Sahand Communications in Mathematical Analysis (SCMA) Vol. 7 No. 1 (2017), 27-37
http://scma.maragheh.ac.ir

ON THE REDUCIBLE M-IDEALS IN BANACH SPACES

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Abstract. The object of the investigation is to study reducible M-ideals in Banach spaces. It is shown that if the number of M-ideals in a Banach space X is $n(<\infty)$, then the number of reducible M-ideals does not exceed of $\frac{(n-2)(n-3)}{2}$. Moreover, given a compact metric space X, we obtain a general form of a reducible M-ideal in the space $C(X)$ of continuous functions on X. The intersection of two M-ideals is not necessarily reducible. We construct a subset of the set of all M-ideals in a Banach space X such that the intersection of any pair of it's elements is reducible. Also, some Banach spaces X and Y for which $K(X,Y)$ is not a reducible M-ideal in $L(X,Y)$, are presented. Finally, a weak version of reducible M-ideal called semi reducible M-ideal is introduced.

1. INTRODUCTION AND PRELIMINARIES

In the theory of Banach spaces, the concept of M-ideal, since its introduction by Alfsen and Effros [1], proves to be an important tool to study geometric and isometric properties of the spaces. The notion of an M-ideal generalizes the two sided ideals in a C^*-algebra; due to the geometric characterization of the ideals in these special algebras the M-ideals have been identified with the two sided ideals [12]. As it is quoted in [6], the fact that Y is an M-ideal in X has a strong impact on both Y and X, since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces. For instance, M-ideals are Hahn-Banach smooth subspaces, i.e., every norm one linear functional on an M-ideal has a unique norm one extension to X. The study of the

2010 *Mathematics Subject Classification.* 46B20.
Key words and phrases. M-ideal, Reducible M-ideal, Maximal M-ideal, M-embedded space, Semi reducible M-ideal.
Received: 19 November 2016, Accepted: 25 January 2017.
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phenomenon of unique Hahn-Banach extensions was initiated by Phelps in [10].

Here, we have some definitions and results which will be needed in the sequel.

Definition 1.1. A closed subspace J of a Banach space X is called M-ideal if there exist a linear projection $P : X^* \to J^\perp$ such that

$$\|f\| = \|Pf\| + \|f - Pf\|,$$

for all $f \in X^*$, where $J^\perp = \{f \in X^* : \ f(x) = 0 \text{ for any } x \in J\}$ is the annihilator of J in X^*. Such a projection is called a L-projection and its range a L-summand.

In [1], Alfsen and Effros proved that a closed subspace J of a Banach space X is an M-ideal if and only if it satisfies in n-ball property ($n \in \mathbb{N}, n \geq 3$). We say that a subspace J of a Banach space X satisfies the n-ball property if given n open balls $B_1, \ldots, B_n$ for which $B_1 \cap \ldots \cap B_n \neq \emptyset$, and $B_i \cap J \neq \emptyset$, $i = 1, \ldots, n$, it follows that $B_1 \cap \ldots \cap B_n \cap J \neq \emptyset$.

Finite intersection of M-ideals is also an M-ideal ([6, Proposition 1.11(b)]); but it need not be true for any family of M-ideals (see [9, P.78 and 81]). Uttersrud proved in [14] that in a G-space, the intersection of any family of M-ideals is an M-ideal. Recall that a real Banach space V is said to be a G-space if there exists a compact Hausdorff space X and a set

$$S = \{(x_\alpha, y_\alpha, \lambda_\alpha)\} \subseteq X \times X \times [-1, 1],$$

such that V is isometric to the space

$$Z = \{f \in C(X) : \ f(x_\alpha) = \lambda_\alpha f(y_\alpha) \text{ for all } (x_\alpha, y_\alpha, \lambda_\alpha) \in S\}.$$

Uttersrud also proved V is a G-space if and only if the Alfsen-Effros topology on the extreme points of the dual ball is Hausdorff. In [11, Theorem 5.2], Roy gave a partial converse of Uttersrud's theorem. In fact, she proved that if in a L^1-predual space V the intersection of any family of M-ideals is an M-ideal, then V is a G-space.

