Sahand Communications in Mathematical Analysis (SCMA) Vol. 7 No. 1 (2017), 75-83
http://scma.maragheh.ac.ir

G-ASYMPTOTIC CONTRACTIONS IN METRIC SPACES WITH A GRAPH AND FIXED POINT RESULTS

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Abstract. In this paper, we discuss the existence and uniqueness of fixed points for G-asymptotic contractions in metric spaces endowed with a graph. The result given here is a new version of Kirk's fixed point theorem for asymptotic contractions in metric spaces endowed with a graph. The given result here is a generalization of fixed point theorem for asymptotic contraction from metric s paces to metric spaces endowed with a graph.

1. Introduction

2. Introduction and Preliminaries

In 2003, Kirk [8] investigated the existence and uniqueness of fixed points for asymptotic contractions in metric spaces but Jachymski and Jóźwik [7] constructed a counterexample showing that the continuity of the self-map is essential in Kirk's theorem. They also established a new result for uniformly continuous asymptotic φ-contractions in metric spaces (see [7, Theorem 2]). Finally, Aghanians et al. [1] generalized Kirk's fixed point theorem for p-continuous E-asymptotic contractions in separated uniform spaces via E-distances.

In 2008, Jachymski [6] entered graphs in metric fixed point theory by endowing the underlying metric space with a directed graph and formulated the Banach contraction principle in a graph language. This idea followed by several authors (see e.g., [2, 5]).

The main purpose of this paper is to investigate the existence and uniqueness of fixed points for Kirk's asymptotic contractions in metric

2010 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 05C20.
Key words and phrases. G-asymptotic contraction, Orbitally G-continuous self-map, Fixed point.
Received: 14 January 2017, Accepted: 30 January 2017.

spaces endowed with a graph. Our main result is a new version of [8, Theorem 2.1] in metric spaces endowed with a graph.

Consistent with Bondy and Murthy as well as Jachymski, the following concepts is needed in the sequel. For a idespread discussion on the theory of graphs, the reader is refered to [3, 6].

Let (X, d) be a metric space and G be a directed graph with vertex set $V(G) = X$ such that the edge set $E(G)$ contains all loops, that is, $(x, x) \in E(G)$ for all $x \in X$. Assume further that G has no parallel edges. Then the graph G can be easily denoted by the ordered pair $(V(G), E(G))$ and it is said that the metric space (X, d) is endowed with G.

The metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \big\{(x, y) : (y, x) \in E(G)\big\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

If $x, y \in X$, then a finite sequence $(x_i)_{i=0}^N$ consisting of $N + 1$ vertices is called a path in G from x to y whenever $x_0 = x$, $x_N = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \dots, N$. The graph G is called connected if there exists a path in G between each two vertices of G.

A graph H is called a subgraph of G if $V(H)$ and $E(H)$ are contained in $V(G)$ and $E(G)$, respectively and also $(x, y) \in E(H)$ implies $x, y \in V(H)$.

Following Petruşel and Rus [9], one can naturally formulate Picard operators in metric spaces as follows:

Definition 2.1 ([6, 9])**.** Let (X, d) be a metric space. A self-map f on X is called a Picard operator if f has a unique fixed point x^* in X and $f^n x \to x^*$ for all $x \in X$.

We also need a weaker type of continuity in metric spaces endowed with a graph. The idea of this definition comes from the definition of orbital continuity defined by Cirić [4].

Definition 2.2 ([6])**.** Let (X, d) be a metric space endowed with a graph G. A self-map f on X is called orbitally G-continuous if for each $x, y \in X$ and each sequence $\{\lambda_n\}$ of positive integers satisfying $(f^{\lambda_n} x, f^{\lambda_{n+1}} x) \in E(G)$ for $n = 1, 2, \dots$ and $f^{\lambda_n} x \to y$, one has $f(f^{\lambda_n} x) \to fy$.

Obviously, a continuous mapping on a metric space is orbitally G-continuous for all graphs G but the converse is not necessarily true. The next example shows the effective role played by the graph to imply a weaker type of continuity.

