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Subspace-diskcyclic sequences of linear operators

Mohammad Reza Azimi

Abstract. A sequence $\{T_n\}_{n=1}^{\infty}$ of bounded linear operators on a separable infinite dimensional Hilbert space $\mathcal{H}$ is called subspace-diskcyclic with respect to the closed subspace $M \subseteq \mathcal{H}$, if there exists a vector $x \in \mathcal{H}$ such that the disk-scaled orbit $\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \cap M$ is dense in M. The goal of this paper is the studying of subspace-diskcyclic sequence of operators like as the well known results in a single operator case. In the first section of this paper, we study some conditions that imply the diskcyclicity of $\{T_n\}_{n=1}^{\infty}$. In the second section, we survey some conditions and subspace-diskcyclicity criterion (analogue the results obtained by some authors in [6, 10, 11]) which are sufficient for the sequence $\{T_n\}_{n=1}^{\infty}$ to be subspace-diskcyclic(subspace-hypercyclic).

1. Introduction and Preliminaries

Let X and Y be separable Banach spaces. The set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$ is called hypercyclic if there exists a vector $x \in X$ such that the set $\{T_n x : n \in \mathbb{N}\}$ is dense in Y. Such a vector is called hypercyclic vector for the sequence of operators $\{T_n\}_{n=1}^{\infty}$. We say that an operator $T : X \to X$ is hypercyclic if the sequence of its iterations $\{T^n\}_{n=1}^{\infty}$ is hypercyclic. Over the last two decades hypercyclic operators have been widely studied. A good survey of hypercyclic operators is the recent book [3]. Furthermore, some survey articles such as [5, 6, 12] and [7] are important references in this subject. The hypercyclicity of sequence of linear operators and its relevant criteria have been studied in [4, 10] and [6]. In [11], B.F. Madore and R.A. Martinez-Avendano introduced the concept of subspace-hypercyclicity for a bounded linear operator T defined on Hilbert space $\mathcal{H}$ and they

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proved a Kitai-like criterion that implies subspace-hypercyclicity. They have also posed some questions which one of them was answered by H. Rezaei in [13]. He showed that $p(T)$ has a relatively dense range for every polynomial p. Moreover R.R. Jiménez-Munguía et al. [8] have answered some questions raised by H. Rezaei. The subspace-diskcyclicity of an operator $T \in \mathcal{L}(\mathcal{H})$ have been studied in [1, 2] in details. All these motivated us to study diskcyclicity and subspace-diskcyclicity of sequences of linear operators.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the field of complex numbers $\mathbb{C}$ and let M be a closed subspace of $\mathcal{H}$. A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ is called subspace-diskcyclic if there exists a vector $x \in X$ such that the intersection of disk scaled orbit of $\{T_n\}_{n=1}^{\infty}$ and M,

$$\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \cap M,$$

is dense in M. Such a vector is called subspace-diskcyclic vector for the sequence of operators $\{T_n\}_{n=1}^{\infty}$ with respect to M. The set of all subspace-diskcyclic vectors for $\{T_n\}_{n=1}^{\infty}$ is denoted by $DC(\{T_n\}_{n=1}^{\infty}, M)$. In particular we say that an operator $T : X \to X$ is subspace-diskcyclic for some $M \subseteq \mathcal{H}$ if the sequence $\{T^n\}_{n=1}^{\infty}$ is subspace-diskcyclic for M, see [14]. For an operator T, the notion of subspace-diskcyclicity has been characterized in [1, 2]. Although the notion of subspace-diskcyclicity can be defined between different separable Banach spaces, nevertheless we prefer to deal with the Hilbert space. Note that if the operator T is hypercyclic then the underlying Banach space X should be separable. In [3] it is shown that an operator $T : X \to X$ is hypercyclic if and only if it is topologically transitive i.e., for any pair U, V of nonempty open subsets of X there exists $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$. In the first section of this paper, we define the notion of the disk topologically transitivity for the sequences of operators $\{T_n\}_{n=1}^{\infty}$ and then we show that this is a necessary and sufficient condition for $\{T_n\}_{n=1}^{\infty}$ to be diskcyclic. Many criteria for hypercyclicity of $\{T_n\}_{n=1}^{\infty}$ have been studied in [4, 6, 9, 10].