We should be repeatedly making use of the following result.

Proposition 1.2 ([4, p.39f.])**.** *Let X be a Banach space, $J \subset Y \subset X$ and Y be an M-ideal in X. Then J is an M-ideal in Y if and only if it is an M-ideal in X.*

Reducible M-ideals are a subclass of M-ideals which introduced by Alfsen [14]. A reducible M-ideal is defined as follow.

Definition 1.3. An M-ideal J in a Banach space X is reducible if there exist M-ideals J_1 and J_2, $J \neq J_1, J_2$, such that $J = J_1 \cap J_2$. An M-ideal is irreducible if it is not reducible.

Theorem 1.4 ([14])**.** *A Banach space X is isometric to a L^1-predual space if every irreducible M-ideal $J \neq X$ is a hyperplane (i.e. $codim J = 1$).*

We recall that a Banach space X whose dual X^* is isometric to $L^1(\mu)$ for some positive measure μ is called an L^1-predual.

Let us fix some more notation. Throughout this paper, X is a real Banach space. $L(X)$ and $K(X)$ denote the spaces of linear continuous and linear compact operators, respectively. We shall always regard X as a subspace of X^{**}. Any unexplained notion can be found in [6]. We hope our results be helpful in the study of the Banach space geometry.

Now we briefly describe the paper. First of all, we prove some elementary results on reducible M-ideals (Proposition 2.1). Then an upper bound for the number of reducible M-ideals in Banach spaces which have finitely many M-ideals is given (Proposition 2.2). To obtain a lower one, we use an interesting method called R-process. Given a compact metric space X, we characterize reducible M-ideals in the space of continuous functions $C(X)$ (Theorem 2.5). An interesting result asserts that the cardinal of the set of all M-ideals in an M-embedded space X which the second dual of every closed subspace of it, is an M-ideal in X^{**} is at least $\aleph_0$ (Theorem 2.14). In Theorem 2.15 we collect some Banach spaces X and Y such that $K(X, Y)$ is not a reducible M-ideal in $L(X, Y)$. Also, the concept of a semi reducible M-ideal is introduced and some results are given (Definition 2.16).

2. REDUCIBLE AND SEMI REDUCIBLE M-IDEALS

By $M{-}ideal(X)$ and $R{-}M{-}ideal(X)$ we mean the set of all M-ideals and reducible M-ideals in a Banach space X, respectively.

In the following, we give some elementary results on reducible M-ideals.

Proposition 2.1. *Let X be a Banach space.*

(i) *If $K \subset J$, $K \in R - M - ideal(J)$ and $J \in M - ideal(X)$, then $K \in R - M - ideal(X)$.*
(ii) *If $K \subset J$, $K \in R - M - ideal(J)$ and $J \in R - M - ideal(X)$, then $K \in R - M - ideal(X)$.*
(iii) *If $J, K \in R - M - ideal(X)$, then $J \oplus K \in R - M - ideal(X)$, where $\oplus$ denotes the direct sum.*

(iv) *Let* $J_1, \ldots, J_n \in R-M-ideal(X)$. *If there exists* k, $1 \leq k \leq n$, *such that* $J_k \not\subset \bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i$ *and* $\bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i \not\subset J_k$, *then* $\bigcap_{i=1}^{n} J_i \in R-M-ideal(X)$.

(v) *Suppose that* X *is a* L^1*-predual* G*-space and* $\{J_i\}_{i\in I} \subseteq R-M-ideal(X)$. *If there exists* $k \in I$ *such that* $\bigcap_{\substack{i\in I\\ i\neq k}} J_i \not\subset J_k$ *and* $J_k \not\subset \bigcap_{\substack{i\in I\\ i\neq k}} J_i$, *then* $\bigcap_{i\in I} J_i \in R-M-ideal(X)$.

