G-ASYMPTOTIC CONTRACTIONS IN METRIC SPACES WITH A GRAPH AND FIXED POINT RESULTS

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Abstract. In this paper, we discuss the existence and uniqueness of fixed points for $G$-asymptotic contractions in metric spaces endowed with a graph. The result given here is a new version of Kirk's fixed point theorem for asymptotic contractions in metric spaces endowed with a graph. The given result here is a generalization of fixed point theorem for asymptotic contraction from metric spaces to metric spaces endowed with a graph.

1. Introduction

2. Introduction and Preliminaries

In 2003, Kirk [8] investigated the existence and uniqueness of fixed points for asymptotic contractions in metric spaces but Jachymski and Jóźwik [7] constructed a counterexample showing that the continuity of the self-map is essential in Kirk's theorem. They also established a new result for uniformly continuous asymptotic $\varphi$-contractions in metric spaces (see [7, Theorem 2]). Finally, Aghanians et al. [1] generalized Kirk's fixed point theorem for $p$-continuous $E$-asymptotic contractions in separated uniform spaces via $E$-distances.

In 2008, Jachymski [3] entered graphs in metric fixed point theory by endowing the underlying metric space with a directed graph and formulated the Banach contraction principle in a graph language. This idea followed by several authors (see e.g., [2, 5]).

The main purpose of this paper is to investigate the existence and uniqueness of fixed points for Kirk's asymptotic contractions in metric spaces.
spaces endowed with a graph. Our main result is a new version of [8, Theorem 2.1] in metric spaces endowed with a graph.

Consistent with Bondy and Murthy as well as Jachymski, the following concepts is needed in the sequel. For a idespread discussion on the theory of graphs, the reader is refered to [3, 6].

Let \((X; d)\) be a metric space and \(G\) be a directed graph with vertex set \(V(G) = X\) such that the edge set \(E(G)\) contains all loops, that is, \((x, x) \in E(G)\) for all \(x \in X\). Assume further that \(G\) has no parallel edges. Then the graph \(G\) can be easily denoted by the ordered pair \((V(G), E(G))\) and it is said that the metric space \((X, d)\) is endowed with \(G\).

The metric space \((X, d)\) can also be endowed with the graphs \(G^{-1}\) and \(\tilde{G}\), where the former is the conversion of \(G\) which is obtained from \(G\) by reversing the directions of the edges, and the latter is an undirected graph obtained from \(G\) by ignoring the directions of the edges. In other words, \(V(G^{-1}) = V(\tilde{G}) = X\), \(E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}\) and \(E(\tilde{G}) = E(G) \cup E(G^{-1})\).

If \(x, y \in X\), then a finite sequence \((x_i)_{i=0}^{N}\) consisting of \(N+1\) vertices is called a path in \(G\) from \(x\) to \(y\) whenever \(x_0 = x\), \(x_N = y\) and \((x_{i-1}, x_i)\) is an edge of \(G\) for \(i = 1, \ldots, N\). The graph \(G\) is called connected if there exists a path in \(G\) between each two vertices of \(G\).

A graph \(H\) is called a subgraph of \(G\) if \(V(H)\) and \(E(H)\) are contained in \(V(G)\) and \(E(G)\), respectively and also \((x, y) \in E(H)\) implies \(x, y \in V(H)\).

Following Petruşel and Rus [9], one can naturally formulate Picard operators in metric spaces as follows:

**Definition 2.1 ([6, 9]).** Let \((X, d)\) be a metric space. A self-map \(f\) on \(X\) is called a Picard operator if \(f\) has a unique fixed point \(x^*\) in \(X\) and \(f^n x \to x^*\) for all \(x \in X\).

We also need a weaker type of continuity in metric spaces endowed with a graph. The idea of this definition comes from the definition of orbital continuity defined by Cirić [11].

**Definition 2.2 ([8]).** Let \((X, d)\) be a metric space endowed with a graph \(G\). A self-map \(f\) on \(X\) is called orbitally \(G\)-continuous if for each \(x, y \in X\) and each sequence \(\{\lambda_n\}\) of positive integers satisfying \((f^{\lambda_n} x, f^{\lambda_{n+1}} x) \in E(G)\) for \(n = 1, 2, \ldots\) and \(f^{\lambda_n} x \to y\), one has \(f(f^{\lambda_n} x) \to f y\).

