ON THE REDUCIBLE $M$-IDEALS IN BANACH SPACES

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Abstract. The object of the investigation is to study reducible $M$-ideals in Banach spaces. It is shown that if the number of $M$-ideals in a Banach space $X$ is $n(< \infty)$, then the number of reducible $M$-ideals does not exceed of $\frac{(n-2)(n-3)}{2}$. Moreover, given a compact metric space $X$, we obtain a general form of a reducible $M$-ideal in the space $C(X)$ of continuous functions on $X$. The intersection of two $M$-ideals is not necessarily reducible. We construct a subset of the set of all $M$-ideals in a Banach space $X$ such that the intersection of any pair of its elements is reducible. Also, some Banach spaces $X$ and $Y$ for which $K(X,Y)$ is not a reducible $M$-ideal in $L(X,Y)$, are presented. Finally, a weak version of reducible $M$-ideal called semi reducible $M$-ideal is introduced.

1. INTRODUCTION AND PRELIMINARIES

In the theory of Banach spaces, the concept of $M$-ideal, since its introduction by Alfsen and Effros [1], proves to be an important tool to study geometric and isometric properties of the spaces. The notion of an $M$-ideal generalizes the two sided ideals in a $C^*$-algebra; due to the geometric characterization of the ideals in these special algebras the $M$-ideals have been identified with the two sided ideals [12]. As it is quoted in [2], the fact that $Y$ is an $M$-ideal in $X$ has a strong impact on both $Y$ and $X$, since there are a number of important properties shared by $M$-ideals, but not by arbitrary subspaces. For instance, $M$-ideals are Hahn-Banach smooth subspaces, i.e., every norm one linear functional on an $M$-ideal has a unique norm one extension to $X$. The study of the
phenomenon of unique Hahn-Banach extensions was initiated by Phelps in [10].

Here, we have some definitions and results which will be needed in the sequel.

**Definition 1.1.** A closed subspace $J$ of a Banach space $X$ is called $M$-ideal if there exist a linear projection $P : X^* \to J^\perp$ such that

$$\|f\| = \|Pf\| + \|f - Pf\|,$$

for all $f \in X^*$, where $J^\perp = \{f \in X^* : f(x) = 0$ for any $x \in J\}$ is the annihilator of $J$ in $X^*$. Such a projection is called a $L$-projection and its range a $L$-summand.

In [1], Alfsen and Effros proved that a closed subspace $J$ of a Banach space $X$ is an $M$-ideal if and only if it satisfies in $n$-ball property ($n \in \mathbb{N}, n \geq 3$). We say that a subspace $J$ of a Banach space $X$ satisfies the $n$-ball property if given $n$ open balls $B_1, \ldots, B_n$ for which $B_1 \cap \ldots \cap B_n \neq \emptyset$, and $B_i \cap J \neq \emptyset, i = 1, \ldots, n$, it follows that $B_1 \cap \ldots \cap B_n \cap J \neq \emptyset$.

Finite intersection of $M$-ideals is also an $M$-ideal ([6, Proposition 1.11(b)]); but it need not be true for any family of $M$-ideals (see [9, P.78 and 81]). Uttersrud proved in [14] that in a $G$-space, the intersection of any family of $M$-ideals is an $M$-ideal. Recall that a real Banach space $V$ is said to be a $G$-space if there exists a compact Hausdorff space $X$ and a set

$$S = \{(x_\alpha, y_\alpha, \lambda_\alpha) \subseteq X \times X \times [-1,1],$$

such that $V$ is isometric to the space

$$Z = \{f \in C(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha)$ for all $(x_\alpha, y_\alpha, \lambda_\alpha) \in S\}.$$

Uttersrud also proved $V$ is a $G$-space if and only if the Alfsen-Effros topology on the extreme points of the dual ball is Hausdorff. In [11, Theorem 5.2], Roy gave a partial converse of Uttersrud’s theorem. In fact, she proved that if in a $L^1$-predual space $V$ the intersection of any family of $M$-ideals is an $M$-ideal, then $V$ is a $G$-space.

We should be repeatedly making use of the following result.

**Proposition 1.2** ([11, p.39f.]). Let $X$ be a Banach space, $J \subset Y \subset X$ and $Y$ be an $M$-ideal in $X$. Then $J$ is an $M$-ideal in $Y$ if and only if it is an $M$-ideal in $X$.