In section 3 we introduce the concept of subspace-disk topologically transitivity for the $\{T_n\}_{n=1}^{\infty}$ and then it shall be shown that $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic if and only if it is subspace-disk topologically transitive. In addition some necessary and sufficient conditions, criterion and other properties concerning the subspace-diskcyclicity of sequences of linear operators $\{T_n\}_{n=1}^{\infty}$ are studied.

2. Diskcyclic sequences of linear operators

In this section we first define the concept of disk topologically transitivity and then we show that a sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, Y)$

is diskcyclic if and only if it is disk topologically transitive. Other equivalent conditions of this concept are also studied.

Definition 2.1. A sequence of operators $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$ is called disk topologically transitive if for any pair U, V of nonempty open subsets of X and Y respectively, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that

$$T_n(\alpha U) \cap V \neq \emptyset.$$

Lemma 2.2. *Let* $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$. *Then*

$$DC(\{T_n\}_{n=1}^{\infty}) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\alpha\in\mathbb{C},|\alpha|\geq 1} T_n^{-1}(\alpha V_k),$$

where $\{V_k\}$ *is a countable open basis for* Y.

Proof. Note that $x \in DC(\{T_n\}_{n=1}^{\infty})$ if and only if for each $k \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C},\ |\alpha| \leq 1$ such that $\alpha T_n x \in V_k$ or $x \in T_n^{-1}(\frac{1}{\alpha} V_k)$. This occurs if and only if $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\alpha\in\mathbb{C},|\alpha|\geq 1} T_n^{-1}(\alpha V_k)$. Hence the set of all diskcyclic vectors for $\{T_n\}_{n=1}^{\infty}$ is a G_δ set. □

Lemma 2.3. *A sequence of operators* $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$ *is diskcyclic if and only if it is disk topologically transitive.*

Proof. Choose open subsets $U \subseteq X$ and $V \subseteq Y$ arbitrarily. It is easy to check that if a sequence $\{T_n\}_{n=1}^{\infty})$ is diskcyclic then $DC(\{T_n\}_{n=1}^{\infty})$ is dense in X. Therefore we have

$$U \cap DC(\{T_n\}_{n=1}^{\infty}) \neq \emptyset.$$

Pick $x \in U \cap DC(\{T_n\}_{n=1}^{\infty})$. Then the set $\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$ is dense in Y and so it must $\{\alpha T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \cap V \neq \emptyset$. Thus, $\alpha T_n x \in V$ for some $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \leq 1$. Eventually $T_n(\alpha U) \cap V \neq \emptyset$.
Conversely suppose that the sequence $\{T_n\}_{n=1}^{\infty}$ is disk topologically transitive. By Bair's category theorem and Lemma 2.2, $DC(\{T_n\}_{n=1}^{\infty})$ is dense in X if and only if every open set

$$W_k = \bigcup_{n=1}^{\infty} \bigcup_{\alpha\in\mathbb{C},|\alpha|\geq 1} T_n^{-1}(\alpha V_k),$$

is dense in X. Indeed, for every nonempty open subset U of X, there are $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $U \cap T_n^{-1}(\alpha V_k) \neq \emptyset$, equivalently $T_n(\frac{1}{\alpha} U) \cap V_k \neq \emptyset$. This completes the proof. □

Proposition 2.4. *Let* $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$ *be a sequences of operators. The following conditions are equivalent:*

(i) *The sequence* $\{T_n\}_{n=1}^{\infty}$ *is disk topologically transitive;*

(ii) *For each nonempty open subset U of X there are $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha|\leq 1} T_n(\alpha U)$ is dense in Y;*

(iii) *For each nonempty open subset V of Y there are $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha|\geq 1} T_n^{-1}(\alpha V)$ is dense in X;*

(iv) *For each $x \in X$, $y \in Y$ and $\epsilon > 0$ there exist $n \in \mathbb{N}$, $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ and $u \in X$ such that $\|u - x\| < \epsilon$ and $\|\alpha T_n u - y\| < \epsilon$.*

Proof. $(i) \Rightarrow (ii)$ Let U be an arbitrary nonempty open subset of X. By (i) there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $T_n(\alpha U) \cap V \neq \emptyset$ for any nonempty open subset V of Y. Hence (ii) is established.

