Subspace-diskcyclic sequences of linear operators

Mohammad Reza Azimi

ABSTRACT. A sequence \( \{T_n\}_{n=1}^{\infty} \) of bounded linear operators on a separable infinite dimensional Hilbert space \( H \) is called subspace-diskcyclic with respect to the closed subspace \( M \subseteq H \), if there exists a vector \( x \in H \) such that the disk-scaled orbit \( \{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \cap M \) is dense in \( M \). The goal of this paper is the studying of subspace-diskcyclic sequence of operators like as the well known results in a single operator case. In the first section of this paper, we study some conditions that imply the diskcyclicity of \( \{T_n\}_{n=1}^{\infty} \). In the second section, we survey some conditions and subspace-diskcyclicity criterion (analogue the results obtained by some authors in \([6,10,11]\)) which are sufficient for the sequence \( \{T_n\}_{n=1}^{\infty} \) to be subspace-diskcyclic(subspace-hypercyclic).

1. Introduction and Preliminaries

Let \( X \) and \( Y \) be separable Banach spaces. The set of all bounded linear operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}(X,Y) \). A sequence of operators \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y) \) is called hypercyclic if there exists a vector \( x \in X \) such that the set \( \{T_n x : n \in \mathbb{N}\} \) is dense in \( Y \). Such a vector is called hypercyclic vector for the sequence of operators \( \{T_n\}_{n=1}^{\infty} \). We say that an operator \( T : X \rightarrow X \) is hypercyclic if the sequence of its iterations \( \{T^n\}_{n=1}^{\infty} \) is hypercyclic. Over the last two decades hypercyclic operators have been widely studied. A good survey of hypercyclic operators is the recent book [3]. Furthermore, some survey articles such as [4, 5, 12] and [11] are important references in this subject. The hypercyclicity of sequence of linear operators and its relevant criteria have been studied in [4, 5, 11] and [6]. In [11], B. F. Madore and R. A. Martinez-Avendano introduced the concept of subspace-hypercyclicity for a bounded linear operator \( T \) defined on Hilbert space \( H \) and they

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proved a Kitai-like criterion that implies subspace-hypercyclicity. They have also posed some questions which one of them was answered by H. Rezaei in [13]. He showed that $p(T)$ has a relatively dense range for every polynomial $p$. Moreover R.R. Jiménez-Munguía et al. [8] have answered some questions raised by H. Rezaei. The subspace-diskcyclicity of an operator $T \in \mathcal{L}(\mathcal{H})$ have been studied in [1, 2] in details. All these motivated us to study diskcyclicity and subspace-diskcyclicity of sequences of linear operators.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the field of complex numbers $\mathbb{C}$ and let $M$ be a closed subspace of $\mathcal{H}$. A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ is called subspace-diskcyclic if there exists a vector $x \in X$ such that the intersection of disk scaled orbit of $\{T_n\}_{n=1}^{\infty}$ and $M$, 

$$\{ \alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} \cap M,$$

is dense in $M$. Such a vector is called subspace-diskcyclic vector for the sequence of operators $\{T_n\}_{n=1}^{\infty}$ with respect to $M$. The set of all subspace-diskcyclic vectors for $\{T_n\}_{n=1}^{\infty}$ is denoted by $\text{DC}(\{T_n\}_{n=1}^{\infty}, M)$. In particular we say that an operator $T : X \to X$ is subspace-diskcyclic for some $M \subseteq \mathcal{H}$ if the sequence $\{T^n\}_{n=1}^{\infty}$ is subspace-diskcyclic for $M$, see [13]. For an operator $T$, the notion of subspace-diskcyclicity has been characterized in [1, 2]. Although the notion of subspace-diskcyclicity can be defined between different separable Banach spaces, nevertheless we prefer to deal with the Hilbert space. Note that if the operator $T$ is hypercyclic then the underlying Banach space $X$ should be separable. In [3] it is shown that an operator $T : X \to X$ is hypercyclic if and only if it is topologically transitive i.e., for any pair $U, V$ of nonempty open subsets of $X$ there exists $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$. In the first section of this paper, we define the notion of the disk topologically transitivity for the sequences of operators $\{T_n\}_{n=1}^{\infty}$ and then we show that this is a necessary and sufficient condition for $\{T_n\}_{n=1}^{\infty}$ to be diskcyclic. Many criteria for hypercyclicity of $\{T_n\}_{n=1}^{\infty}$ have been studied in [1, 3, 6, 10, 11].