Reducible $M$-ideals are a subclass of $M$-ideals which introduced by Alfsen [14]. A reducible $M$-ideal is defined as follow.
Definition 1.3. An $M$-ideal $J$ in a Banach space $X$ is reducible if there exist $M$-ideals $J_1$ and $J_2$, $J \neq J_1, J_2$, such that $J = J_1 \cap J_2$. An $M$-ideal is irreducible if it is not reducible.

Theorem 1.4 ([14]). A Banach space $X$ is isometric to a $L^1$-predual space if every irreducible $M$-ideal $J \neq X$ is a hyperplane (i.e. $\text{codim}J = 1$).

We recall that a Banach space $X$ whose dual $X^*$ is isometric to $L^1(\mu)$ for some positive measure $\mu$ is called an $L^1$-predual.

Let us fix some more notation. Throughout this paper, $X$ is a real Banach space. $L(X)$ and $K(X)$ denote the spaces of linear continuous and linear compact operators, respectively. We shall always regard $X$ as a subspace of $X^{**}$. Any unexplained notion can be found in [11]. We hope our results be helpful in the study of the Banach space geometry.

Now we briefly describe the paper. First of all, we prove some elementary results on reducible $M$-ideals (Proposition 2.1). Then an upper bound for the number of reducible $M$-ideals in Banach spaces which have finitely many $M$-ideals is given (Proposition 2.2). To obtain a lower one, we use an interesting method called $R$-process. Given a compact metric space $X$, we characterize reducible $M$-ideals in the space of continuous functions $C(X)$ (Theorem 2.5). An interesting result asserts that the cardinal of the set of all $M$-ideals in an $M$-embedded space $X$ which the second dual of every closed subspace of it, is an $M$-ideal in $X^{**}$ is at least $\aleph_0$ (Theorem 2.13). In Theorem 2.15 we collect some Banach spaces $X$ and $Y$ such that $K(X,Y)$ is not a reducible $M$-ideal in $L(X,Y)$. Also, the concept of a semi reducible $M$-ideal is introduced and some results are given (Definition 2.16).

2. REDUCIBLE AND SEMI REDUCIBLE $M$-IDEALS

By $M-\text{ideal}(X)$ and $R-M-\text{ideal}(X)$ we mean the set of all $M$-ideals and reducible $M$-ideals in a Banach space $X$, respectively.

In the following, we give some elementary results on reducible $M$-ideals.

Proposition 2.1. Let $X$ be a Banach space.

(i) If $K \subset J$, $K \in R-M-\text{ideal}(J)$ and $J \in M-\text{ideal}(X)$, then $K \in R-M-\text{ideal}(X)$.

(ii) If $K \subset J$, $K \in R-M-\text{ideal}(J)$ and $J \in R-M-\text{ideal}(X)$, then $K \in R-M-\text{ideal}(X)$.

(iii) If $J, K \in R-M-\text{ideal}(X)$, then $J \oplus K \in R-M-\text{ideal}(X)$, where $\oplus$ denotes the direct sum.
(iv) Let $J_1, \ldots, J_n \in R - M - \text{ideal}(X)$. If there exists $k$, $1 \leq k \leq n$, such that $J_k \not\subset \bigcap_{i=1, i \neq k}^n J_i$ and $\bigcap_{i=1}^n J_i \not\subset J_k$, then $\bigcap_{i=1}^n J_i \in R - M - \text{ideal}(X)$.

(v) Suppose that $X$ is a $L^1$-predual $G$-space and $\{J_i\}_{i \in I} \subseteq R - M - \text{ideal}(X)$. If there exists $k \in I$ such that $\bigcap_{i \in I, i \neq k} J_i \not\subset J_k$ and $J_k \not\subset \bigcap_{i \in I, i \neq k} J_i$, then $\bigcap_{i \in I, i \neq k} J_i \in R - M - \text{ideal}(X)$.

Proof. (i) Firstly, Proposition 1.2 implies that $K \in M - \text{ideal}(X)$. Now, since $K \in R - M - \text{ideal}(J)$, there exist $K_1, K_2 \in M - \text{ideal}(J)$ such that $K = K_1 \cap K_2$, $K \neq K_1, K_2$. Notice that $K_1, K_2 \in M - \text{ideal}(X)$, by Proposition 1.2. This finishes the proof of this part.

