Sahand Communications in Mathematical Analysis (SCMA) Vol. 7 No. 1 (2017), 17-26
http://scma.maragheh.ac.ir

ON SOME RESULTS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES USING THEIR RELATIVE LOWER ORDER

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ABSTRACT. Some basic properties relating to relative lower order of entire functions of two complex variables are discussed in this paper.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let f be any entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in view of maximum principal and Hartogs's theorem ([3], p.21, p.51), $M_f(r_1, r_2)$ is increasing function of r_1, r_2 and for given two entire functions f and g of two complex variables, the ratio $\frac{M_f(r_1,r_2)}{M_g(r_1,r_2)}$ as $r_1, r_2 \to \infty$ is called the growth of f with respect to g. The order ${}_{v_2}\rho_f$ ([3], p.339, see also [1]) of an entire function f of two complex variables which is generally used in computational purposes, is defined in terms of the growth of f respect to the exponencial function $\exp(z_1 z_2)$ as

$$\begin{aligned} {}_{v_2}\rho_f &= \limsup_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log M_{\exp(z_1 z_2)}(r_1, r_2)} \\ &= \limsup_{r_1, r_2 \to \infty} \frac{\log \log M_f(r_1, r_2)}{\log (r_1 r_2)} \ . \end{aligned}$$

2010 *Mathematics Subject Classification.* 32A15, 30D20.

Key words and phrases. Entire functions of two complex variables, Relative lower order of two complex variables, Property(A).

Received: 18 June 2016, Accepted: 29 January 2017.

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Banerjee and Datta [2] introduced the notion of relative order between two entire functions of two complex variables to avoid comparing growth just with $\exp(z_1 z_2)$ in the following manner:

$$\begin{aligned} {}_{v_2}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r_1, r_2) < M_g\left(r_1^{\mu}, r_2^{\mu}\right); r_1 \geq R(\mu), r_2 \geq R(\mu)\right\} \\ &= \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}, \end{aligned}$$

where g is an entire function holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\},$$

and the definition coincides with the classical one [2] if $g(z) = \exp(z_1 z_2)$. Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_2}\lambda_g(f)$ as follows :

$${}_{v_2}\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

In this connection the following definition is relevant:

Definition 1.1 ([1])**.** A non-constant entire function f of two complex variables is said have the property (A) if for any $\sigma > 1$ and for all sufficiently large r_1, r_2, $[M_f(r_1, r_2)]^2 \leq M_f(r_1^{\sigma}, r_2^{\sigma})$ holds. For exapmles of functions with or without the Property (A), one may see [2].

In this paper we wish to investigate some basic properties of relative lower order of entire functions of two complex variables. We do not explain the standard definitions and notations in the theory of entire function of two complex variables as those are available in [3].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([2])**.** *Suppose f is an entire function of two complex variables, $\alpha > 1, 0 < \beta < \alpha$, $s > 1$, $0 < \mu < \lambda$ and n is a positive integer. Then*

(a) $M_f(\alpha r_1, \alpha r_2) > \beta M_f(r_1, r_2)$.

(b) *There exists $K = K(s, f) > 0$ such that*

$$(M_f(r_1, r_2))^s \leq K M_f(r_1^s, r_2^s), \quad \textit{for } r > 0.$$

(c)

$$\lim_{r_1, r_2 \to \infty} \frac{M_f(r_1^s, r_2^s)}{M_f(r_1, r_2)} = \infty = \lim_{r_1, r_2 \to \infty} \frac{M_f\left(r_1^{\lambda}, r_2^{\lambda}\right)}{M_f\left(r_1^{\mu}, r_2^{\mu}\right)}.$$

Lemma 2.2 ([2])**.** *Let f be an entire function satisfying the Property (A),.Then for any positive integer n and for all large r_1, r_2,*

$$[M_f(r_1, r_2)]^n \leq M_f(r_1^{\sigma}, r_2^{\sigma}),$$

holds where $\delta > 1$.