Example 2.3. Consider the set $X = \{\frac{1}{n} : n \geq 1\} \cup \{0\}$ equipped with the usual metric and define a self-map $f : X \to X$ by $f(0) = 0$ and $f(x) = \frac{2}{3}$ for all $x \in X$ with $x \neq 0$. Then it is clear that f is not continuous at $x = 0$, and so on the whole set X. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and

$$E(G) = \{(x,x) : x \in X\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{n+1}\right) : n \geq 1 \right\} \cup \left\{ \left(\frac{1}{n}, 0\right) : n \geq 1 \right\}.$$

If $x, y \in X$ and $\{\lambda_n\}$ is a sequence of positive integers satisfying $(f^{\lambda_n}x, f^{\lambda_{n+1}}x) \in E(G)$ for $n = 1, 2, \dots$ and $f^{\lambda_n}x \to y$, then $\{f^{\lambda_n}x\}$ is necessarily a constant sequence and either $y = 0$ or $y = \frac{2}{3}$. Thus, $f^{\lambda_n}x = y$ for all $n \geq 1$ and so $f(f^{\lambda_n}x) \to fy$. Hence f is orbitally G-continuous on X.

3. Main Results

In this section, we assume that (X, d) is a metric space endowed with a graph G. We denote by $\mathrm{Fix}(f)$ the set of all fixed points of a self-map f on X, and we use C_f to denote the set of all points $x \in X$ such that $(f^m x, f^n x)$ is an edge of $\widetilde{G}$ for all $m, n \in \mathbb{N} \cup 0$. In other words,

$$C_f = \left\{ x \in X : (f^m x, f^n x) \in E(\widetilde{G}), \quad m, n = 0, 1, \cdots \right\}.$$

We further assume that Ψ is the class of all continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$ for all $t > 0$. Clearly, if $\psi \in \Psi$, then

$$0 \leq \psi(0) = \lim_{t \to 0^+} \psi(t) \leq \lim_{t \to 0^+} t = 0,$$

that is, $\psi(0) = 0$. Therefore, $\psi(t) \leq t$ for all $t \geq 0$.

Now, we are ready to define G-asymptotic contractions in metric spaces endowed with a graph. This definition is motivated from [7, Definition 2.1] and [8, Definition 2.1].

Definition 3.1. Let (X, d) be metric space endowed with garph G. We say that a self-map f on X is a G-asymptotic contraction if

(A1) f preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;

(A2) there exists a sequence $\psi_n : [0, +\infty) \to [0, +\infty)$ converging uniformly to a $\psi \in \Psi$ on the range of d such that

$$d(f^n x, f^n y) \leq \psi_n\big(d(x, y)\big) \tag{3.1}$$

for all $n \geq 1$ and all $x, y \in X$ with $(x, y) \in E(G)$.

We next give some examples of G-asymptptic contractions in metric spaces endowed with a graph.

Example 3.2. Suppose that (X,d) is a metric space and G_0 is the complete graph with the vertex set X, that is, $V(G_0)=X$ and $E(G_0)=X\times X$. Then (A1) holds trivially, and (A2) means that

$$d(f^n x, f^n y) \le \psi_n\left(d(x,y)\right), \quad (x,y\in X),$$

where $\psi_n : [0,+\infty)\to[0,+\infty)$ converges uniformly to a $\psi\in\Psi$ on the range of d. Hence G_0-asymptotic contractions are precicely the asymptotic contractions introduced by Kirk.

Example 3.3. Suppose that $(X,\preceq)$ is a partially ordered set and d is a metric on X. Consider the poset graph G_1 with $V(G_1)=X$ and $E(G_1)=\{(x,y)\in X\times X : x\preceq y\}$. Then (A1) holds if and only if f is nondecreasing with respect to $\preceq$, and (A2) means that

$$d(f^n x, f^n y) \le \psi_n\big(d(x,y)\big)$$

for all comparable elements $x,y\in X$, where $\psi_n : [0,+\infty)\to[0,+\infty)$ converges uniformly to a $\psi\in\Psi$ on the range of d.