In section 3 we introduce the concept of subspace-disk topologically transitivity for the $\{T_n\}_{n=1}^{\infty}$ and then it shall be shown that $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic if and only if it is subspace-disk topologically transitive. In addition some necessary and sufficient conditions, criterion and other properties concerning the subspace-diskcyclicity of sequences of linear operators $\{T_n\}_{n=1}^{\infty}$ are studied.

2. Diskcyclic sequences of linear operators

In this section we first define the concept of disk topologically transitivity and then we show that a sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, Y)$
is diskcyclic if and only if it is disk topologically transitive. Other equivalent conditions of this concept are also studied.

**Definition 2.1.** A sequence of operators \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X, Y) \) is called disk topologically transitive if for any pair \( U, V \) of nonempty open subsets of \( X \) and \( Y \) respectively, there are \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \) such that

\[
T_n(\alpha U) \cap V \neq \emptyset.
\]

**Lemma 2.2.** Let \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X, Y) \). Then

\[
DC(\{T_n\}_{n=1}^\infty) = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \bigcup_{\alpha \in \mathbb{C}, |\alpha| \geq 1} T_n^{-1}(\alpha V_k),
\]

where \( \{V_k\} \) is a countable open basis for \( Y \).

**Proof.** Note that \( x \in DC(\{T_n\}_{n=1}^\infty) \) if and only if for each \( k \in \mathbb{N} \), there exist \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \leq 1 \) such that \( \alpha T_n x \in V_k \) or \( x \in T_n^{-1}(1/2 V_k) \). This occurs if and only if \( x \in \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \bigcup_{\alpha \in \mathbb{C}, |\alpha| \geq 1} T_n^{-1}(\alpha V_k) \). Hence the set of all diskcyclic vectors for \( \{T_n\}_{n=1}^\infty \) is a \( G_\delta \) set. \( \square \)

**Lemma 2.3.** A sequence of operators \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X, Y) \) is diskcyclic if and only if it is disk topologically transitive.

**Proof.** Choose open subsets \( U \subseteq X \) and \( V \subseteq Y \) arbitrarily. It is easy to check that if a sequence \( \{T_n\}_{n=1}^\infty \) is diskcyclic then \( DC(\{T_n\}_{n=1}^\infty) \) is dense in \( X \). Therefore we have

\[
U \cap DC(\{T_n\}_{n=1}^\infty) \neq \emptyset.
\]

Pick \( x \in U \cap DC(\{T_n\}_{n=1}^\infty) \). Then the set \( \{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \) is dense in \( Y \) and so it must \( \{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \cap V \neq \emptyset \). Thus, \( \alpha T_n x \in V \) for some \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \leq 1 \). Eventually \( T_n(\alpha U) \cap V \neq \emptyset \).

Conversely suppose that the sequence \( \{T_n\}_{n=1}^\infty \) is disk topologically transitive. By Bair’s category theorem and Lemma 2.2, \( DC(\{T_n\}_{n=1}^\infty) \) is dense in \( X \) if and only if every open set

\[
W_k = \bigcup_{n=1}^\infty \bigcup_{\alpha \in \mathbb{C}, |\alpha| \geq 1} T_n^{-1}(\alpha V_k),
\]

is dense in \( X \). Indeed, for every nonempty open subset \( U \) of \( X \), there are \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \geq 1 \) such that \( U \cap T_n^{-1}(\alpha V_k) \neq \emptyset \), equivalently \( T_n(1/\alpha U) \cap V_k \neq \emptyset \). This completes the proof. \( \square \)

**Proposition 2.4.** Let \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X, Y) \) be a sequences of operators. The following conditions are equivalent:

(i) The sequence \( \{T_n\}_{n=1}^\infty \) is disk topologically transitive;
For each nonempty open subset $U$ of $X$ there are $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha| \leq 1} T_n(\alpha U)$ is dense in $Y$;

(iii) For each nonempty open subset $V$ of $Y$ there are $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha| \geq 1} T_n^{-1}(\alpha V)$ is dense in $X$;

(iv) For each $x \in X, y \in Y$ and $\epsilon > 0$ there exist $n \in \mathbb{N}$, $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ and $u \in X$ such that $\|u - x\| < \epsilon$ and $\|\alpha T_n u - y\| < \epsilon$. 