(ii) It follows from (i).

(iii) Let $J = J_1 \cap J_2$, $J_1, J_2 \neq J$ and $K = K_1 \cap K_2$, $K \neq K_1, K_2$ where $J_i, K_i \in M - \text{ideal}(X)$, $i = 1, 2$. It is routine to check that $J \oplus K \in M - \text{ideal}(X)$. On the other hand, $J \oplus K = (K_1 \oplus (J_1 \cap J_2)) \cap (K_2 \oplus (J_1 \cap J_2))$ and $J \oplus K \neq K_1 \oplus (J_1 \cap J_2)$, $K_2 \oplus (J_1 \cap J_2)$. (This follows from the fact that if $L, M$ and $N$ are subspaces such that $L \oplus M = L \oplus N$, then $M = N$).

(iv) It follows from [12] that $\bigcap_{i=1}^n J_i \in M - \text{ideal}(X)$. We have

$$\bigcap_{i=1}^n J_i = J_k \cap \left( \bigcap_{i=1}^n \bigcap_{i \neq k} J_i \right).$$

If $\bigcap_{i=1}^n J_i = J_k$, then $J_k \subset \bigcap_{i=1}^n J_i$ which contradicts with the assumption. Similarly, if $\bigcap_{i=1}^n J_i = \bigcap_{i \neq k} J_i$, again, we get a contradiction. This shows that $\bigcap_{i=1}^n J_i \neq J_k$, $\bigcap_{i \neq k} J_i$, as desired.

(v) Since $X$ is a $L^1$-predual $G$-space, according to Theorem 10 of [12], $\bigcap_{i \in I} J_i \in M - \text{ideal}(X)$. The rest is similar to the proof of (iv).
The following result provides an upper bound for the number of reducible $M$-ideals in Banach spaces which have finitely many $M$-ideals. To observe a lower bound of the number of reducible $M$-ideals see Theorem 2.10 of this paper. Here, $|M - \text{ideal}(X)|$ denotes the number of $M$-ideals in $X$.

**Proposition 2.2.** Let $|M - \text{ideal}(X)| = n$.

(i) If $n = 2$, then $X$ has no reducible $M$-ideal.

(ii) If $n \geq 3$, then $|R - M - \text{ideal}(X)| \leq \frac{(n-2)(n-3)}{2}$.

**Proof.**

(i) Suppose that $|M - \text{ideal}(X)| = 2$, i.e., $X$ has no non-trivial $M$-ideal. Therefore, $X$ has no reducible $M$-ideal.

(ii) Since $X$ is not reducible in itself, hence the number of pairwise selections of $M$-ideals which may be reducible in $X$, is $\frac{(n-2)(n-3)}{2}$.

**Corollary 2.3.** If $0$ is a reducible $M$-ideal in $X$, then $|M - \text{ideal}(X)| > 3$.

**Corollary 2.4.** Let $1 < p < \infty$. Then $|R - M - \text{ideal}(L(l_p))| = 0$.

**Proof.** According to [13], $|M - \text{ideal}(X)| = 3$. Then Proposition 2.2(ii) yields that $L(l_p)$ has no reducible $M$-ideal.

The following result determines the general form of a reducible $M$-ideal in the space $C(X)$ of all continuous functions on a compact metric space $X$.

**Theorem 2.5.** Let $X$ be a compact metric space and $J \subset C(X)$ be an $M$-ideal. Then $J$ is reducible if and only if there exist closed subsets $F, Z$ and $W$ of $X$ such that $F = Z \cap W$, $F \neq Z, W$ and

$$J = J_F = K_Z \cap R_W,$$

where $J_F = \{x \in C(X) : x(s) = 0 \text{ for all } s \in F\}$, $K_Z, R_W \in M - \text{ideal}(C(X))$ and are defined as $J_F$.