Example 3.4. Suppose that $(X,\preceq)$ is a partially ordered set and d is a metric on X. Consider the poset graph G_2 defined by $V(G_2)=X$ and $E(G_2)=\{(x,y)\in X\times X : x\preceq y \ \vee\ y\preceq x\}$. Then (A1) holds if and only if f maps comparable elements of X onto comparable elements of X, and (A2) means that

$$d(f^n x, f^n y) \le \psi_n\big(d(x,y)\big)$$

for all comparable elements $x,y\in X$, where $\psi_n : [0,+\infty)\to[0,+\infty)$ converges uniformly to a $\psi\in\Psi$ on the range of d, as in Example 3.2.

Example 3.5. Suppose that (X,d) is a metric space and ε is a fixed positive real number. Two elements $x,y\in X$ are called ε-close whenever $d(x,y)<\varepsilon$. Define a graph G_3 by $V(G_3)=X$ and $E(G_3)=\{(x,y)\in X\times X :\ d(x,y)<\varepsilon\}$. Then (A1) holds if and only if f maps ε-close elements of X onto ε-close elements, and (A2) means that

$$d(f^n x, f^n y) \le \psi_n\big(d(x,y)\big)$$

for all ε-close elements $x,y\in X$, where $\psi_n : [0,+\infty)\to[0,+\infty)$ converges uniformly to a $\psi\in\Psi$ on the range of d.

To prove our main theorem, we need the next lemma.

Lemma 3.6. *Let (X,d) be a metric space endowed with a graph G and $f:X\to X$ be a G-asymptotic contraction such that the functions ψ_n in (3.1) are continuous on $[0,+\infty)$ for sufficiently large indices n. Then $\{f^n x\}$ is a Cauchy sequence for all $x\in C_f$.*

Proof. Let $x_0 \in C_f$. Then $(f^n x_0, f^{n+1}x_0) \in E(\widetilde{G})$ for all $n \geq 0$. Moreover, f is clearly a $\widetilde{G}$-asymptotic contraction. So from (3.1) we get

$$\begin{aligned} \limsup_{n\to\infty} d(f^n x_0, f^{n+1}x_0) &\leq \lim_{n\to\infty} \psi_n\big(d(x_0, fx_0)\big) \\ &= \psi\big(d(x_0, fx_0)\big) \\ &\leq d(x_0, fx_0) < \infty. \end{aligned} \tag{3.2}$$

On the other hand, if

$$\limsup_{n\to\infty} d(f^n x_0, f^{n+1}x_0) = \eta > 0,$$

then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $d(f^{n_k}x_0, f^{n_k+1}x_0) \to \eta$, and so by the continuity of ψ, we obtain

$$\psi\big(d(f^{n_k}x_0, f^{n_k+1}x_0)\big) \to \psi(\eta) < \eta.$$

Hence there is a positive integer k_0 with $\psi\left(d(f^{n_{k_0}}x_0, f^{n_{k_0}+1}x_0)\right) < \eta$ and thus, by (3.1) we get

$$\begin{aligned} \eta &= \limsup_{n\to\infty} d(f^n x_0, f^{n+1}x_0) \\ &= \limsup_{n\to\infty} d\big(f^n(f^{n_{k_0}}x_0), f^n(f^{n_{k_0}+1}x_0)\big) \\ &\leq \lim_{n\to\infty} \psi_n\big(d(f^{n_{k_0}}x_0, f^{n_{k_0}+1}x_0)\big) \\ &= \psi\big(d(f^{n_{k_0}}x_0, f^{n_{k_0}+1}x_0)\big) < \eta, \end{aligned}$$

which is a contradiction. Therefore,

$$\limsup_{n\to\infty} d(f^n x_0, f^{n+1}x_0) = 0.$$

Consequently,

$$0 \leq \liminf_{n\to\infty} d(f^n x_0, f^{n+1}x_0) \leq \limsup_{n\to\infty} d(f^n x_0, f^{n+1}x_0) = 0,$$

that is,

$$\lim_{n\to\infty} d(f^n x_0, f^{n+1}x_0) = 0. \tag{3.3}$$

Now, if the sequence $\{f^n x_0\}$ is not Cauchy, then there exist an $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$m_k > n_k \geq k \quad \text{and} \quad d(f^{m_k}x_0, f^{n_k}x_0) \geq \varepsilon, \quad k = 1, 2, \dots .$$