Lemma 2.3. *Let f, g and h are any three entire functions of two complex variables. Then for $M_g(r_1, r_2) \leq M_h(r_1, r_2)$ and all sufficiently large values of r_1, r_2,*

$$_{v_2}\lambda_h(f) \leq {}_{v_2}\lambda_g(f),$$

where $l \geq 1$.

Proof. As $M_g(r_1, r_2) \leq M_h(r_1, r_2)$ and $M_f(r_1, r_2)$ is an increasing function of r_1, r_2, we get for all sufficiently large values of r_1, r_2 that

$$M_h^{-1}(r_1, r_2) \leq M_g^{-1}(r_1, r_2),$$

i.e.,

$$M_h^{-1} M_f(r_1, r_2) \leq M_g^{-1} M_f(r_1, r_2),$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)},$$

i.e.,

$$_{v_2}\lambda_h(f) \leq {}_{v_2}\lambda_g(f).$$

This proves the lemma. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *If $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ and g are entire functions of two complex variables, then*

$$_{v_2}\lambda_f(g) \geq {}_{v_2}\lambda_{f_i}(g),$$

where

$$f = f_1 \pm \sum_{k=2}^{n} f_k,$$

and $_{v_2}\lambda_{f_i}(g) = \min\{_{v_2}\lambda_{f_k}(g) \mid k = 1, 2, \ldots, n\}$. The sign of equality holds when $_{v_2}\lambda_{f_i}(g) \neq \{_{v_2}\lambda_{f_k}(g) \mid k = 1, 2, \ldots, n$ and $k \neq i\}$.

Proof. If $_{v_2}\lambda_f(g) = \infty$ then the result is obvious. So we suppose that $_{v_2}\lambda_f(g) < \infty$.
We can clearly assume that $_{v_2}\lambda_{f_i}(g)$ is finite. Also suppose that

$$_{v_2}\lambda_{f_i}(g) \leq_{v_2} \lambda_{f_k}(g),$$

where $k = 1, 2, \ldots, i, \ldots, n$.

Now for any arbitrary $\varepsilon > 0$, we get for all sufficiently large values of r_1, r_2 that

$$M_{f_k}\left(r_1^{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}, r_2^{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}\right) < M_g\left(r_1, r_2\right),$$

where $k = 1, 2, \ldots, n$, i.e.,

$$M_{f_k}\left(r_1, r_2\right) < M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}\right),$$

where $k = 1, 2, \ldots, n$, i.e.,

$$M_{f_k}\left(r_1, r_2\right) \le M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}\right), \tag{3.1}$$

where $k = 1, 2, \ldots, n$.

Now for all sufficiently large values of r_1, r_2,

$$M_f\left(r_1, r_2\right) < \sum_{k=1}^{n} M_{f_k}\left(r_1, r_2\right),$$

i.e.,

$$M_f\left(r_1, r_2\right) < \sum_{k=1}^{n} M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}\right),$$

i.e.,

$$M_f\left(r_1, r_2\right) < nM_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}\right). \tag{3.2}$$

Now in view of the first part of Lemma 2.1, we obtain from (3.2) for all sufficiently large values of r_1, r_2 that

$$M_f\left(r_1, r_2\right) < M_g\left((n+1)r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}, (n+1)r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}}\right),$$

i.e.,

$$M_f\left[\left(\frac{r_1}{n+1}\right)^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}, \left(\frac{r_2}{n+1}\right)^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}\right] < M_g\left(r_1, r_2\right),$$

i.e.,

$$\left(\frac{r_1}{n+1}\right)^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)} \cdot \left(\frac{r_2}{n+1}\right)^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)} < M_f^{-1} M_g\left(r_1, r_2\right),$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}(g) - \varepsilon\right) \log\left(\frac{r_1}{n+1} \cdot \frac{r_2}{n+1}\right) < \log M_f^{-1} M_g\left(r_1, r_2\right),$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}(g) - \varepsilon\right) < \frac{\log M_f^{-1} M_g(r_1, r_2)}{\log(r_1 r_2) + O(1)},$$

i.e.,

$$\frac{\log M_f^{-1} M_g(r_1, r_2)}{\log(r_1 r_2) + O(1)} > \left({}_{v_2}\lambda_{f_i}(g) - \varepsilon\right).$$