**Proof.** Suppose that $J \subset C(X)$ is a reducible $M$-ideal. There exist $K, R \in M - \text{ideal}(C(X))$ so that $J \neq K, R$ and $J = K \cap R$. As $X$ is compact, $C_0(X) = C(X)$ and so by applying Example 1.4(a) of [13], we obtain closed subsets $F, Z$ and $W$ of $X$ such that $J = J_F, K = K_Z$ and $R = R_W$. Therefore, $J_F = K_Z \cap R_W$. Hence for every $f \in C(X)$ with
\( F \subseteq Z(f) \) we have \( Z \cap W \subseteq Z(f) \) and vice versa, where \( Z(f) \) denotes the zero set of \( f \). Now by compactness of \( X \) (and so completely regularity), one can deduce that \( F = Z \cap W \) ([13]). Moreover, \( F \neq Z, W \) follows from \( J \neq K, R \).

Conversely, let \( J = J_F = K_Z \cap R_W \), where \( F, Z \) and \( W \) are closed subsets of \( X \) such that \( F = Z \cap W \), \( F \neq Z, W \). We shall show that \( J_F \neq K_Z \). In exactly the same way, \( J_F \neq R_W \).

Let \( X \) be a compact metric space. We know from ([9]) that \( C(X) \) is an \( L^1 \)-predual space. Is there a closed subset \( D \) of \( X \) such that \( X \setminus D \) is not dense in \( X \) and \( J_D \) is irreducible in \( C(X) \)? If the answer is positive, then since \( X \setminus D \) is not dense in \( X \), Corollary 2.7 of [2] infers that \( J_D \) is not a \( VN \)-subspace. Now Corollary 3.7 of [2] implies that it is not a hyperplane. Hence we may give the following conjecture.

Conjecture 2.6. The converse of Theorem 1.4 does not hold.

It follows from the definition of a reducible \( M \)-ideal that the intersection of two \( M \)-ideals need not be reducible. But there is a subclass of \( M \)-ideals so-called maximal \( M \)-ideals which their finite intersections are reducible.

Definition 2.7. An nontrivial \( M \)-ideal \( J \) in \( X \) is called to be maximal \( M \)-ideal if when \( K \) is a nontrivial \( M \)-ideal in \( X \) containing \( J \), then \( K = J \).

Example 2.8. (i) Let \( 1 < p < \infty \). Then \( K(l_p) \) is only maximal \( M \)-ideal in \( L(l_p) \). (See [13]).

(ii) \( X = C[0, 1] \) has no maximal \( M \)-ideal. To see this, suppose that \( J \in M - \text{ideal}(X) \). Then there exist a closed subset \( F \) of \([0, 1]\) such that \( J = J_F \), where \( J_F \) is as Theorem 4.1. Now, let \( E \subseteq F \) be a closed subset of \([0, 1]\). Then \( K = K_E \in M - \text{ideal}(X) \) (see Example 1.4(a) of [2]). Finally, an application of Urysohn’s lemma yields that \( J \subseteq K \).

It is evident from Definition 2.7 that a maximal \( M \)-ideal cannot be reducible; but the intersection of two maximal \( M \)-ideals is reducible.

Proposition 2.9. Let \( K \) and \( J \) be maximal \( M \)-ideals in \( X \). Then \( K \cap J \in R - \text{ideal}(X) \).
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Proof. It follows from the definition of a maximal $M$-ideal.

The previous proposition motivates us to construct a process that produces a subset of $M$-ideals in $X$ such that every element of it, is maximal. For this purpose, let $J \in M - \text{ideal}(X)$. If there exist a nontrivial $M$-ideal $K$ in $X$ which contain $J$, then remove $J$ and repeat the process for other $M$-ideals in $X$. We call this $R$-process. The set of all $M$-ideals in $X$ which remained from the $R$-process denoted by $M - \text{ideal}(X)$.

We now give some useful results on $M - \text{ideal}(X)$.

Theorem 2.10. (i) The elements of $M - \text{ideal}(X)$ are precisely maximal $M$-ideals in $X$.

(ii) Let $J_1, \ldots, J_n \in M - \text{ideal}(X)$. If there exist $k$, $1 \leq k \leq n$, such that $\bigcap_{i=1, i \neq k}^{n} J_i \not\subset J_k$, then $\bigcap_{i=1}^{n} J_i \in R - M - \text{ideal}(X)$.