Obviously, a continuous mapping on a metric space is orbitally \(G\)-continuous for all graphs \(G\) but the converse is not necessarily true. The next example shows the effective role played by the graph to imply a weaker type of continuity.
Example 2.3. Consider the set $X = \left\{ \frac{1}{n} : n \geq 1 \right\} \cup \{0\}$ equipped with the usual metric and define a self-map $f : X \to X$ by $f(0) = 0$ and $f(x) = \frac{2}{3}$ for all $x \in X$ with $x \neq 0$. Then it is clear that $f$ is not continuous at $x = 0$, and so on the whole set $X$. Now assume that $X$ is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \left\{ (x,x) : x \in X \right\}$. Then it is clear that $f$ is not continuous at $x = 0$, and so on the whole set $X$. Now assume that $X$ is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \left\{ (x,x) : x \in X \right\}$. If $x, y \in X$ and $\{ \lambda_n \}$ is a sequence of positive integers satisfying $(f^{\lambda_n}x, f^{\lambda_{n+1}}x) \in E(G)$ for $n = 1, 2, \ldots$ and $f^{\lambda_n}x \to y$, then $\{f^{\lambda_n}x\}$ is necessarily a constant sequence and either $y = 0$ or $y = \frac{2}{3}$. Thus, $f^{\lambda_n}x = y$ for all $n \geq 1$ and so $f(f^{\lambda_n}x) \to fy$. Hence $f$ is orbitally $G$-continuous on $X$.

3. Main Results

In this section, we assume that $(X,d)$ is a metric space endowed with a graph $G$. We denote by $\text{Fix}(f)$ the set of all fixed points of a self-map $f$ on $X$, and we use $C_f$ to denote the set of all points $x \in X$ such that $(f^m x, f^n x)$ is an edge of $\tilde{G}$ for all $m, n \in \mathbb{N} \cup 0$. In other words,

$$C_f = \{ x \in X : (f^m x, f^n x) \in E(\tilde{G}), \quad m, n = 0, 1, \ldots \}.$$ 

We further assume that $\Psi$ is the class of all continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$ for all $t > 0$. Clearly, if $\psi \in \Psi$, then

$$0 \leq \psi(0) = \lim_{t \to 0^+} \psi(t) \leq \lim_{t \to 0^+} t = 0,$$

that is, $\psi(0) = 0$. Therefore, $\psi(t) \leq t$ for all $t \geq 0$.

Now, we are ready to define $G$-asymptotic contractions in metric spaces endowed with a graph. This definition is motivated from [R, Definition 2.1] and [R, Definition 2.1].

Definition 3.1. Let $(X,d)$ be metric space endowed with a graph $G$. We say that a self-map $f$ on $X$ is a $G$-asymptotic contraction if

(A1) $f$ preserves the edges of $G$, that is, $(x, y) \in E(G)$ implies $(f(x), f(y)) \in E(G)$ for all $x, y \in X$;

(A2) there exists a sequence $\psi_n : [0, +\infty) \to [0, +\infty)$ converging uniformly to a $\psi \in \Psi$ on the range of $d$ such that

$$d(f^n x, f^n y) \leq \psi_n(d(x, y))$$

for all $n \geq 1$ and all $x, y \in X$ with $(x, y) \in E(G)$.

We next give some examples of $G$-asymptotic contractions in metric spaces endowed with a graph.
Example 3.2. Suppose that \((X, d)\) is a metric space and \(G_0\) is the complete graph with the vertex set \(X\), that is, \(V(G_0) = X\) and \(E(G_0) = \{(x, y) \in X \times X : x \neq y\}\). Then (A1) holds trivially, and (A2) means that
\[
d(f^n x, f^n y) \leq \psi_n(d(x, y)), \quad (x, y \in X),
\]
where \(\psi_n : [0, +\infty) \to [0, +\infty)\) converges uniformly to a \(\psi \in \Psi\) on the range of \(d\). Hence \(G_0\)-asymptotic contractions are precisely the asymptotic contractions introduced by Kirk.

Example 3.3. Suppose that \((X, \preceq)\) is a partially ordered set and \(d\) is a metric on \(X\). Consider the poset graph \(G_1\) with \(V(G_1) = X\) and \(E(G_1) = \{(x, y) \in X \times X : x \preceq y \lor y \preceq x\}\). Then (A1) holds if and only if \(f\) is nondecreasing with respect to \(\preceq\), and (A2) means that
\[
d(f^n x, f^n y) \leq \psi_n(d(x, y))
\]
for all comparable elements \(x, y \in X\), where \(\psi_n : [0, +\infty) \to [0, +\infty)\) converges uniformly to a \(\psi \in \Psi\) on the range of \(d\).