\textbf{Proof.} (i) Let $U$ be an arbitrary nonempty open subset of $X$. By (i) there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $T_n(\alpha U) \cap V \neq \emptyset$. So $T_n^{-1}(\frac{1}{\alpha} V) \cap U \neq \emptyset$ which implies that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha| \geq 1} T_n^{-1}(\alpha V)$ is dense in $X$, since $U$ was chosen arbitrary.

(iii) \Rightarrow \ (i) We have $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha| \geq 1} T_n^{-1}(\alpha V) \cap U \neq \emptyset$ for every nonempty open subset $U$ of $X$. Therefore, $U \cap T_n^{-1}(\alpha V) \neq \emptyset$ or $T_n(\frac{1}{\alpha} U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and the sequence $\{T_n\}_{n=1}^{\infty}$ is disk topologically transitive.

By Definition 2.4 it can be easily verified that the statements (i) and (iv) are equivalent. 

\[ \blacksquare \]

\textbf{Lemma 2.5.} Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, Y)$ and $c_n \geq 0$ for $n = 1, 2, \ldots$. If $\{c_n T_n\}_{n=1}^{\infty}$ is diskcyclic then the sequence $\{k_n T_n\}_{n=1}^{\infty}$ is diskcyclic for all $\{k_n\}_{n=1}^{\infty}$ with $k_n \geq c_n$ ($n = 1, 2, \ldots$).

\textbf{Proof.} Without loss of generality we may assume that $k_n > 0$ for each $n \in \mathbb{N}$. Let $x$ be a diskcyclic vector for $\{c_n T_n\}_{n=1}^{\infty}$. To establish the result, it is enough to show that

$$\{\alpha c_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \subseteq \{\alpha k_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}.$$

Take $y \in \{\alpha c_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$. Then there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \leq 1$ such that $y = \alpha c_n T_n x$. One may write $y = \alpha \frac{k_n}{c_n} k_n T_n x = \alpha' k_n T_n x$ where $\alpha \in \mathbb{C}, |\alpha'| \leq 1$. This follows that $y \in \{\alpha k_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$. 

\[ \blacksquare \]
3. Subspace-diskcyclic sequences of linear operators

From now on \( \mathcal{H} \) denotes a separable infinite dimensional Hilbert space over the field of complex numbers \( \mathbb{C} \).

**Definition 3.1.** Let \( M \) be a nontrivial closed subspace of \( \mathcal{H} \). A sequence \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) is called **subspace-diskcyclic** sequence of linear operators for \( M \) if there exists \( x \in \mathcal{H} \) such that the set
\[
\{\alpha T_n x : n \in \mathbb{N}, \ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M,
\]
is dense in \( M \). We call \( x \) a **subspace-diskcyclic** vector for \( \{T_n\}_{n=1}^{\infty} \). The set of all subspace-diskcyclic vectors for \( \{T_n\}_{n=1}^{\infty} \) in a subspace \( M \) is denoted by \( DC(\{T_n\}_{n=1}^{\infty}, M) \). In a single case see [1].

**Example 3.2.** One may consider that the subspace-diskcyclicity does not imply diskcyclicity in general. Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) be a diskcyclic sequence with the diskcyclic vector \( x \) and let \( I \) be the identity operator on \( \mathcal{H} \). Then the sequence \( \{T_n \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}\}_{n=1}^{\infty} \) is subspace-diskcyclic for the subspace \( M = \mathcal{H} \oplus \{0\} \) with the subspace-diskcyclic vector \( x \oplus 0 \), while \( \{T_n \oplus I\}_{n=1}^{\infty} \) is not diskcyclic.

**Theorem 3.3.** Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). Then
\[
DC(\{T_n\}_{n=1}^{\infty}, M) = \bigcap_{k=1}^{\infty} \bigcup_{|\alpha| \geq 1} \bigcup_{n=1}^{\infty} T_n^{-1}(\alpha B_k),
\]
where \( \{B_k\}_{k=1}^{\infty} \) is a countable open basis for the relatively topology of \( M \) as a subspace of \( \mathcal{H} \).