$(ii) \Rightarrow (iii)$ Let $U \subseteq X$ and $V \subseteq Y$ be any nonempty open subsets. By (ii) there are $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that

$$\bigcup_{n=1}^{\infty} \bigcup_{|\alpha|\leq 1} T_n(\alpha U) \bigcap V \neq \emptyset.$$

Thus, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $T_n(\alpha U) \cap V \neq \emptyset$. So $T_n^{-1}(\frac{1}{\alpha} V) \cap U \neq \emptyset$ which implies that $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha|\geq 1} T_n^{-1}(\alpha V)$ is dense in X, since U was chosen arbitrary.

$(iii) \Rightarrow (i)$ We have $\bigcup_{n=1}^{\infty} \bigcup_{|\alpha|\geq 1} T_n^{-1}(\alpha V) \bigcap U \neq \emptyset$ for every nonempty open subset U of X. Therefore, $U \cap T_n^{-1}(\alpha V) \neq \emptyset$ or $T_n(\frac{1}{\alpha} U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and the sequence $\{T_n\}_{n=1}^{\infty}$ is disk topologically transitive.

By Definition 2.1 it can be easily verified that the statements (i) and (iv) are equivalent.

□

Lemma 2.5. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, Y)$ and $c_n \geq 0$ for $n = 1, 2, \ldots$. If $\{c_n T_n\}_{n=1}^{\infty}$ is diskcyclic then the sequence $\{k_n T_n\}_{n=1}^{\infty}$ is diskcyclic for all $\{k_n\}_{n=1}^{\infty}$ with $k_n \geq c_n$ $(n = 1, 2, \ldots)$.*

Proof. Without loss of generality we may assume that $k_n > 0$ for each $n \in \mathbb{N}$. Let x be a diskcyclic vector for $\{c_n T_n\}_{n=1}^{\infty}$. To establish the result, it is enough to show that

$$\{\alpha c_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\} \subseteq \{\alpha k_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}.$$

Take $y \in \{\alpha c_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$. Then there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \leq 1$ such that $y = \alpha c_n T_n x$. One may write $y = \alpha \frac{c_n}{k_n} k_n T_n x = \alpha' k_n T_n x$ where $\acute{\alpha} \in \mathbb{C}, |\alpha'| \leq 1$. This follows that $y \in \{\alpha k_n T_n x : n \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$. □

3. Subspace-diskcyclic sequences of linear operators

From now on $\mathcal{H}$ denotes a separable infinite dimensional Hilbert space over the field of complex numbers $\mathbb{C}$.

Definition 3.1. Let M be a nontrivial closed subspace of $\mathcal{H}$. A sequence $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ is called *subspace-diskcyclic* sequence of linear operators for M if there exists $x \in \mathcal{H}$ such that the set

$$\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M,$$

is dense in M. We call x a *subspace-diskcyclic* vector for $\{T_n\}_{n=1}^{\infty}$. The set of all subspace-diskcyclic vectors for $\{T_n\}_{n=1}^{\infty}$ in a subspace M is denoted by $DC(\{T_n\}_{n=1}^{\infty}, M)$. In a single case T see [1].

Example 3.2. One may consider that the subspace-diskcyclicity does not imply diskcyclicity in general. Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a diskcyclic sequence with the diskcyclic vector x and let I be the identity operator on $\mathcal{H}$. Then the sequence $\{T_n \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}\}_{n=1}^{\infty}$ is subspace-diskcyclic for the subspace $M = \mathcal{H} \oplus \{0\}$ with the subspace-diskcyclic vector $x \oplus 0$, while $\{T_n \oplus I\}_{n=1}^{\infty}$ is not diskcyclic.