Proof. (i) Firstly, Proposition 1.2 implies that $K \in M-ideal(X)$. Now, since $K \in R-M-ideal(J)$, there exist $K_1, K_2 \in M-ideal(J)$ such that $K = K_1 \cap K_2$, $K \neq K_1, K_2$. Notice that $K_1, K_2 \in M-ideal(X)$, by Proposition 1.2. This finishes the proof of this part.

(ii) It follows from (i).

(iii) Let $J = J_1 \cap J_2$, $J_1, J_2 \neq J$ and $K = K_1 \cap K_2$, $K \neq K_1, K_2$ where $J_i, K_i \in M-ideal(X)$, $i = 1, 2$. It is routine to check that $J \oplus K \in M-ideal(X)$. On the other hand, $J \oplus K = (K_1 \oplus (J_1 \cap J_2)) \cap (K_2 \oplus (J_1 \cap J_2))$ and $J \oplus K \neq K_1 \oplus (J_1 \cap J_2)$, $K_2 \oplus (J_1 \cap J_2)$. (This follows from the fact that if L, M and N are subspaces such that $L \oplus M = L \oplus N$, then $M = N$).

(iv) It follows from [6] that $\bigcap_{i=1}^{n} J_i \in M-ideal(X)$. We have

$$\bigcap_{i=1}^{n} J_i = J_k \bigcap \left(\bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i \right).$$

If $\bigcap_{i=1}^{n} J_i = J_k$, then $J_k \subset \bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i$ which contradicts with the assumption. Similarly, if $\bigcap_{i=1}^{n} J_i = \bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i$, again, we get a contradiction. This shows that $\bigcap_{i=1}^{n} J_i \neq J_k, \bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i$, as desired.

(v) Since X is a L^1-predual G-space, according to Theorem 10 of [14], $\bigcap_{i\in I} J_i \in M-ideal(X)$. The rest is similar to the proof of (iv).

□

The following result provides an upper bound for the number of reducible M-ideals in Banach spaces which have finitely many M-ideals. To observe a lower bound of the number of reducible M-ideals see Theorem 2.10 of this paper. Here, $|M - ideal(X)|$ denotes the number of M-ideals in X.

Proposition 2.2. *Let* $|M - ideal(X)| = n$.

(i) *If* $n = 2$*, then* X *has no reducible* M*-ideal.*
(ii) *If* $n \geq 3$*, then* $|R - M - ideal(X)| \leq \frac{(n-2)(n-3)}{2}$.

Proof. (i) Suppose that $|M - ideal(X)| = 2$, i.e., X has no non-trivial M-ideal. Therefore, X has no reducible M-ideal.

(ii) Since X is not reducible in itself, hence the number of pairwise selections of M-ideals which may be reducible in X, is $\frac{(n-2)(n-3)}{2}$.

□

Corollary 2.3. *If* 0 *is a reducible* M*-ideal in* X*, then*

$$|M - ideal(X)| > 3.$$

Corollary 2.4. *Let* $1 < p < \infty$*. Then* $|R - M - ideal(L(l_p))| = 0$.

Proof. According to [13], $|M - ideal(X)| = 3$. Then Proposition 2.2(ii) yields that $L(l_p)$ has no reducible M-ideal. □

The following result determines the general form of a reducible M-ideal in the space $C(X)$ of all continuous functions on a compact metric space X.

Theorem 2.5. *Let* X *be a compact metric space and* $J \subset C(X)$ *be an* M*-ideal. Then* J *is reducible if and only if there exist closed subsets* F, Z *and* W *of* X *such that* $F = Z \cap W$, $F \neq Z, W$ *and*

$$J = J_F = K_Z \cap R_W,$$

where $J_F = \{x \in C(X) : x(s) = 0 \text{ for all } s \in F\}$, $K_Z, R_W \in M - ideal(C(X))$ *and are defined as* J_F.