**Proof.** Note that \( x \in DC(\{T_n\}_{n=1}^{\infty}, M) \) if and only if \( \{\alpha T_n x : n \in \mathbb{N}, \ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M \) is dense in \( M \). Equivalently, for each \( k \), there are \( n \in \mathbb{N}, \ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \) such that \( \alpha T_n x \in B_k \). But the last is equivalent to that \( x \in T_n^{-1}(\frac{1}{\alpha} B_k) \) for each \( k \), and hence
\[
x \in \bigcap_{k=1}^{\infty} \bigcup_{|\alpha| \geq 1} \bigcup_{n=1}^{\infty} T_n^{-1}(\alpha B_k).
\]

\( \square \)

**Definition 3.4.** Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). We say that a sequence of linear operators \( \{T_n\}_{n=1}^{\infty} \) is **subspace-disk topologically transitive** with respect to \( M \) if for all nonempty sets \( U \subseteq M \) and \( V \subseteq M \), both relatively open, there exist \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \geq 1 \) such that \( T_n^{-1}(\alpha U) \cap V \) contains a relatively open nonempty subset of \( M \).
Theorem 3.5. Let \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(\mathcal{H}) \) be a sequence of linear operators and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). Then the followings are equivalent:

(i) The sequence of linear operators \( \{T_n\}_{n=1}^\infty \) is subspace-disk topologically transitive with respect to \( M \);

(ii) for all nonempty sets \( U \subseteq M \) and \( V \subseteq M \), both relatively open, there exist \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C} \), \( |\alpha| \geq 1 \) such that \( T_n^{-1}(\alpha U) \cap V \neq \emptyset \) and \( T_nM \subseteq M \);

(iii) for all nonempty sets \( U \subseteq M \) and \( V \subseteq M \), both relatively open, there exist \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C} \), \( |\alpha| \geq 1 \) such that \( T_n^{-1}(\alpha U) \cap V \) is nonempty open subset of \( M \).

Proof. \((i) \Rightarrow (ii)\) : Let \( U \subseteq M \) and \( V \subseteq M \) be nonempty open subsets and let \( W \) be the nonempty open subset of \( T_n^{-1}(\alpha U) \cap V \) for some \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C} \), \( |\alpha| \geq 1 \). Then \( \frac{1}{\alpha}T_nW \subseteq M \). Take \( x \in M \) and \( x_0 \in W \). Since \( W \) is open subset we may claim that \( x_0 + rx \in W \) for sufficiently small \( r > 0 \). Hence

\[
\frac{1}{\alpha}T_n(x_0) + \frac{1}{\alpha}T_n(rx) = \frac{1}{\alpha}T_n(x_0 + rx) \in M.
\]

Since \( \frac{1}{\alpha}T_n(x_0) \in M \), it is easily inferred that \( T_n(x) \in M \) and the proof is complete.

The implication \((iii) \Rightarrow (i)\) is obvious. \((ii) \Rightarrow (iii)\) is also obvious, since the sequence of operators \( \{T_n|_M\}_{n=1}^\infty \) is still continuous. \( \square \)

Corollary 3.6. Let \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(\mathcal{H}) \) be a sequence of linear operators and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). Assume that \( \{T_n\}_{n=1}^\infty \) is subspace-disk topologically transitive with respect to \( M \). Then \( DC(\{T_n\}_{n=1}^\infty, M) \) is a dense subset of \( M \).

Proof. Let \( \{B_i\} \) be a countable open basis for the relative topology of \( M \) as a subspace of \( \mathcal{H} \). By Theorem 3.3, for each \( i, j \), there exist \( n_{i,j} \in \mathbb{N} \) and \( \alpha_{i,j} \in \mathbb{C} \) with \( |\alpha_{i,j}| \geq 1 \) such that the set \( T_{n_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j \) is a nonempty open subset of \( M \). Hence the set

\[
A_i = \bigcup_j T_{n_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j,
\]

is a nonempty, open and dense set in \( M \). By Bair’s category theorem

\[
\bigcap_i A_i = \bigcap_i \bigcup_j T_{n_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j,
\]

remains still dense set in \( M \). But by Theorem 3.3, we know that

\[
DC(\{T_n\}_{n=1}^\infty, M) = \bigcap_i \bigcup_n \bigcup_{|\alpha| \geq 1} T^{-1}(\alpha B_i),
\]

and the result is obtained. \( \square \)
Corollary 3.7. If \( \{T_n\}_{n=1}^{\infty} \) is subspace-disk topologically transitive for a subspace \( M \), then \( \{T_n\}_{n=1}^{\infty} \) is diskcyclic for \( M \).