(iii) Let $X$ be a $G$-space and $\{J_i\}_{i \in I} \subseteq M - \text{ideal}(X)$. If there exist $k \in I$ such that $\bigcap_{i \in I, i \neq k} J_i \not\subset J_k$, then $\bigcap_{i \in I} J_i \in R - M - \text{ideal}(X)$.

Proof. (i) It is immediate from $R$-process.

(ii) Suppose $J_1, \ldots, J_n \in M - \text{ideal}(X)$. We have

$$\bigcap_{i=1}^{n} J_i = J_k \bigcap_{i=1, i \neq k}^{n} J_i.$$ 

Since $J_k \in M - \text{ideal}(X)$, $J_k \not\subset \bigcap_{i=1, i \neq k}^{n} J_i$. Using this and the assumption, we get the desired conclusion.

(iii) Let $\{J_i\}_{i \in I}$ be a family of $M$-ideals in $X$ belong to $M - \text{ideal}(X)$. Theorem 10 of [13] yields that $\bigcap_{i \in I} J_i \in M - \text{ideal}(X)$.

The rest is similar to the proof of (ii).

As an interesting consequence we infer that

Corollary 2.11. Let $X$ be such that every nontrivial $J \in M - \text{ideal}(X)$ be maximal and $|M - \text{ideal}(X)| = 4$. Then $|R - M - \text{ideal}(X)| = 1$. 

In general, the number of pairwise intersections of elements of \( M - M\text{-ideal}(X) \) does not equal to the number of all reducible \( M \)-ideals in \( X \) (for example if \( J \) is an \( M \)-ideal removed during \( R \)-process, and \( K \in M - M\text{-ideal}(X) \) such that \( K \cap J \neq K \cap L \) for every \( L \in M - M\text{-ideal}(X) \), and \( K \cap J \in R - M\text{-ideal}(X) \), then \( K \cap J \) has not derived from taking intersections of elements of \( M - M\text{-ideal}(X) \).) but it provides a lower bound for \( \text{Card}(R - M\text{-ideal}(X)) \). Therefore, we reach to the following easy corollaries. Given a \( J \in M - M\text{-ideal}(X) \), we denote by \( M_J(X) \), the set of all \( M \)-ideals in \( X \) which contained in \( J \).

**Corollary 2.12.** \( R - M\text{-ideal}(X) = A \cup B \cup C \), where

\[
A = \{ J \cap K : J, K \in M - M\text{-ideal}(X) \},
\]
\[
B = \{ L \cap N : L, N \in M - M\text{-ideal}(X) \setminus M - M\text{-ideal}(X), N \notin M_L(X) \text{ and } L \notin M_N(X) \},
\]
\[
C = \{ J \cap L : J \in M - M\text{-ideal}(X), L \in M - M\text{-ideal}(X) \setminus M - M\text{-ideal}(X) \text{ and } L \notin M_J(X) \}.
\]

**Corollary 2.13.** \( \text{Card}(R - M\text{-ideal}(X)) \geq \text{Card}A \), where \( A \) is as Corollary 2.12.

Notice that if \( X \) has no maximal \( M \)-ideal or its maximal \( M \)-ideals set is singleton, then the previous corollary does not give useful information on \( \text{Card}(R - M\text{-ideal}(X)) \). For instance, \( C[0,1] \) has no maximal \( M \)-ideal (Example 2.12) whereas it has many reducible \( M \)-ideals (Theorem 2.13). Furthermore, if every \( M \)-ideal in \( X \) is maximal, then \( \text{Card}(R - M\text{-ideal}(X)) = \text{Card}A \).

We shall use the shorthand phrase “\( X \) is \( M \)-embedded” to indicate that \( X \) is an \( M \)-ideal in its bidual. See [2], for several examples of such spaces and their geometric properties.

Next we establish a result on the cardinal of reducible \( M \)-ideals in a special \( M \)-embedded Banach space.