Example 3.4. Suppose that \((X, \preceq)\) is a partially ordered set and \(d\) is a metric on \(X\). Consider the poset graph \(G_2\) defined by \(V(G_2) = X\) and \(E(G_2) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}\). Then (A1) holds if and only if \(f\) maps comparable elements of \(X\) onto comparable elements of \(X\), and (A2) means that
\[
d(f^n x, f^n y) \leq \psi_n(d(x, y))
\]
for all comparable elements \(x, y \in X\), where \(\psi_n : [0, +\infty) \to [0, +\infty)\) converges uniformly to a \(\psi \in \Psi\) on the range of \(d\), as in Example 3.2.

Example 3.5. Suppose that \((X, d)\) is a metric space and \(\varepsilon\) is a fixed positive real number. Two elements \(x, y \in X\) are called \(\varepsilon\)-close whenever \(d(x, y) < \varepsilon\). Define a graph \(G_3\) by \(V(G_3) = X\) and \(E(G_3) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}\). Then (A1) holds if and only if \(f\) maps \(\varepsilon\)-close elements of \(X\) onto \(\varepsilon\)-close elements, and (A2) means that
\[
d(f^n x, f^n y) \leq \psi_n(d(x, y))
\]
for all \(\varepsilon\)-close elements \(x, y \in X\), where \(\psi_n : [0, +\infty) \to [0, +\infty)\) converges uniformly to a \(\psi \in \Psi\) on the range of \(d\).

To prove our main theorem, we need the next lemma.

Lemma 3.6. Let \((X, d)\) be a metric space endowed with a graph \(G\) and \(f : X \to X\) be a \(G\)-asymptotic contraction such that the functions \(\psi_n\) in (3.1) are continuous on \([0, +\infty)\) for sufficiently large indices \(n\). Then \(\{f^n x\}\) is a Cauchy sequence for all \(x \in \text{C}_{f}\).
Proof. Let $x_0 \in C_f$. Then $(f^n x_0, f^{n+1} x_0) \in E(\bar{G})$ for all $n \geq 0$. Moreover, $f$ is clearly a $\bar{G}$-asymptotic contraction. So from (3.1) we get

$$
\limsup_{n \to \infty} d(f^n x_0, f^{n+1} x_0) \leq \lim_{n \to \infty} \psi_n(d(x_0, f x_0))
= \psi(d(x_0, f x_0))
\leq d(x_0, f x_0) < \infty.
$$

On the other hand, if

$$
\limsup_{n \to \infty} d(f^n x_0, f^{n+1} x_0) = \eta > 0,
$$

then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $d(f^{n_k} x_0, f^{n_k+1} x_0) \to \eta$, and so by the continuity of $\psi$, we obtain

$$
\psi(d(f^{n_k} x_0, f^{n_k+1} x_0)) \to \psi(\eta) < \eta.
$$

Hence there is a positive integer $k_0$ with $\psi(d(f^{n_k_0} x_0, f^{n_k_0+1} x_0)) < \eta$ and thus, by (3.1) we get

$$
\eta = \limsup_{n \to \infty} d(f^n x_0, f^{n+1} x_0)
= \limsup_{n \to \infty} d(f^n(f^{n_k_0} x_0), f^n(f^{n_k_0+1} x_0))
\leq \lim_{n \to \infty} \psi_n(d(f^{n_k_0} x_0, f^{n_k_0+1} x_0))
= \psi(d(f^{n_k_0} x_0, f^{n_k_0+1} x_0)) < \eta,
$$

which is a contradiction. Therefore,

$$
\limsup_{n \to \infty} d(f^n x_0, f^{n+1} x_0) = 0.
$$

Consequently,

$$
0 \leq \liminf_{n \to \infty} d(f^n x_0, f^{n+1} x_0) \leq \limsup_{n \to \infty} d(f^n x_0, f^{n+1} x_0) = 0,
$$

that is,

$$
\lim_{n \to \infty} d(f^n x_0, f^{n+1} x_0) = 0.
$$

Now, if the sequence $\{f^n x_0\}$ is not Cauchy, then there exist an $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$
m_k > n_k \geq k \quad \text{and} \quad d(f^{m_k} x_0, f^{m_k} x_0) \geq \varepsilon, \quad k = 1, 2, \ldots.
$$