Proof. This is an immediate consequence of Theorem 3.5. \( \square \)

Theorem 3.8. Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) be a sequence of linear operators and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). Assume that there exist \( X \) and \( Y \), dense subsets of \( M \) and an increasing sequence of positive integers \( \{n_k\}_{k=1}^{\infty} \) such that

(i) \( T_{n_k}x \to 0 \) for all \( x \in X \);

(ii) for any \( y \in Y \), there exists a sequence \( \{x_k\} \) in \( M \) such that \( x_k \to 0 \) and \( T_{n_k}x_k \to y \);

(iii) \( T_{n_k}M \subseteq M \) for each \( k \in \mathbb{N} \).

Then \( \{T_n\}_{n=1}^{\infty} \) is subspace-topologically transitive with respect to \( M \) and hence \( \{T_n\}_{n=1}^{\infty} \) is subspace-hypercyclic for \( M \).

Proof. The sketch of the proof is well-known and we follow it same as used in [11]. Let \( U \subseteq M \) and \( V \subseteq M \) be nonempty open subsets. By Theorem 3.3 we should only show that there exists \( k \in \mathbb{N} \) such that \( T_{n_k}^{-1}(U) \cap V \) is nonempty. Since \( X \) and \( Y \) are dense in \( M \), there exists \( u \in U \cap Y \) and \( v \in V \cap X \). Moreover, one may catch \( \delta > 0 \) such that the \( M \)-ball centered at \( u \) of radius \( \delta \), denoted by \( B_M(u, \delta) \), is contained in \( U \) and \( B_M(v, \delta) \subseteq V \). Now by (ii), we can choose \( k \) large enough such that there exists \( x_k \in M \) with

\[
\|T_{n_k}v\| < \frac{\delta}{2}, \quad \|x_k\| < \delta, \quad \|T_{n_k}x_k - u\| < \frac{\delta}{2}
\]

We know that \( v + x_k \in M \) and

\[
\|v + x_k - v\| = \|x_k\| < \delta,
\]

which follows that

\[
v + x_k \in B_M(v, \delta) \subseteq V.
\]

In addition, \( T_{n_k} \) leaves \( M \) invariant, so \( T_{n_k}(v + x_k) \in M \) and

\[
\|T_{n_k}(v + x_k) - u\| \leq \|T_{n_k}v\| + \|T_{n_k}x_k - u\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

It follows that

\[
T_{n_k}(v + x_k) \in B_M(u, \delta) \subseteq U.
\]

Eventually, the above arguments imply that

\[
v + x_k \in T_{n_k}^{-1}(U) \cap V,
\]

and the result is followed. \( \square \)
Theorem 3.9. Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) be a sequence of linear operators and let \( M \) be a nontrivial subspace of \( \mathcal{H} \). Assume that there exist \( X \) and \( Y \), dense subsets of \( M \) and an increasing sequence of positive integers \( \{n_k\}_{k=1}^{\infty} \) such that

(i) \( \alpha T_{n_k} x \to 0 \) for all \( x \in X \);
(ii) for any \( y \in Y \), there exist a sequence \( \{x_k\} \) in \( M \) and \( \alpha \in \mathbb{C}, |\alpha| \leq 1 \) such that \( x_k \to 0 \) and \( \alpha T_{n_k} x_k \to y \);
(iii) \( T_{n_k} M \subseteq M \) for each \( k \in \mathbb{N} \).

Then \( \{T_n\}_{n=1}^{\infty} \) is subspace-disk topologically transitive with respect to \( M \) and hence \( \{T_n\}_{n=1}^{\infty} \) is subspace-diskcyclic for \( M \).