So

$${}_{v_2}\lambda_f(g) = \liminf_{r_1, r_2 \to \infty} \frac{\log M_f^{-1} M_g(r_1, r_2)}{\log(r_1 r_2) + O(1)} \geq {}_{v_2}\lambda_{f_i}(g) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$ {}_{v_2}\lambda_f(g) \geq {}_{v_2}\lambda_{f_i}(g). \tag{3.3}$$

Next let ${}_{v_2}\lambda_{f_i}(g) < {}_{v_2}\lambda_{f_k}(g)$ where $k = 1, 2, \ldots, n$ and $k \neq i$.

As $\varepsilon\,(> 0)$ is arbitrary, from the definition of generalized lower order it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_g(r_1, r_2) < M_{f_i}\left(r_1^{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}, r_2^{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}\right),$$

i.e.,

$$M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}\right) < M_{f_i}(r_1, r_2). \tag{3.4}$$

Since ${}_{v_2}\lambda_{f_i}(g) < {}_{v_2}\lambda_{f_k}(g)$ where $k = 1, 2, \ldots, n$ and $k \neq i$, then in view of the third part of Lemma 2.1 we obtain that

$$\lim_{r_1, r_2 \to \infty} \frac{M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}\right)}{M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}\right)} = \infty, \tag{3.5}$$

where $k = 1, 2, \ldots, n$ and $k \neq i$.

Therefore from (3.5) we obtain for all sufficiently large values of r_1, r_2 that

(3.6)

$$M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}}\right) > n M_g\left(r_1^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left({}_{v_2}\lambda_{f_k}(g)-\varepsilon\right)}}\right),$$

where $k = 1, 2, \ldots, n$ and $k \neq i$.

Thus from (3.1), (3.4) and (3.6) we get for a sequence of values of r_1, r_2 tending to infinity that

$$M_{f_i}(r_1, r_2) > M_g\left(r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}\right),$$

i.e.,

$$M_{f_i}(r_1, r_2) > nM_g\left(r_1^{\frac{1}{\left(v_2\lambda_{f_k}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_k}(g)+\varepsilon\right)}}\right),$$

i.e.,

$$M_{f_i}(r_1, r_2) > nM_{f_k}(r_1, r_2), \tag{3.7}$$

where $k = 1, 2, \ldots, n$ and $k \neq i$.

So from (3.4) and (3.7) and in view of the first part of Lemma 2.1 it follows for a sequence of values of r_1, r_2 tending to infinity that

$$M_f(r_1, r_2) \geq M_{f_i}(r_1, r_2) \sum_{\substack{k=1 \\ k \neq i}}^{n} M_{f_k}(r_1, r_2),$$

i.e.,

$$M_f(r_1, r_2) \geq M_{f_i}(r_1, r_2) - \frac{1}{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} M_{f_i}(r_1, r_2),$$

i.e.,

$$M_f(r_1, r_2) > M_{f_i}(r_1, r_2) - \left(\frac{n-1}{n}\right) M_{f_i}(r_1, r_2),$$

i.e.,

$$M_f(r_1, r_2) > \left(\frac{1}{n}\right) M_{f_i}(r_1, r_2)$$

i.e.,

$$M_f(r_1, r_2) > \left(\frac{1}{n}\right) M_g\left[r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}\right],$$

i.e.,

$$M_f(r_1, r_2) > M_g\left[\frac{r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}}{n+1}, \frac{r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}}}{n+1}\right].$$