Keeping the integer $n_k$ fixed for sufficiently large $k$, say $k \geq k_0$, we can assume without loss of generality that $m_k$ is the smallest integer greater than $n_k$ with $d(f^{m_k} x_0, f^{m_k} x_0) \geq \varepsilon$, that is,

$$
d(f^{m_k-1} x_0, f^{m_k} x_0) < \varepsilon, \quad (k \geq k_0).
$$
Hence for each \( k \geq k_0 \), we have

\[
\varepsilon \leq d(f^{m_k}x_0, f^{n_k}x_0)
\leq d(f^{m_k}x_0, f^{m_k-1}x_0) + d(f^{m_k-1}x_0, f^{n_k}x_0)
\leq d(f^{m_k}x_0, f^{m_k-1}x_0) + \varepsilon.
\]

If \( k \to \infty \), then from (3.3) we have \( d(f^{m_k}x_0, f^{m_k-1}x_0) \to 0 \), and so it follows from the squeeze theorem that \( d(f^{m_k}x_0, f^{n_k}x_0) \to \varepsilon \).

We next show by induction that

\[
\limsup_{k \to \infty} d(f^{m_k+i}x_0, f^{n_k+i}x_0) \geq \varepsilon, \quad i = 1, 2, \ldots .
\]

To this end, note first that from (3.3) we get

\[
\varepsilon = \lim_{k \to \infty} d(f^{m_k}x_0, f^{n_k}x_0)
\leq \limsup_{k \to \infty} \left[ d(f^{m_k}x_0, f^{m_k+1}x_0) + d(f^{m_k+1}x_0, f^{n_k+1}x_0) + d(f^{n_k+1}x_0, f^{n_k}x_0) \right]
\leq \limsup_{k \to \infty} d(f^{m_k}x_0, f^{m_k+1}x_0) + \limsup_{k \to \infty} d(f^{m_k+1}x_0, f^{n_k+1}x_0)
+ \limsup_{k \to \infty} d(f^{n_k+1}x_0, f^{n_k}x_0)
= \limsup_{k \to \infty} d(f^{m_k+1}x_0, f^{n_k+1}x_0),
\]

that is, (3.4) holds for \( i = 1 \). If (3.4) is true for a positive integer \( i \), then using (3.3) once more we find

\[
\varepsilon \leq \limsup_{k \to \infty} d(f^{m_k+i}x_0, f^{n_k+i}x_0)
\leq \limsup_{k \to \infty} \left[ d(f^{m_k+i}x_0, f^{m_k+i+1}x_0) + d(f^{m_k+i+1}x_0, f^{n_k+i+1}x_0) + d(f^{n_k+i+1}x_0, f^{n_k+i}x_0) \right]
\leq \limsup_{k \to \infty} d(f^{m_k+i+1}x_0, f^{n_k+i+1}x_0).
\]
Consequently, from (3.1) and uniform convergence of \( \{ \psi_n \} \) to \( \psi \) on the range of \( d \) we have

\[
\psi(\varepsilon) = \lim_{k \to \infty} \psi(d(f^{m_k}x_0, f^{m_k}x_0)) \\
= \lim_{k \to \infty} \lim_{i \to \infty} \psi_i(d(f^{m_k}x_0, f^{m_k}x_0)) \\
= \lim_{i \to \infty} \lim_{k \to \infty} \psi_i(d(f^{m_k}x_0, f^{m_k}x_0)) \\
\geq \limsup_{i \to \infty} \limsup_{k \to \infty} d(f^{m_k+i}x_0, f^{m_k+i}x_0) \\
\geq \varepsilon,
\]

which is a contradiction. Therefore, the sequence \( \{ f^n x \} \) is Cauchy. 

Now we are ready to prove our main theorem on the fixed point of \( G \)-asymptotic contractions.

**Theorem 3.7.** Let \( (X, d) \) be a complete metric space endowed with a graph \( G \) and \( f : X \to X \) be an orbitally \( G \)-continuous \( G \)-asymptotic contraction such that the functions \( \psi_n \) in (3.1) are continuous on \( [0, +\infty) \) for sufficiently large indices \( n \). Then \( f \) has a fixed point in \( X \) if and only if \( C_f \neq \emptyset \). Moreover, if the subgraph of \( G \) with the vertex set \( \text{Fix}(f) \) is connected, then the restriction of \( f \) to \( C_f \) is a Picard operator.