**Theorem 2.14.** Let \( X \) be an \( M \)-embedded Banach space such that \( Y^{**} \in M - M\text{-ideal}(X^{**}) \), for every closed subspace \( Y \) of \( X \). Then

\[
\text{Card}(R - M - M\text{-ideal}(X)) \geq \aleph_0.
\]

**Proof.** Since \( X \) is an \( M \)-ideal in its bidual, every closed subspace \( Y \) of \( X \) is an \( M \)-ideal in \( Y^{**} \) ([3, Theorem 3.4(a)]). Therefore, one can find a sequence \( \{Y_n\}_{n \in \mathbb{N}} \) of closed subspaces of \( X \) such that \( Y_n \subseteq Y_{n+1} \) and any \( Y_i, i \in \mathbb{N} \), is an \( M \)-ideal in \( Y_i^{**} \) and a closed proper subspace of \( Y_{i+1} \). Therefore, there exists a closed subspace \( Z_i \) of \( X \) such that \( Z_i \cap Y_i = Y_{i-1} \). Without loss of generality, assume that \( Z_i \neq Y_{i-1} \). On the other hand,
by assumption, $Z_i$ and $Y_i$ are $M$-ideals in $X^{**}$ and so by Proposition 1.2, $Z_i$ and $Y_i$ are also $M$-ideals in $X$. Therefore, $Y_{i-1} \in M$-ideal$(X)$. Thus $Y_{i-1} \in R - M$-ideal$(X)$. Hence the result. □

Let us mention some situations where $K(X, Y)$ is not a reducible $M$-ideal in $L(X, Y)$.

**Theorem 2.15.** Let

(i) $X = c_0$ and $Y = l_q$ for $1 \leq q < \infty$.
(ii) $X = c_0$ and $Y = L^1$.
(iii) $X = c_0$ and $Y = L^q$ for $1 < q < \infty$.
(iv) $X = l_\infty$ and $Y = l_q$ for $q < 2$.
(v) $X = l_p$ and $Y = l_q$ for $p > q$.
(vi) $X = l_p$ and $Y = L^1$ for $p > 2$.
(vii) $X = l_p$ and $Y = L^q$ for $p > q \vee 2$.
(viii) $X = L^p$ and $Y = l_q$ for $q < p \wedge 2$.
(ix) $X = C(E)$ and $Y = l_q$ when $C(E)^*$ is isomorphic to $l_1(E)$ or $q < 2$.
(x) $X = C(E)$ and $Y = L^1$ when $C(E)^*$ is isomorphic to $l_1(E)$.
(xi) $X = C(E)$ and $Y = L^q$ ($1 < q < \infty$) when $C(E)^*$ is isomorphic to $l_1(E)$.

Then in any of the above cases, $K(X, Y) \not\in R - M$-ideal$(L(X, Y))$.

**Proof.** It follows from [7] that $|M$-ideal$(L(X, Y))| = 2$ in any of (i)-(xi). Now by Proposition 2.2, $|R - M$-ideal$(L(X, Y))| = 0$. □

Since the finite intersection of $M$-ideals is an $M$-ideal, one may think that if an $M$-ideal $J$ is as the intersection of two subspaces, then any of intersection factors must be $M$-ideal. This need not be true. This motivates us to introduce the notion of a semi reducible $M$-ideal.

**Definition 2.16.** An $M$-ideal $J$ in a Banach space $X$ is semi reducible if there exist an $M$-ideal $K$ and a closed subspace $F$ such that $K, F \neq J$ and $J = K \cap F$.

We denote by $S - R - M$-ideal$(X)$ the set of all semi reducible $M$-ideals in $X$.

**Example 2.17.** Let $J, Z \in M$-ideal$(X)$ and $P$ be the corresponding $L$-projection from $X^*$ onto $J^\perp$. Suppose in addition that $Y$ is a closed subspace of $X$ such that $J \cap Y \neq J$, $J \cap Y \subset Z \subset Y$ and $P(Y^\perp) \subset Y^\perp$. Then $Y \cap J \in S - R - M$-ideal$(X)$. (See [7, Proposition 1.16]).
An argument similar to the one used to prove Proposition 2.1 gives the following.

**Proposition 2.18.** Let $X$ be a Banach space.

(i) If $J \in S - R - M - \text{ideal}(K)$ and $K \in M - \text{ideal}(X)$, then $J \in S - R - M - \text{ideal}(X)$.

(ii) If $K \subset J$, $K \in S - R - M - \text{ideal}(J)$ and $J \in S - R - M - \text{ideal}(X)$, then $K \in S - R - M - \text{ideal}(X)$.

We conclude the section with the following question.

**Question 2.19.** Under which condition(s) a semi reducible $M$-ideal is reducible?

**References**

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