Proof. Let \( U \) and \( V \) be nonempty relatively open subsets of \( M \). By Theorem 3.5, it is enough to prove that there exist \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \geq 1 \) such that \( T_{n_k}^{-1}(\alpha U) \cap V \) is nonempty. For each \( \epsilon > 0 \), choose \( k \) large enough such that there exist \( x_k \in M \) and \( \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \) where

\[
\|T_{n_k} x\| < \frac{\epsilon}{2}, \quad \|x_k\| < \epsilon, \quad \|\alpha T_{n_k} x_k - y\| < \frac{\epsilon}{2},
\]

hold for every \( x \in X \) and \( y \in Y \). As mentioned in the proof of the previous theorem, \( \alpha u \in U \cap Y \), \( v \in V \cap X \) and \( \delta > 0 \) are easily found on which

\[
B_M(\alpha u, \delta) \subseteq U, \quad B_M(v, \delta) \subseteq V.
\]

Hence the above inequalities can be rewritten as follows

\[
\|T_{n_k} v\| < \frac{\delta}{2|\alpha|}, \quad \|x_k\| < \delta, \quad \|\alpha T_{n_k} x_k - \alpha u\| < \frac{\delta}{2}.
\]

But \( v + x_k \in B_M(v, \delta) \subseteq V \) and \( T_{n_k}(v + x_k) \in M \), since \( T_{n_k} \) leaves \( M \) invariant. Moreover

\[
\|\alpha T_{n_k} (v + x_k) - \alpha u\| \leq \|\alpha T_{n_k} v\| + \|\alpha T_{n_k} x_k - \alpha u\|
\]

\[
< \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

which follows that

\[
\alpha T_{n_k} (v + x_k) \in B_M(\alpha u, \delta) \subseteq U.
\]

Therefore \( T_{n_k}^{-1}(\frac{1}{\alpha} U) \cap V \neq \emptyset \) and the proof is completed. \( \square \)

Theorem 3.10. Let \( \{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}) \) be a subspace-diskcyclic (subspace-hypercyclic) sequence of mutually commuting linear operators for a nontrivial subspace \( M \) of \( \mathcal{H} \). Suppose that \( N \supseteq M \) is an invariant subspace sequence for \( \{T_n\}_{n=1}^{\infty} \) i.e., \( T_n N \subseteq N \) for each \( n \in \mathbb{N} \). Then there exists \( k \in \mathbb{N} \) such that \( \{T_n|_N : N \to N\}_{n=1}^{\infty} \) is a subspace-diskcyclic (subspace-hypercyclic) for \( T_k M \).
Proof. Suppose that
\[ \{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \} \cap M, \]
is dense in $M$ for a subspace-diskcyclic vector $x$. Take $\alpha T_k x$ in the above intersection for some $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. It follows that $T_k x \in M$. Hence $T_n T_k x \in N$ for each $n \in \mathbb{N}$, since $N$ is invariant subspace for $\{T_n\}_{n=1}^{\infty}$. Now note that
\[ \{\alpha T_n(T_k x) : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \} \cap T_k M \]
\[ = T_k(\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \}) \cap T_k M \]
\[ \supseteq T_k(\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \} \cap M). \]
Consequently $\{T_n|_N : N \to N\}_{n=1}^{\infty}$ is a subspace-diskcyclic for $T_k M$. □

**Theorem 3.11.** Let $\mathcal{H} = M \oplus N$ and $P$ be the projection onto $M$ along $N$. Let $T_n N \subseteq N$ for each $n \in \mathbb{N}$. If $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ is subspace-diskcyclic for some $L \subseteq M$, then $\{PT_n|_M\}_{n=1}^{\infty}$ is subspace-diskcyclic for $L$.

**Proof.** Suppose that $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic for $L \subseteq M$ with diskcyclic vector $x \in L$. Then
\[ \{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \} \cap L \]
\[ \subseteq P(\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \}) \cap L. \]
But we have $PT_n P = PT_n$, since every $T_n$ leaves $N$ invariant. This implies that
\[ P(\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \}) \]
\[ = \{\alpha PT_n|_M x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \}. \]
Therefore, $\{\alpha PT_n|_M x : n \in \mathbb{N}, \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1 \}$ is dense in $L$. □

**Corollary 3.12.** Let $P$ be an orthogonal projection onto a reducible subspace $M$ for $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$. If $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic (subspace-hypercyclic) for some $L \subseteq M$, then $\{T_n|_M\}_{n=1}^{\infty}$ is subspace-diskcyclic (subspace-hypercyclic) for $L$.

**Proof.** By following the proof of Theorem 3.11 and using the fact that $PT_n = T_n P$, the desired result is established. □

**References**