This gives for a sequence of values of r_1, r_2 tending to infinity that

$$M_f\left[\{(n+1)\, r_1\}^{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}, \{(n+1)\, r_2\}^{\left(v_2\lambda_{f_i}(g)+\varepsilon\right)}\right] > M_g(r_1, r_2),$$

i.e.,

$$\{(n+1)\,r_1\}^{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)}\cdot\{(n+1)\,r_2\}^{\left({}_{v_2}\lambda_{f_i}(g)+\varepsilon\right)} > M_f^{-1}M_g\,(r_1,r_2)\,,$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}\,(g)+\varepsilon\right) > \frac{\log M_f^{-1}M_g\,(r_1,r_2)}{\log\{(n+1)\,r_1\cdot(n+1)\,r_1r_2\}},$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}\,(g)+\varepsilon\right) > \frac{\log M_f^{-1}M_g\,(r_1,r_2)}{\log(r_1r_2)+O(1)},$$

i.e.,

$${}_{v_2}\lambda_{f_i}\,(g) \geq \liminf_{r\to\infty}\frac{\log M_f^{-1}M_g\,(r_1,r_2)}{\log(r_1r_2)+O(1)},$$

i.e.,

$$ {}_{v_2}\lambda_f\,(g) = \liminf_{r\to\infty}\frac{\log M_f^{-1}M_g\,(r_1,r_2)}{\log(r_1r_2)} \leq_{v_2} \lambda_{f_i}\,(g)\,. \tag{3.8}$$

So from (3.3) and (3.8), we finally obtain that

$${}_{v_2}\lambda_f\,(g) = {}_{v_2}\lambda_{f_i}\,(g)\,,$$

when ${}_{v_2}\lambda_{f_i}\,(g) \neq \{{}_{v_2}\lambda_{f_k}\,(g) \mid k=1,2,\ldots,n \text{ and } k\neq i\}$. □

Theorem 3.2. *If $f_1, f_2, \ldots, f_n$ $(n \geq 2)$, g are entire functions of two complex variables and g has the Property (A), then*

$${}_{v_2}\lambda_f\,(g) \geq {}_{v_2}\lambda_{f_i}\,(g)\,,$$

where

$$f = \prod_{k=1}^{n} f_k,$$

and

$${}_{v_2}\lambda_{f_i}\,(g) = \min\left\{{}_{v_2}\lambda_{f_k}\,(g) \mid k=1,2,\ldots,n\right\}.$$

Proof. Suppose that ${}_{v_2}\lambda_f\,(g) < \infty$. Otherwise if ${}_{v_2}\lambda_f\,(g) = \infty$ then the result is obvious.

We can clearly assume that ${}_{v_2}\lambda_{f_i}\,(g)$ is finite. Also suppose that

$${}_{v_2}\lambda_{f_i}\,(g) \leq_{v_2} \lambda_{f_k}\,(g)\,,$$

where $k = 1, 2, \ldots, n$.

Now for any arbitrary $\varepsilon > 0$, we have for all sufficiently large values of r_1, r_2 that

$$M_{f_k}\left(r_1^{\left({}_{v_2}\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}, r_2^{\left({}_{v_2}\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}\right) < M_g\,(r_1,r_2)\,,$$

where $k = 1, 2, \ldots, n$, i.e.,

$$M_{f_k}(r_1, r_2) < M_g\left[r_1^{\frac{1}{\left(v_2\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}}\right],$$

where $k = 1, 2, \ldots, n$, i.e.,

$$M_{f_k}(r_1, r_2) \le M_g\left[r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}\right]. \tag{3.9}$$

where $k = 1, 2, \ldots, n$.

Now we consider the expression

$$\frac{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)\log(r_1r_2)}{\left(v_2\lambda_{f_i}(g)-\varepsilon\right)\log(r_1r_2)},$$

for all sufficiently large values of r_1, r_2.

Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r_1,r_2, say $r_1 \ge r_{11} \ge r_{10}$, $r_2 \ge r_{21} \ge r_{20}$ that

$$\frac{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)\log(r_{10}r_{20})}{\left(v_2\lambda_{f_i}(g)-\varepsilon\right)\log(r_{10}r_{20})} = \delta. \tag{3.10}$$

Now from (3.9) we have for all sufficiently large values of r_1, r_2 that

$$M_f(r_1, r_2) < \prod_{k=1}^{n} M_{f_k}(r_1, r_2),$$

i.e.,

$$M_f(r_1, r_2) < \prod_{k=1}^{n} M_g\left(r_1^{\frac{1}{\left(v_2\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_k}(g)-\frac{\varepsilon}{2}\right)}}\right),$$

i.e.,

$$M_f(r_1, r_2) < \left[M_g\left(r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}\right)\right]^n. \tag{3.11}$$