**Proof.** Because \( \text{Fix}(f) \subseteq C_f \), it follows that if \( f \) has a fixed point, then \( C_f \) is nonempty. Now let \( x_0 \in C_f \). By Lemma 3.4, the sequence \( \{ f^n x_0 \} \) is Cauchy. Since \( (X, d) \) is complete, there exists an \( x^* \in X \) such that \( f^n x_0 \to x^* \). We are going to show that \( x^* \) is a fixed point for \( f \). To this end, note that from \( x_0 \in C_f \) we have \( (f^n x_0, f^{n+1} x_0) \in E(G) \) for all \( n \geq 0 \). So by orbital \( G \)-continuity of \( f \), we get \( f^{n+1} x_0 \to f x^* \). Therefore, \( f x^* = x^* \).

Next, suppose that the subgraph of \( G \) with the vertex set \( \text{Fix}(f) \) is connected and \( x^{**} \in X \) is a fixed point of \( f \). Then there exists a path \((x_i)_{i=0}^N \) in \( G \) from \( x^* \) to \( x^{**} \) such that \( x_1, \ldots, x_{N-1} \in \text{Fix}(f) \), that is, \( x_0 = x^*, x_N = x^{**} \) and \( (x_{i-1}, x_i) \in E(G) \) for \( i = 1, \ldots, N \). Since \( f \) is a \( G \)-asymptotic contraction, for each \( i = 1, \ldots, N \), it follows that

\[
d(x_{i-1}, x_i) = \lim_{n \to \infty} d(f^n x_{i-1}, f^n x_i) \\
\leq \lim_{n \to \infty} \psi_n(d(x_{i-1}, x_i)) \\
= \psi(d(x_{i-1}, x_i)),
\]

which is a contradiction unless \( x_{i-1} = x_i \). Hence

\[
x^* = x_0 = x_1 = \cdots = x_{N-1} = x_N = x^{**}.
\]

Consequently, the fixed point of \( f \) is unique and the restriction of \( f \) to \( C_f \) is a Picard operator.
We now give four simple consequences of Theorem 3.7. If we set $G = G_0$ in Theorem 3.7, then it is clear that $G_0$-orbital continuity is precisely the orbital continuity introduced by Ćirić [4] and the set $C_f$ related to any self-map $f$ on $X$ coincides with the whole set $X$. So we have the following version of [8, Theorem 2.1].

**Corollary 3.8.** Let $(X, d)$ be a complete metric space and $f : X \to X$ be an orbitally continuous asymptotic contraction such that the functions $\psi_n$ in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices $n$. Then $f$ is a Picard operator.

Secondly, we consider a partially ordered set $(X, \preceq)$ and put $G = G_1$ in Theorem 3.7. Hence we get a partially ordered version of [8, Theorem 2.1] in complete metric spaces equipped with a partial order.

**Corollary 3.9.** Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Let $f : X \to X$ be an orbitally $G_1$-continuous $G_1$-asymptotic contraction such that the functions $\psi_n$ in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices $n$. Then $f$ has a fixed point in $X$ if and only if there exists an $x_0 \in X$ such that $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Similarly, by considering a partially ordered set $(X, \preceq)$ and setting $G = G_2$ in Theorem 3.7, another partially ordered version of [8, Theorem 2.1] in complete metric spaces equipped with a partial order is obtained.

**Corollary 3.10.** Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Let $f : X \to X$ be an orbitally $G_2$-continuous $G_2$-asymptotic contraction such that the functions $\psi_n$ in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices $n$. Then $f$ has a fixed point in $X$ if and only if there exists an $x_0 \in X$ such that $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

In particular, if every two elements of $\text{Fix}(f)$ are comparable, then the restriction of $f$ to the set of all $x \in X$ whose every two iterates under $f$ are comparable elements of $(X, \preceq)$ is a Picard operator.

Finally, we put $G = G_3$ in Theorem 3.7 and we get the following version of [8, Theorem 2.1]:

**Corollary 3.11.** Let $(X, d)$ be a complete metric space and $\varepsilon$ be a fixed positive real number. Let $f : X \to X$ be an orbitally $G_3$-continuous $G_3$-asymptotic contraction such that the functions $\psi_n$ in (3.1) are continuous on $[0, +\infty)$ for sufficiently large indices $n$. Then $f$ has a fixed point in $X$ if and only if there exists an $x_0 \in X$ such that $f^m x$ and $f^n x$ are $\varepsilon$-close elements of $(X, d)$ for all $m, n \in \mathbb{N} \cup \{0\}$. 
Moreover, if every two elements of Fix(\(f\)) are \(\varepsilon\)-close, then the restriction of \(f\) to the set of all \(x \in X\) whose every two iterates under \(f\) are \(\varepsilon\)-close elements of \((X, d)\) is a Picard operator.

Acknowledgment. The author is thankful to the Payame Noor University for supporting this research.

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