Keeping the integer n_k fixed for sufficiently large k, say $k \geq k_0$, we can assume without loss of generality that m_k is the smallest integer greater than n_k with $d(f^{m_k}x_0, f^{n_k}x_0) \geq \varepsilon$, that is,

$$d(f^{m_k-1}x_0, f^{n_k}x_0) < \varepsilon, \quad (k \geq k_0).$$

Hence for each $k \geq k_0$, we have

$$\begin{aligned}\varepsilon &\leq d(f^{m_k}x_0, f^{n_k}x_0)\\ &\leq d(f^{m_k}x_0, f^{m_k-1}x_0) + d(f^{m_k-1}x_0, f^{n_k}x_0)\\ &< d(f^{m_k}x_0, f^{m_k-1}x_0) + \varepsilon.\end{aligned}$$

If $k \to \infty$, then from (3.3) we have $d(f^{m_k}x_0, f^{m_k-1}x_0) \to 0$, and so it follows from the squeeze theorem that $d(f^{m_k}x_0, f^{n_k}x_0) \to \varepsilon$.

We next show by induction that

$$\limsup_{k\to\infty} d(f^{m_k+i}x_0, f^{n_k+i}x_0) \geq \varepsilon, \quad i = 1, 2, \ldots . \tag{3.4}$$

To this end, note first that from (3.3) we get

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} d(f^{m_k}x_0, f^{n_k}x_0)\\ &\leq \limsup_{k\to\infty} \Big[d(f^{m_k}x_0, f^{m_k+1}x_0) + d(f^{m_k+1}x_0, f^{n_k+1}x_0)\\ &\quad + d(f^{n_k+1}x_0, f^{n_k}x_0)\Big]\\ &\leq \limsup_{k\to\infty} d(f^{m_k}x_0, f^{m_k+1}x_0) + \limsup_{k\to\infty} d(f^{m_k+1}x_0, f^{n_k+1}x_0)\\ &\quad + \limsup_{k\to\infty} d(f^{n_k+1}x_0, f^{n_k}x_0)\\ &= \limsup_{k\to\infty} d(f^{m_k+1}x_0, f^{n_k+1}x_0),\end{aligned}$$

that is, (3.4) holds for $i = 1$. If (3.4) is true for a positive integer i, then using (3.3) once more we find

$$\begin{aligned}\varepsilon &\leq \limsup_{k\to\infty} d(f^{m_k+i}x_0, f^{n_k+i}x_0)\\ &\leq \limsup_{k\to\infty} \Big[d(f^{m_k+i}x_0, f^{m_k+i+1}x_0) + d(f^{m_k+i+1}x_0, f^{n_k+i+1}x_0)\\ &\quad + d(f^{n_k+i+1}x_0, f^{n_k+i}x_0)\Big]\\ &\leq \limsup_{k\to\infty} d(f^{m_k+i+1}x_0, f^{n_k+i+1}x_0).\end{aligned}$$

Consequently, from (3.1) and uniform convergence of $\{\psi_n\}$ to ψ on the range of d we have

$$\begin{aligned}\psi(\varepsilon) &= \lim_{k\to\infty} \psi\big(d(f^{m_k}x_0, f^{n_k}x_0)\big)\\ &= \lim_{k\to\infty}\lim_{i\to\infty} \psi_i\big(d(f^{m_k}x_0, f^{n_k}x_0)\big)\\ &= \lim_{i\to\infty}\lim_{k\to\infty} \psi_i\big(d(f^{m_k}x_0, f^{n_k}x_0)\big)\\ &\geq \limsup_{i\to\infty}\limsup_{k\to\infty} d(f^{m_k+i}x_0, f^{n_k+i}x_0)\\ &\geq \varepsilon,\end{aligned}$$

which is a contradiction. Therefore, the sequence $\{f^n x\}$ is Cauchy. □

Now we are ready to prove our main theorem on the fixed point of G-asymptotic contractions.