Theorem 3.3. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ and let M be a nontrivial subspace of $\mathcal{H}$. Then*

$$DC(\{T_n\}_{n=1}^{\infty}, M) = \bigcap_{k=1}^{\infty} \bigcup_{|\alpha|\geq 1} \bigcup_{n=1}^{\infty} T_n^{-1}(\alpha B_k),$$

where $\{B_k\}_{k=1}^{\infty}$ is a countable open basis for the relatively topology of M as a subspace of $\mathcal{H}$.

Proof. Note that $x \in DC(\{T_n\}_{n=1}^{\infty}, M)$ if and only if $\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M$ is dense in M. Equivalently, for each k, there are $n \in \mathbb{N},\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ such that $\alpha T_n x \in B_k$. But the last is equivalent to that $x \in T_n^{-1}(\frac{1}{\alpha} B_k)$ for each k, and hence

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{|\alpha|\geq 1} \bigcup_{n=1}^{\infty} T_n^{-1}(\alpha B_k).$$

□

Definition 3.4. Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ and let M be a nontrivial subspace of $\mathcal{H}$. We say that a sequence of linear operators $\{T_n\}_{n=1}^{\infty}$ is subspace-disk topologically transitive with respect to M if for all nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $T_n^{-1}(\alpha U) \cap V$ contains a relatively open nonempty subset of M.

Theorem 3.5. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a sequence of linear operators and let M be a nontrivial subspace of $\mathcal{H}$. Then the followings are equivalent:*

(i) *The sequence of linear operators $\{T_n\}_{n=1}^{\infty}$ is subspace-disk topologically transitive with respect to M;*

(ii) *for all nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $T_n^{-1}(\alpha U) \cap V \neq \emptyset$ and $T_n M \subseteq M$;*

(iii) *for all nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $T_n^{-1}(\alpha U) \cap V$ is nonempty open subset of M.*

Proof. $(i) \Rightarrow (ii)$: Let $U \subseteq M$ and $V \subseteq M$ be nonempty open subsets and let W be the nonempty open subset of $T_n^{-1}(\alpha U) \cap V$ for some $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$. Then $\frac{1}{\alpha} T_n W \subseteq M$. Take $x \in M$ and $x_0 \in W$. Since W is open subset we may claim that $x_0 + rx \in W$ for sufficiently small $r > 0$. Hence

$$\frac{1}{\alpha} T_n(x_0) + \frac{1}{\alpha} T_n(rx) = \frac{1}{\alpha} T_n(x_0 + rx) \in M.$$

Since $\frac{1}{\alpha} T_n(x_0) \in M$, it is easily inferred that $T_n(x) \in M$ and the proof is complete.

The implication $(iii) \Rightarrow (i)$ is obvious. $(ii) \Rightarrow (iii)$ is also obvious, since the sequence of operators $\{T_n|_M\}_{n=1}^{\infty}$ is still continuous. □

Corollary 3.6. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a sequence of linear operators and let M be a nontrivial subspace of $\mathcal{H}$. Assume that $\{T_n\}_{n=1}^{\infty}$ is subspace-disk topologically transitive with respect to M. Then $DC(\{T_n\}_{n=1}^{\infty}, M)$ is a dense subset of M.*

Proof. Let $\{B_i\}$ be a countable open basis for the relative topology of M as a subspace of $\mathcal{H}$. By Theorem 3.5, for each i, j, there exist $n_{i,j} \in \mathbb{N}$ and $\alpha_{i,j} \in \mathbb{C}$ with $|\alpha_{i,j}| \geq 1$ such that the set $T_{n_{i,j}}^{-1}(\alpha_{i,j} B_i) \cap B_j$ is a nonempty open subset of M. Hence the set

$$A_i = \bigcup_j T_{n_{i,j}}^{-1}(\alpha_{i,j} B_i) \cap B_j,$$

is a nonempty, open and dense set in M. By Bair's category theorem

$$\bigcap_i A_i = \bigcap_i \bigcup_j T_{n_{i,j}}^{-1}(\alpha_{i,j} B_i) \cap B_j,$$

remains still dense set in M. But by Theorem 3.3, we know that

$$DC(\{T_n\}_{n=1}^{\infty}, M) = \bigcap_i \bigcup_n \bigcup_{|\alpha| \geq 1} T^{-1}(\alpha B_i),$$

and the result is obtained. □

Corollary 3.7. *If $\{T_n\}_{n=1}^{\infty}$ is subspace-disk topologically transitive for a subspace M, then $\{T_n\}_{n=1}^{\infty}$ is diskcyclic for M.*