Proof. Suppose that $J \subset C(X)$ is a reducible M-ideal. There exist $K, R \in M - ideal(C(X))$ so that $J \neq K, R$ and $J = K \cap R$. As X is compact, $C_0(X) = C(X)$ and so by applying Example 1.4(a) of [6], we obtain closed subsets F, Z and W of X such that $J = J_F$, $K = K_Z$ and $R = R_W$. Therefore, $J_F = K_Z \cap R_W$. Hence for every $f \in C(X)$ with

$F \subseteq Z(f)$ we have $Z \cap W \subseteq Z(f)$ and vice versa, where $Z(f)$ denotes the zero set of f. Now by compactness of X (and so completely regularity), one can deduce that $F = Z \cap W$ ([15]). Moreover, $F \neq Z, W$ follows from $J \neq K, R$.
Conversely, let $J = J_F = K_Z \cap R_W$, where F, Z and W are closed subsets of X such that $F = Z \cap W$, $F \neq Z, W$. We shall show that $J_F \neq K_Z, R_W$. To see this, we define $x \in C(X)$ by $x(t) = d(t, F) = \inf\{d(t, r) : \ r \in F\}$. It is clear that $x|_F = 0$. Assume that $t_0 \in Z \backslash F$. If $x(t_0) = 0$, then there is a sequence $\{r_n\}_{n \in \mathbb{N}}$ of elements of F such that $r_n \to t_0$ as $n \to \infty$. Then $t_0 \in F$, since F is closed. This contradiction shows that $x|_Z \neq 0$. Therefore, $J_F \neq K_Z$. In exactly the same way, $J_F \neq R_W$. □

Let X be a compact metric space. We know from ([9]) that $C(X)$ is an L^1-predual space. Is there a closed subset D of X such that $X \backslash D$ is not dense in X and J_D is irreducible in $C(X)$? If the answer is positive, then since $X \backslash D$ is not dense in X, Corollary 2.7 of [2] infers that J_D is not a VN-subspace. Now Corollary 3.7 of [2] implies that it is not a hyperplane. Hence we may give the following conjecture.

Conjecture 2.6. *The converse of Theorem 1.4 does not hold.*

It follows from the definition of a reducible M-ideal that the intersection of two M-ideals need not be reducible. But there is a subclass of M-ideals so-called maximal M-ideals which their finite intersections are reducible.

Definition 2.7. An nontrivial M-ideal J in X is called to be maximal M-ideal if when K is a nontrivial M-ideal in X containing J, then $K = J$.

Example 2.8.
(i) Let $1 < p < \infty$. Then $K(l_p)$ is only maximal M-ideal in $L(l_p)$. (See [13]).
(ii) $X = C[0, 1]$ has no maximal M-ideal. To see this, suppose that $J \in M - ideal(X)$. Then there exist a closed subset F of $[0, 1]$ such that $J = J_F$, where J_F is as Theorem 2.5. Now, let $E \subsetneq F$ be a closed subset of $[0, 1]$. Then $K = K_E \in M - ideal(X)$ (see Example 1.4(a) of [6]). Finally, an application of Urysohn's lemma yields that $J \subsetneq K$.

It is evident from Definition 2.7 that a maximal M-ideal cannot be reducible; but the intersection of two maximal M-ideals is reducible.

Proposition 2.9. *Let K and J be maximal M-ideals in X. Then $K \cap J \in R - M - ideal(X)$.*

Proof. It follows from the definition of a maximal M-ideal. □

The previous proposition motivates us to construct a process that produces a subset of M-ideals in X such that every element of it, is maximal. For this purpose, let $J \in M - ideal(X)$. If there exist a nontrivial M-ideal K in X which contain J, then remove J and repeat the process for other M-ideals in X. We call this R-process. The set of all M-ideals in X which remained from the R-process denoted by $M - M - ideal(X)$.

We now give some useful results on $M - M - ideal(X)$.

