\Lambda_j : j \in J\}$ is the operator

$$T^* : \mathcal{H} \rightarrow \ell^2(A^2),$$

defined by

$$T^*f = \{\Lambda_j f\}_{j \in J} = \{(\langle f, \varphi_j \rangle, \langle f, \psi_j \rangle)\}_{j \in J},$$

for all $f \in \mathcal{H}$.

Also the g -frame operator for $\{\Lambda_j : j \in J\}$ is the operator

$$S_\Lambda : \mathcal{H} \rightarrow \mathcal{H},$$

defined by

$$\begin{aligned} S_\Lambda f &= TT^*f \\ &= \sum_{j \in J} (\langle f, \varphi_j \rangle \varphi_j + \langle f, \psi_j \rangle \psi_j) \\ &= S_\varphi f + S_\psi f, \end{aligned}$$

for all $f \in \mathcal{H}$.

In the following, we check super Hilbert modules by different space and different C^* -algebra.

Proposition 3.7. *Let A and B be unital C^* -algebras, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be sequences in Hilbert A -module \mathcal{H} and Hilbert B -module \mathcal{K} , respectively. Then $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames in Hilbert A -module \mathcal{H} and Hilbert B -module \mathcal{K} respectively if and only if $\{\Lambda_j : j \in J\}$ is a g -frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$ with respect to $A \oplus B$ where $\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)$ for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.*

Proof. Since

$$\begin{aligned} \langle \Lambda_j(f, g), \Lambda_j(f, g) \rangle &= \langle (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle), (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle) \rangle \\ &= \langle \langle f, \varphi_j \rangle, \langle f, \varphi_j \rangle \rangle + \langle \langle g, \psi_j \rangle, \langle g, \psi_j \rangle \rangle \\ &= \langle f, \varphi_j \rangle \langle \varphi_j, f \rangle + \langle g, \psi_j \rangle \langle \psi_j, g \rangle, \end{aligned}$$

then $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames in Hilbert A -module \mathcal{H} and Hilbert B -module \mathcal{K} respectively if and only if $\{\Lambda_j : j \in J\}$ is a g -frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$ with respect to $A \oplus B$. \square

In this case, we have the following proposition.

Proposition 3.8. *Let A and B be unital C^* -algebras, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be frames in Hilbert A -module \mathcal{H} and Hilbert B -module \mathcal{K} respectively and $\{\Lambda_j : j \in J\}$ be a g -frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$ with respect to $A \oplus B$ where $\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)$ for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator*

$$T : \ell^2(A \oplus B) \rightarrow \mathcal{H} \oplus \mathcal{K},$$

defined by

$$\begin{aligned} T(\{(a_j, b_j)\}_{j \in J}) &= \sum_{j \in J} \Lambda_j^*(a_j, b_j) \\ &= \left(\sum_{j \in J} a_j \varphi_j, \sum_{j \in J} b_j \psi_j \right), \end{aligned}$$

for all $\{(a_j, b_j) : j \in J\} \subseteq A \oplus B$.

The analysis operator for $\{\Lambda_j : j \in J\}$ is the operator

$$T^* : \mathcal{H} \oplus \mathcal{K} \rightarrow \ell^2(A \oplus B),$$

defined by

$$\begin{aligned} T^*(f, g) &= \{\Lambda_j(f, g)\}_{j \in J} \\ &= \{\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle\}_{j \in J}, \end{aligned}$$

for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.

Also the g -frame operator for $\{\Lambda_j : j \in J\}$ is the operator

$$S_\Lambda : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K},$$

defined by

$$\begin{aligned} S_\Lambda(f, g) &= TT^*(f, g) \\ &= \left(\sum_{j \in J} \langle f, \varphi_j \rangle \varphi_j, \sum_{j \in J} \langle g, \psi_j \rangle \psi_j \right) \\ &= (S_\varphi f, S_\psi g), \end{aligned}$$

for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.

Now we state the following propositions for the case of super Hilbert modules by different spaces and same C^* -algebra.