Since g has the Property (A), in view of Lemma 2.2 and (3.10) we obtain from (3.11) for all sufficiently large values of r_1, r_2 that

$$M_f(r_1, r_2) < M_g\left(r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\frac{\varepsilon}{2}\right)}}\right)^{\delta},$$

i.e.,

$$M_f(r_1, r_2) < M_g\left[r_1^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\varepsilon\right)}}, r_2^{\frac{1}{\left(v_2\lambda_{f_i}(g)-\varepsilon\right)}}\right],$$

i.e.,

$$M_f\left[r_1^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}, r_2^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)}\right] < M_g\left(r_1, r_2\right),$$

i.e.,

$$r_1^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)} \cdot r_2^{\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)} < M_f^{-1} M_g\left(r_1, r_2\right),$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right)\log\left(r_1 r_2\right) < \log M_f^{-1} M_g\left(r_1, r_2\right),$$

i.e.,

$$\left({}_{v_2}\lambda_{f_i}(g)-\varepsilon\right) < \frac{\log M_f^{-1} M_g\left(r_1, r_2\right)}{\log\left(r_1 r_2\right)}.$$

So

$$\begin{aligned} {}_{v_2}\lambda_f(g) &= \liminf_{r_1, r_2 \to \infty} \frac{\log M_f^{-1} M_g\left(r_1, r_2\right)}{\log\left(r_1 r_2\right)} \\ &\geq {}_{v_2}\lambda_{f_i}(g)-\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$ {}_{v_2}\lambda_f(g) \geq {}_{v_2}\lambda_{f_i}(g). \tag{3.12}$$

Thus the theorem follows from (3.12). □

Theorem 3.3. *If $n > 1$ is a positive integer, then*

$$\frac{1}{n}\,{}_{v_2}\lambda_f(g) \leq {}_{v_2}\lambda_{f^n}(g) \leq {}_{v_2}\lambda_f(g).$$

Proof. From the first and second part of Lemma 2.1, we obtain that

$$\begin{aligned} \left\{M_f\left(r_1, r_2\right)\right\}^n &\leq K M_f\left(r_1^n, r_2^n\right) \\ &< M_f\left(\left(K+1\right) r_1^n, \left(K+1\right) r_2^n\right), \end{aligned} \tag{3.13}$$

for $n > 1, r_1 > 0$ and $r_2 > 0$ where $K = K(n, f) > 0$.
Therefore from (3.13) we obtain that

$$M_f^{-1}\left(r_1^n, r_2^n\right) < \left(K+1\right)\left\{M_f^{-1}\left(r_1, r_2\right)\right\}^n,$$

i.e.,

$$\frac{1}{(K+1)} M_f^{-1}\left(r_1^n, r_2^n\right) < \left\{M_f^{-1}\left(r_1, r_2\right)\right\}^n.$$

So

$${}_{v_2}\lambda_{f^n}(g) \geq \frac{\log \frac{1}{(K+1)} M_f^{-1} M_g\left(r_1^n, r_2^n\right)}{\log r^n},$$

i.e.,

$${}_{v_2}\lambda_{f^n}(g) \geq \frac{1}{n}\,{}_{v_2}\lambda_f(g). \tag{3.14}$$

On the other hand since $\{M_f(r_1, r_2)\}^n > M_f(r_1, r_2)$ for all sufficiently large values of r, we have by Lemma 2.3

$$ {}_{v_2}\lambda f^n(g) \leq_{v_2} \lambda_f(g). \tag{3.15}$$

Thus the theorem follows from (3.14) and (3.15). □

Corollary 3.4. *If $n > 1$ is a positive integer, then*

$$\frac{1}{n}{}_{v_2}\rho_f(g) \leq {}_{v_2}\rho_{f^n}(g) \leq {}_{v_2}\rho_f(g).$$

The proof is omitted as it can be carried out in the line of Theorem 3.3.

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