Theorem 2.10. (i) *The elements of $M - M - ideal(X)$ are precisely maximal M-ideals in X.*

(ii) *Let $J_1, \dots, J_n \in M - M - ideal(X)$. If there exist k, $1 \le k \le n$, such that $\bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i \not\subset J_k$, then $\bigcap_{i=1}^{n} J_i \in R - M - ideal(X)$.*

(iii) *Let X be a G-space and $\{J_i\}_{i\in I} \subseteq M - M - ideal(X)$. If there exist $k \in I$ such that $\bigcap_{\substack{i\in I\\ i\neq k}} J_i \not\subset J_k$, then*

$$\bigcap_{i\in I} J_i \in R - M - ideal(X).$$

Proof. (i) It is immediate from R-process.

(ii) Suppose $J_1, \dots, J_n \in M - M - ideal(X)$. We have

$$\bigcap_{i=1}^{n} J_i = J_k \bigcap \left(\bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i \right).$$

Since $J_k \in M - M - ideal(X)$, $J_k \not\subset \bigcap_{\substack{i=1\\ i\neq k}}^{n} J_i$. Using this and the assumption, we get the desired conclusion.

(iii) Let $\{J_i\}_{i\in I}$ be a family of M-ideals in X belong to $M - M - ideal(X)$. Theorem 10 of [14] yields that $\bigcap_{i\in I} J_i \in M - ideal(X)$. The rest is similar to the proof of (ii).

□

As an interesting consequence we infer that

Corollary 2.11. *Let X be such that every nontrivial $J \in M - ideal(X)$ be maximal and $|M - ideal(X)| = 4$. Then $|R - M - ideal(X)| = 1$.*

In general, the number of pairwise intersections of elements of $M-M-ideal(X)$ does not equal to the number of all reducible M-ideals in X (for example if J is an M-ideal removed during R-process, and $K \in M-M-ideal(X)$ such that $K \cap J \neq K \cap L$ for every $L \in M-M-ideal(X)$, and $K \cap J \in R-M-ideal(X)$, then $K \cap J$ has not derived from taking intersections of elements of $M-M-ideal(X)$.); but it provides a lower bound for $Card(R-M-ideal(X))$. Therefore, we reach to the following easy corollaries. Given a $J \in M-ideal(X)$, we denote by $M_J(X)$, the set of all M-ideals in X which contained in J.

Corollary 2.12. *$R-M-ideal(X) = A \cup B \cup C$, where*
$A = \{J \cap K : \ J, K \in M-M-ideal(X)\}$,
$B = \{L \cap N : \ L, N \in M-ideal(X) \backslash M-M-ideal(X), N \notin M_L(X) \text{ and } L \notin M_N(X)\}$,
$C = \{J \cap L : \ J \in M-M-ideal(X), L \in M-ideal(X) \backslash M-M-ideal(X) \text{ and } L \notin M_J(X)\}$.

Corollary 2.13. *$Card(R-M-ideal(X)) \geq CardA$, where A is as Corollary 2.12.*

Notice that if X has no maximal M-ideal or its maximal M-ideals set is singleton, then the previous corollary does not give useful information on $Card(R-M-ideal(X))$. For instance, $C[0,1]$ has no maximal M-ideal (Example 2.8) whereas it has many reducible M-ideals (Theorem 2.5). Furthermore, if every M-ideal in X is maximal, then $Card(R-M-ideal(X)) = CardA$.

We shall use the shorthand phrase "X is M-embedded" to indicate that X is an M-ideal in its bidual. See [5], for several examples of such spaces and their geometric properties.

Next we establish a result on the cardinal of reducible M-ideals in a special M-embedded Banach space.