Proposition 3.9. *Let A be an unital C^* -algebra, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be sequences in Hilbert A -modules \mathcal{H} and \mathcal{K} respectively. Then $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames in Hilbert A -modules \mathcal{H} and \mathcal{K} respectively if and only if $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$.*

Proposition 3.10. *Let A be an unital C^* -algebra, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be sequences in Hilbert A -modules \mathcal{H} and \mathcal{K} respectively. Then $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are frames in Hilbert A -modules \mathcal{H} and \mathcal{K} respectively if and only if $\{\Lambda_j : j \in J\}$ is a g -frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$ with respect to A^2 where $\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)$ for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.*

Proposition 3.11. *Let A be an unital C^* -algebra, $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ be frames in Hilbert A -module \mathcal{H} and \mathcal{K} respectively and $\{\Lambda_j : j \in J\}$ be a g -frame in super Hilbert C^* -module $\mathcal{H} \oplus \mathcal{K}$ with respect to A where $\Lambda_j(f, g) = (\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle)$ for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Then the synthesis operator for $\{\Lambda_j : j \in J\}$ is the operator*

$$T : \ell^2(A^2) \rightarrow \mathcal{H} \oplus \mathcal{K},$$

defined by

$$\begin{aligned} T(\{(a_j, b_j)\}_{j \in J}) &= \sum_{j \in J} \Lambda_j^*(a_j, b_j) \\ &= \left(\sum_{j \in J} a_j \varphi_j, \sum_{j \in J} b_j \psi_j \right), \end{aligned}$$

for all $\{(a_j, b_j) : j \in J\} \subseteq A^2$. The analysis operator for $\{\Lambda_j : j \in J\}$ is the operator

$$T^* : \mathcal{H} \oplus \mathcal{K} \rightarrow \ell^2(A^2),$$

defined by

$$\begin{aligned} T^*(f, g) &= \{\Lambda_j(f, g)\}_{j \in J} \\ &= \{\langle f, \varphi_j \rangle, \langle g, \psi_j \rangle\}_{j \in J}, \end{aligned}$$

for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Also the g -frame operator for $\{\Lambda_j : j \in J\}$ is the operator

$$S_\Lambda : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K},$$

defined by

$$\begin{aligned} S_\Lambda(f, g) &= TT^*(f, g) \\ &= \left(\sum_{j \in J} \langle f, \varphi_j \rangle \varphi_j, \sum_{j \in J} \langle g, \psi_j \rangle \psi_j \right) \\ &= (S_\varphi f, S_\psi g), \end{aligned}$$

for all $(f, g) \in \mathcal{H} \oplus \mathcal{K}$.

Let \mathcal{H} and \mathcal{K} be Hilbert C^* -modules on an unital C^* -algebra A . We say that the frame pairs $(\{\varphi_j\}, \{\psi_j\}) \subset \mathcal{H} \oplus \mathcal{K}$ and $(\{\mu_j\}, \{\nu_j\}) \subset \mathcal{H} \oplus \mathcal{K}$ are similar if there are bounded invertible operators $T_1 \in L(\mathcal{H})$ and $T_2 \in L(\mathcal{K})$ such that $T_1 \varphi_j = \mu_j$ and $T_2 \psi_j = \nu_j$ for all $j \in J$. A pair of frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a frame for super Hilbert module $\mathcal{H} \oplus \mathcal{K}$. A pair of parseval frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called strongly disjoint if $\{(\varphi_j, \psi_j) : j \in J\}$ is a parseval frame for $\mathcal{H} \oplus \mathcal{K}$, and a pair of general

frames $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ is called strongly disjoint if it is similar to a strongly disjoint pair of parseval frames.

Han and Larson in [18] have proved that $\{\varphi_j : j \in J\}$ is a parseval frame in a Hilbert space \mathcal{H} if and only if there are a Hilbert space \mathcal{K} and a parseval frame $\{\psi_j : j \in J\}$ in \mathcal{K} such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \oplus \mathcal{K}$. We extend this result for frames in Hilbert C^* -modules.

The following proposition in Hilbert C^* -module setting may be proved in much the same way as Proposition 1.1 in [18]. Also, the following results are related to Theorem 4.1, Propositions 5.1 and 5.2 in reference [10].