Proof. This is an immediate consequence of Theorem 3.5. □

Theorem 3.8. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a sequence of linear operators and let M be a nontrivial subspace of $\mathcal{H}$. Assume that there exist X and Y, dense subsets of M and an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that*

(i) $T_{n_k}x \to 0$ *for all* $x \in X$;
(ii) *for any* $y \in Y$,*there exists a sequence* $\{x_k\}$ *in* M *such that* $x_k \to 0$ *and* $T_{n_k}x_k \to y$;
(iii) $T_{n_k}M \subseteq M$ *for each* $k \in \mathbb{N}$.

Then $\{T_n\}_{n=1}^{\infty}$ is subspace-topologically transitive with respect to M and hence $\{T_n\}_{n=1}^{\infty}$ is subspace-hypercyclic for M.

Proof. The sketch of the proof is well-known and we follow it same as used in [11]. Let $U \subseteq M$ and $V \subseteq M$ be nonempty open subsets. By Theorem 3.3 we should only show that there exists $k \in \mathbb{N}$ such that $T_{n_k}^{-1}(U) \cap V$ is nonempty. Since X and Y are dense in M, there exists $u \in U \cap Y$ and $v \in V \cap X$. Moreover, one may catch $\delta > 0$ such that the M-ball centered at u of radius δ, denoted by $B_M(u, \delta)$, is contained in U and $B_M(v, \delta) \subseteq V$. Now by (ii), we can choose k large enough such that there exists $x_k \in M$ with

$$\|T_{n_k}v\| < \frac{\delta}{2}, \quad \|x_k\| < \delta, \qquad \|T_{n_k}x_k - u\| < \frac{\delta}{2}.$$

We know that $v + x_k \in M$ and

$$\|v + x_k - v\| = \|x_k\| < \delta,$$

which follows that

$$v + x_k \in B_M(v, \delta) \subseteq V.$$

In addition, T_{n_k} leaves M invariant, so $T_{n_k}(v + x_k) \in M$ and

$$\|T_{n_k}(v + x_k) - u\| \leq \|T_{n_k}v\| + \|T_{n_k}x_k - u\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

It follows that

$$T_{n_k}(v + x_k) \in B_M(u, \delta) \subseteq U.$$

Eventually, the above arguments imply that

$$v + x_k \in T_{n_k}^{-1}(U) \cap V,$$

and the result is followed. □

Theorem 3.9. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a sequence of linear operators and let M be a nontrivial subspace of $\mathcal{H}$. Assume that there exist X and Y, dense subsets of M and an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that*

(i) *$\alpha T_{n_k} x \to 0$ for all $x \in X$;*
(ii) *for any $y \in Y$,there exist a sequence $\{x_k\}$ in M and $\alpha \in \mathbb{C}, |\alpha| \leq 1$ such that $x_k \to 0$ and $\alpha T_{n_k} x_k \to y$;*
(iii) *$T_{n_k} M \subseteq M$ for each $k \in \mathbb{N}$.*

Then $\{T_n\}_{n=1}^{\infty}$ is subspace-disk topologically transitive with respect to M and hence $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic for M.