Theorem 3.7. *Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be an orbitally G-continuous G-asymptotic contraction such that the functions ψ_n in* (3.1) *are continuous on* $[0, +\infty)$ *for sufficiently large indices n. Then f has a fixed point in X if and only if $C_f \neq \emptyset$. Moreover, if the subgraph of G with the vertex set* $\mathrm{Fix}(f)$ *is connected, then the restriction of f to C_f is a Picard operator.*

Proof. Because $\mathrm{Fix}(f) \subseteq C_f$, it follows that if f has a fixed point, then C_f is nonempty. Now let $x_0 \in C_f$. By Lemma 3.6, the sequence $\{f^n x_0\}$ is Cauchy. Since (X, d) is complete, there exists an $x^* \in X$ such that $f^n x_0 \to x^*$. We are going to show that x^* is a fixed point for f. To this end, note that from $x_0 \in C_f$ we have $(f^n x_0, f^{n+1}x_0) \in E(G)$ for all $n \geq 0$. So by orbital G-continuity of f, we get $f^{n+1}x_0 \to fx^*$. Therefore, $fx^* = x^*$.

Next, suppose that the subgraph of G with the vertex set $\mathrm{Fix}(f)$ is connected and $x^{**} \in X$ is a fixed point of f. Then there exists a path $(x_i)_{i=0}^N$ in G from x^* to x^{**} such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(f)$, that is, $x_0 = x^*$, $x_N = x^{**}$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. Since f is a G-asymptotic contraction, for each $i = 1, \ldots, N$, it follows that

$$\begin{aligned}d(x_{i-1}, x_i) &= \lim_{n\to\infty} d(f^n x_{i-1}, f^n x_i)\\ &\leq \lim_{n\to\infty} \psi_n\big(d(x_{i-1}, x_i)\big)\\ &= \psi\big(d(x_{i-1}, x_i)\big),\end{aligned}$$

which is a contradiction unless $x_{i-1} = x_i$. Hence

$$x^* = x_0 = x_1 = \cdots = x_{N-1} = x_N = x^{**}.$$

Consequently, the fixed point of f is unique and the restriction of f to C_f is a Picard operator. □

We now give four simple consequences of Theorem 3.7. If we set $G = G_0$ in Theorem 3.7, then it is clear that G_0-orbital continuity is precisely the orbital continuity introduced by Ćirić [4] and the set C_f related to any self-map f on X coincides with the whole set X. So we have the following version of [8, Theorem 2.1].

Corollary 3.8. *Let (X, d) be a complete metric space and $f : X \to X$ be an orbitally continuous asymptotic contraction such that the functions ψ_n in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f is a Picard operator.*

Secondly, we consider a partially ordered set $(X, \preceq)$ and put $G = G_1$ in Theorem 3.7. Hence we get a partially ordered version of [8, Theorem 2.1] in complete metric spaces equipped with a partial order.

Corollary 3.9. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an orbitally G_1-continuous G_1-asymptotic contraction such that the functions ψ_n in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.*

Smilarly, by considering a partially ordered set $(X, \preceq)$ and setting $G = G_2$ in Theorem 3.7, another partially ordered version of [8, Theorem 2.1] in complete metric spaces equipped with a partial order is obtained.

Corollary 3.10. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an orbitally G_2-continuous G_2-asymptotic contraction such that the functions ψ_n in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.*

In particular, if every two elements of $\mathrm{Fix}(f)$ are comparable, then the restriction of f to the set of all $x \in X$ whose every two iterates under f are comparable elements of $(X, \preceq)$ is a Picard operator.

Finally, we put $G = G_3$ in Theorem 3.7 and we get the following version of [8, Theorem 2.1]:

Corollary 3.11. *Let (X, d) be a complete metric space and ε be a fixed positive real number. Let $f : X \to X$ be an orbitally G_3-continuous G_3-asymptotic contraction such that the functions ψ_n in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are ε-close elements of (X, d) for all $m, n \in \mathbb{N} \cup \{0\}$.*

Moreover, if every two elements of $\mathrm{Fix}(f)$ *are* ε*-close, then the restriction of* f *to the set of all* $x \in X$ *whose every two iterates under* f *are* ε*-close elements of* (X, d) *is a Picard operator.*

Acknowledgment. The author is thankful to the Payame Noor University for supporting this research.

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