Proposition 3.12. *Let \mathcal{H} be Hilbert C^* -module on unital C^* -algebra A . Suppose that $\{\varphi_j : j \in J\}$ is a Parseval frame for \mathcal{H} . Then there exist a Hilbert C^* -module $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_j : j \in J\}$ for \mathcal{K} such that $\varphi_j = Pe_j$, where P is the orthogonal projection from \mathcal{K} to \mathcal{H} .*

Proof. Let $\mathcal{K} = \ell^2(A)$ and let $\theta : \mathcal{H} \rightarrow \mathcal{K}$ be defined by

$$\theta(f) = \{\langle f, \varphi_j \rangle : j \in J\},$$

for all $f \in \mathcal{H}$. Since $\{\varphi_j : j \in J\}$ is a Parseval frame for Hilbert C^* -module \mathcal{H} , we have

$$\|\theta(f)\|^2 = \sum_{j \in J} |\langle f, \varphi_j \rangle|^2 = \|f\|^2.$$

Thus θ is well defined and is an isometry. So we can embed \mathcal{H} into \mathcal{K} by identifying \mathcal{H} with $\theta(\mathcal{H})$. Let P be the orthogonal projection from \mathcal{K} onto $\theta(\mathcal{H})$. Denote the standard orthonormal basis for \mathcal{K} by $\{e_j : j \in J\}$. We claim that $Pe_j = \theta(\varphi_j)$. For any $m \in J$, we have

$$\begin{aligned} \langle \theta(\varphi_m), Pe_j \rangle &= \langle P\theta(\varphi_m), e_j \rangle \\ &= \langle \theta(\varphi_m), e_n \rangle \\ &= \langle \varphi_m, \varphi_j \rangle \\ &= \langle \theta(\varphi_m), \theta(\varphi_j) \rangle. \end{aligned}$$

Since the vectors $\theta(\varphi_j)$ span $\theta(\mathcal{H})$, it follows that $Pe_j - \theta(\varphi_j) \perp \theta(\mathcal{H})$. But $\text{ran}(P) = \theta(\mathcal{H})$. Hence $Pe_j - \theta(\varphi_j) = 0$, as required. \square

Corollary 3.13. *A sequence $\{\varphi_j : j \in J\}$ is a parseval frame for Hilbert C^* -module \mathcal{H} if and only if there exist a Hilbert C^* -module \mathcal{M} and a parseval frame $\{\psi_j : j \in J\}$ for \mathcal{M} such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \oplus \mathcal{M}$.*

Proof. By Proposition 3.12 there is a Hilbert C^* -module $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_j : j \in J\}$ of \mathcal{K} such that $\varphi_j = Pe_j$, where P is the

projection from \mathcal{K} onto \mathcal{H} . Let $M = (I - P)K$ and $\psi_j = (I - P)e_j$, $j \in J$. \square

Since every orthonormal basis is Parseval frame, the pair of $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ in Corollary 3.13 is strongly disjoint.

Proposition 3.14. *The extension of a tight frame to an orthonormal basis described in the statement of Corollary 3.13 is unique up to unitary equivalence. That is if \mathcal{N} is another Hilbert C^* -module and $\{\phi_j : j \in J\}$ is a tight frame for \mathcal{N} such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for $\mathcal{H} \oplus \mathcal{N}$, then there is an unitary transformation U mapping M onto N such that $U\psi_j = \phi_j$ for all $j \in J$. In particular, $\dim \mathcal{M} = \dim \mathcal{N}$.*

Proof. The proof is similar to Proposition 1.4. in [18] for Hilbert space. \square

If $\{\varphi_j : j \in J\}$ is a parseval frame, we will call any normalized tight frame $\{\psi_j : j \in J\}$ such that $\{(\varphi_j, \psi_j) : j \in J\}$ is an orthonormal basis for the direct sum space, as in Proposition 3.14, a strong complementary frame (or strong complement) to $\{\varphi_j : j \in J\}$. The above result says that every parseval frame has a strong complement which is unique up to unitary equivalence.

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REFERENCES

1. A. Abdollahi and E. Rahimi, *Generalized frames on super Hilbert spaces*, Bull. Malays. Math. Sci. Soc., 35 (2012), pp. 807-818.
2. A. Arefijamaal and E. Zekae, *Image processing by alternate dual Gabor frames*, Bull. Iranian Math. Soc. to appear (2016).
3. A. Arefijamaal and E. Zekae, *Signal processing by alternate dual Gabor frames*, Appl. Comput. Harmon. Anal., 35 (2013), pp. 535-540.
4. R. Balan, *Density and Redundancy of the Noncoherent Weyl-Heisenberg Superframes*, Contemp. Math., 247 1999.
5. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, Boston, 2003.
6. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27 (1986), pp. 1271-1283.
7. B. DeWitt, *Supermanifolds*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, New York, NY, USA, 2nd edition, (1992).
8. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), pp. 341-366.