Proof. Let U and V be nonempty relatively open subsets of M. By Theorem 3.5, it is enough to prove that there exist $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $T_{n_k}^{-1}(\alpha U) \cap V$ is nonempty. For each $\epsilon > 0$, choose k large enough such that there exist $x_k \in M$ and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$ where

$$\|T_{n_k} x\| < \frac{\epsilon}{2}, \quad \|x_k\| < \epsilon, \qquad \|\alpha T_{n_k} x_k - y\| < \frac{\epsilon}{2},$$

hold for every $x \in X$ and $y \in Y$. As mentioned in the proof of the previous theorem, $\alpha u \in U \cap Y$, $v \in V \cap X$ and $\delta > 0$ are easily found on which

$$B_M(\alpha u, \delta) \subseteq U, \qquad B_M(v, \delta) \subseteq V.$$

Hence the above inequalities can be rewritten as follows

$$\|T_{n_k} v\| < \frac{\delta}{2|\alpha|}, \quad \|x_k\| < \delta, \qquad \|\alpha T_{n_k} x_k - \alpha u\| < \frac{\delta}{2}.$$

But $v + x_k \in B_M(v, \delta) \subseteq V$ and $T_{n_k}(v + x_k) \in M$, since T_{n_k} leaves M invariant. Moreover

$$\begin{aligned}\|\alpha T_{n_k}(v + x_k) - \alpha u\| &\leq \|\alpha T_{n_k} v\| + \|\alpha T_{n_k} x_k - \alpha u\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta,\end{aligned}$$

which follows that

$$\alpha T_{n_k}(v + x_k) \in B_M(\alpha u, \delta) \subseteq U.$$

Therefore $T_{n_k}^{-1}(\frac{1}{\alpha} U) \cap V \neq \emptyset$ and the proof is completed. □

Theorem 3.10. *Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ be a subspace-diskcyclic (subspace-hypercyclic) sequence of mutually commuting linear operators for a nontrivial subspace M of $\mathcal{H}$. Suppose that $N \supseteq M$ is an invariant subspace sequence for $\{T_n\}_{n=1}^{\infty}$ i.e., $T_n N \subseteq N$ for each $n \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $\{T_n|_N : N \to N\}_{n=1}^{\infty}$ is a subspace-diskcyclic (subspace-hypercyclic) for $T_k M$.*

Proof. Suppose that

$$\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M,$$

is dense in M for a subspace-diskcyclic vector x. Take $\alpha T_k x$ in the above intersection for some $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. It follows that $T_k x \in M$. Hence $T_n T_k x \in N$ for each $n \in \mathbb{N}$, since N is invariant subspace for $\{T_n\}_{n=1}^{\infty}$. Now note that

$$\begin{aligned}&\{\alpha T_n(T_k x) : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap T_k M \\ &\quad = T_k(\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\}) \cap T_k M \\ &\quad \supseteq T_k(\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap M).\end{aligned}$$

Consequently $\{T_n|_N : N \to N\}_{n=1}^{\infty}$ is a subspace-diskcyclic for $T_k M$. □

Theorem 3.11. *Let $\mathcal{H} = M \oplus N$ and P be the projection onto M along N. Let $T_n N \subseteq N$ for each $n \in \mathbb{N}$. If $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$ is subspace-diskcyclic for some $L \subseteq M$, then $\{PT_n|_M\}_{n=1}^{\infty}$ is subspace-diskcyclic for L.*

Proof. Suppose that $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic for $L \subseteq M$ with diskcyclic vector $x \in L$. Then

$$\begin{aligned}&\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\} \cap L \\ &\quad \subseteq P(\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\}) \cap L.\end{aligned}$$

But we have $PT_nP = PT_n$, since every T_n leaves N invariant. This implies that

$$\begin{aligned}&P(\{\alpha T_n x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\}) \\ &\quad = \{\alpha PT_n|_M x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\}.\end{aligned}$$

Therefore, $\{\alpha PT_n|_M x : n \in \mathbb{N}\ ,\ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1\}$ is dense in L. □

Corollary 3.12. *Let P be an orthogonal projection onto a reducible subspace M for $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H})$. If $\{T_n\}_{n=1}^{\infty}$ is subspace-diskcyclic (subspace-hypercyclic) for some $L \subseteq M$, then $\{T_n|_M\}_{n=1}^{\infty}$ is subspace-diskcyclic (subspace-hypercyclic) for L.*

Proof. By following the proof of Theorem 3.11 and using the fact that $PT_n = T_nP$, the desired result is established. □

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Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran.

E-mail address: `mhr.azimi@maragheh.ac.ir`