Theorem 2.14. *Let X be an M-embedded Banach space such that $Y^{**} \in M-ideal(X^{**})$, for every closed subspace Y of X. Then*

$$Card(R-M-ideal(X)) \geq \aleph_0.$$

Proof. Since X is an M-ideal in its bidual, every closed subspace Y of X is an M-ideal in Y^{**} ([5, Theorem 3.4(a)]). Therefore, one can find a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of closed subspaces of X such that $Y_n \subsetneq Y_{n+1}$ and any Y_i, $i \in \mathbb{N}$, is an M-ideal in Y_i^{**} and a closed proper subspace of Y_{i+1}. Therefore, there exists a closed subspace Z_i of X such that $Z_i \cap Y_i = Y_{i-1}$. Without loss of generality, assume that $Z_i \neq Y_{i-1}$. On the other hand, by assumption, Z_i and Y_i are M-ideals in X^{**} and so by Proposition 1.2, Z_i and Y_i are also M-ideals in X. Therefore, $Y_{i-1} \in M - ideal(X)$. Thus $Y_{i-1} \in R - M - ideal(X)$. Hence the result. □

Let us mention some situations where $K(X,Y)$ is not a reducible M-ideal in $L(X,Y)$.

Theorem 2.15. *Let*

(i) $X = c_0$ *and* $Y = l_q$ *for* $1 \leq q < \infty$.
(ii) $X = c_0$ *and* $Y = L^1$.
(iii) $X = c_0$ *and* $Y = L^q$ *for* $1 < q < \infty$.
(iv) $X = l_\infty$ *and* $Y = l_q$ *for* $q < 2$.
(v) $X = l_p$ *and* $Y = l_q$ *for* $p > q$.
(vi) $X = l_p$ *and* $Y = L^1$ *for* $p > 2$.
(vii) $X = l_p$ *and* $Y = L^q$ *for* $p > q \vee 2$.
(viii) $X = L^p$ *and* $Y = l_q$ *for* $q < p \wedge 2$.
(ix) $X = C(E)$ *and* $Y = l_q$ *when* $C(E)^*$ *is isomorphic to* $l_1(E)$ *or* $q < 2$.
(x) $X = C(E)$ *and* $Y = L^1$ *when* $C(E)^*$ *is isomorphic to* $l_1(E)$.
(xi) $X = C(E)$ *and* $Y = L^q$ $(1 < q < \infty)$ *when* $C(E)^*$ *is isomorphic to* $l_1(E)$.

Then in any of the above cases, $K(X,Y) \notin R - M - ideal(L(X,Y))$.

Proof. It follows from [7] that $|M - ideal(L(X,Y))| = 2$ in any of (i)-(xi). Now by Proposition 2.2, $|R - M - ideal(L(X,Y))| = 0$. □

Since the finite intersection of M-ideals is an M-ideal, one may think that if an M-ideal J is as the intersection of two subspaces, then any of intersection factors must be M-ideal. This need not be true. This motivates us to introduce the notion of a semi reducible M-ideal.

Definition 2.16. An M-ideal J in a Banach space X is semi reducible if there exist an M-ideal K and a closed subspace F such that $K, F \neq J$ and $J = K \cap F$.

We denote by $S - R - M - ideal(X)$ the set of all semi reducible M-ideals in X.

Example 2.17. Let $J, Z \in M - ideal(X)$ and P be the corresponding L-projection from X^* onto $J^\perp$. Suppose in addition that Y is a closed subspace of X such that $J \cap Y \neq J$, $J \cap Y \subset Z \subsetneq Y$ and $P\left(Y^\perp\right) \subset Y^\perp$. Then $Y \cap J \in S - R - M - ideal(X)$. (See [6, Proposition 1.16]).

An argument similar to the one used to prove Proposition 2.1 gives the following.

Proposition 2.18. *Let X be a Banach space.*

(i) *If $J \in S-R-M-ideal(K)$ and $K \in M-ideal(X)$, then $J \in S-R-M-ideal(X)$.*
(ii) *If $K \subset J$, $K \in S-R-M-ideal(J)$ and $J \in S-R-M-ideal(X)$, then $K \in S-R-M-ideal(X)$.*

We conclude the section with the following question.

Question 2.19. *Under which condition(s) a semi reducible M-ideal is reducible?*

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