9. A.M. El Gradechi and L.M. Nieto, *Supercoherent states, super Kaler geometry and geometric quantization*, Comm. Math. Phys., 175 (1996), pp. 521-563.
10. M. Frank and D.R. Larson, *Frames in Hilbert C^* -modules and C^* -algebras*, J. Operator Theory, 48 (2002), pp. 273-314.
11. M. Frank and D.R. Larson, *A module frame concept for Hilbert C^* -modules*, Functional and Harmonic Analysis of Wavelets, San Antonio, TX, January 1999, Contemp. Math. 247, Amer. Math. Soc., Providence, RI 2000, 207-233.
12. A. Ghaani Farashahi, *Abstract harmonic analysis of wave-packet transforms over locally compact abelian groups*, Banach J. Math. Anal., 11 (2017), pp. 50-71.
13. A. Ghaani Farashahi, *Continuous partial Gabor transform for semi-direct product of locally compact groups*, Bulletin of the Malaysian Mathematical Sciences Society, 38 (2015), pp. 779-803.
14. A. Ghaani Farashahi and R. Kamyabi-Gol, *Continuous Gabor transform for a class of non-Abelian groups*, Bulletin of the Belgian Mathematical Society-Simon Stevin, 19 (2012), pp. 683-701.
15. Q. Gu and D. Han, *Super-wavelets and decomposable wavelet frames*, The Journal of Fourier Analysis and Applications, 11 (2005), pp. 683-696.
16. D. Han, W. Jing, D. Larson, and R. Mohapatra, *Riesz bases and their dual modular frames in Hilbert C^* -modules*, J. Math. Anal. Appl. 343 (2008), pp. 246-256.
17. D. Han, W. Jing, and R. Mohapatra, *Perturbation of frames and Riesz bases in Hilbert C^* -modules*, Linear Algebra Appl. 431 (2009), pp. 746-759.
18. D. Han and D. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. 697 (2000).
19. D. Hua and Y. Huang, *The characterization and stability of g -Riesz frames for super Hilbert space*, J. Function Spaces, 2015 (2015), pp. 1-9.
20. W. Jing, *Frames in Hilbert C^* -modules*, Ph. D. thesis, University of Central Florida Orlando, Florida, 2006.
21. A. Khosravi and B. Khosravi, *g -frames and modular Riesz bases in Hilbert C^* -modules*, Int. J. Wavelets Multiresolut. Inf. Process., 10 (2012), pp. 1-12.
22. A. Khosravi and B. Khosravi, *Frames and bases in tensor products of Hilbert spaces and Hilbert C^* -modules*, Proc. Indian Acad. Sci. Math. Sci. 117 (2007), pp. 1-12.
23. A. Khrennikov and A. Yu, *The Hilbert super space*, Soviet Physics-Doklady, 36 (1991), pp. 759-760.

24. E.C. Lance, *Hilbert C^* -Modules: A Toolkit for Operator Algebras*, London Math. Soc. Lecture Note Ser. 210, Cambridge Univ. Press, 1995.
25. R. Oliver, *Super Hilbert spaces*, Communications in Mathematical Physics, 214 (2000), pp. 449-467.
26. M. Rashidi-Kouchi and A. Nazari, *Equivalent continuous g -frames in Hilbert C^* -modules*, Bull. Math. Anal. Appl., 4 (2012), pp. 91-98.
27. M. Rashidi-Kouchi and A. Nazari, *Continuous g -frame in Hilbert C^* -modules*, Abst. Appl. Anal., 2011 (2011), pp. 1-20.
28. M. Rashidi-Kouchi, A. Nazari, and M. Amini, *On stability of g -frames and g -Riesz bases in Hilbert C^* -modules*, Int. J. Wavelets Multiresolut. Inf. Process, 12 (2014), pp. 1-16.
29. W. Sun, *g -Frames and g -Riesz bases*, J. Math. Anal. Appl. 322 (2006), pp. 437-452.
30. L. Zang, W. Sun, and D. Chen, *Excess of a class of g -frames*, J. Math. Anal. Appl., 352 (2009), pp. 711-717.

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