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Quasicompact and Riesz unital endomorphisms of real Lipschitz algebras of complex-valued functions

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Abstract. We first show that a bounded linear operator T on a real Banach space E is quasicompact (Riesz, respectively) if and only if $T' : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ is quasicompact (Riesz, respectively), where the complex Banach space $E_{\mathbb{C}}$ is a suitable complexification of E and T' is the complex linear operator on $E_{\mathbb{C}}$ associated with T. Next, we prove that every unital endomorphism of real Lipschitz algebras of complex-valued functions on compact metric spaces with Lipschitz involutions is a composition operator. Finally, we study some properties of quasicompact and Riesz unital endomorphisms of these algebras.

1. Introduction and preliminaries

The symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let E be an infinite dimentional Banach space over $\mathbb{K}$. We denote by $\mathcal{B}_{\mathbb{K}}(E)$ and $\mathcal{K}_{\mathbb{K}}(E)$ the set of all bounded linear operators and compact linear operators over $\mathbb{K}$ on E, respectively. It is known that $\mathcal{B}_{\mathbb{K}}(E)$ with the operator norm is a unital Banach algebra and $\mathcal{K}_{\mathbb{K}}(E)$ is a closed ideal of $\mathcal{B}_{\mathbb{K}}(E)$ over $\mathbb{K}$. The essential norm $\|T\|_e$ of $T \in \mathcal{B}_{\mathbb{K}}(E)$ is the norm of $T + \mathcal{K}_{\mathbb{K}}(E)$ in the Calkin algebra $\mathcal{B}_{\mathbb{K}}(E)/\mathcal{K}_{\mathbb{K}}(E)$, i.e.,

$$\begin{aligned} \|T\|_e &= \|T + \mathcal{K}_{\mathbb{K}}(E)\| \\ &= \inf\{\|T - S\| : S \in \mathcal{K}_{\mathbb{K}}(E)\}. \end{aligned}$$

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The essential spectral radius $r_e(T)$ of $T \in \mathcal{B}_{\mathbb{K}}(E)$ is the spectral radius of $T + \mathcal{K}_{\mathbb{K}}(E)$ in $\mathcal{B}_{\mathbb{K}}(E)/\mathcal{K}_{\mathbb{K}}(E)$, i.e.,

$$\begin{aligned} r_e(T) &= \lim_{n\to\infty} (\|T^n\|_e)^{\frac{1}{n}} \\ &= \inf\left\{(\|T^n\|_e)^{\frac{1}{n}} : n \in \mathbb{N}\right\}. \end{aligned}$$

The linear operator $T \in \mathcal{B}_{\mathbb{K}}(E)$ is called power compact if $T^N \in \mathcal{K}_{\mathbb{K}}(E)$ for some $N \in \mathbb{N}$, Riesz if $r_e(T) = 0$ and quasicompact if $r_e(T) < 1$. Clearly, $T \in \mathcal{K}_{\mathbb{K}}(E)$ if and only if $\|T\|_e = 0$, and T is quasicompact if and only if $\|T^n\|_e < 1$ for some $n \in \mathbb{N}$. Moreover, T is quasicompact if T is a Riesz operator and T is Riesz operator if T is power compact.

Let A be a unital commutative Banach algebra with unit e_A over $\mathbb{K}$. A linear map $T : A \to A$ is called an endomorphism of A if $T(fg) = (Tf)(Tg)$ for all $f, g \in A$. An endomorphism T of A is unital if $T(e_A) = e_A$.

Feinstein and Kamowitz studied quasicompact and Riesz unital endomorphisms of commutative semisimple unital complex Banach algebras in [5]. They investigated their studies in [6] whenever Banach algebras considered semiprime. We recall that a complex algebra A is semiprime if $J = \{0\}$ is the only ideal of A with $J^2 = \{0\}$.

Let E be a linear space over $\mathbb{K}$ and $T : E \to E$ be a linear operator of E. We denote by $\sigma_p(T)$ and $\sigma(T)$, the set of all eigenvalues of T and the spectrum of T in $\mathbb{K}$, respectively, i.e.,

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not injective}\}, \\ \sigma(T) &= \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not invertible}\}. \end{aligned}$$

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued continuous functions on X. It is known that $C_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ with unit 1_X, the constant function on X with value 1, where equipped with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\}, \quad (f \in C_{\mathbb{K}}(X)).$$

We write $C(X)$ instead of $C_{\mathbb{C}}(X)$. Let A be a unital subalgebra of $C_{\mathbb{K}}(X)$ over $\mathbb{K}$ which is a Banach algebra under an algebra norm $\|\cdot\|$. If $\varphi : X \to X$ is a self-map of X such that $f\circ\varphi \in A$ for each $f \in A$, then the map $T : A \to A$ defined by $Tf = f\circ\varphi$ $(f \in A)$, is a unital endomorphism of A which is called the composition endomorphism of A induced by φ. Recall that if $T : A \to A$ is the composition endomorphism of A induced by a self-map $\varphi : X \to X$, then $T^n : A \to A$ is the composition endomorphism of A induced by the self-map $\varphi_n : X \to X$, where φ_n is the nth iterate of φ. We also consider φ_0 the identity self-map of X.

Let (X, d) and (Y, ρ) be metric spaces such that X and Y have infinitely many points. A map $\varphi : X \to Y$ is called a Lipschitz mapping from

(X, d) into (Y, ρ) if there exists a constant C such that $\rho(\varphi(x), \varphi(y)) \leq Cd(x, y)$ for all $x, y \in X$. The constant Lipschitz of a map $\varphi : X \to Y$ is denoted by $p(\varphi)$ and defined by

$$p(\varphi) = \sup \left\{ \frac{\rho(\varphi(x), \varphi(y))}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

Thus $\varphi : X \to Y$ is a Lipschitz mapping from (X, d) into (Y, ρ) if and only if $p(\varphi) < \infty$. A map $\varphi : X \to Y$ is called a supercontractive mapping from (X, d) into (Y, ρ) if

$$\lim_{d(x,y)\to 0} \frac{\rho(\varphi(x), \varphi(y))}{d(x, y)} = 0.$$

Let (X, d) be a metric space such that X has infinitely many points. A self-map $\varphi : X \to X$ is called a Lipschitz mapping on (X, d) if φ is a Lipschitz mapping from (X, d) into (X, d). A function $f : X \to \mathbb{K}$ is called a $\mathbb{K}$-valued Lipschitz function on (X, d) if f is a Lipschitz mapping from (X, d) into the Euclidean metric space $\mathbb{K}$. We denote by $p_{(X,d)}(f)$ the constant Lipschitz of $f : X \to \mathbb{K}$, i.e.,

$$p_{(X,d)}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

A function $f : X \to \mathbb{K}$ is called a $\mathbb{K}$-valued supercontractive function on (X, d) if f is a supercontractive mapping from (X, d) into the Euclidean space $\mathbb{K}$. For $\alpha \in (0, 1]$, the formula $d^\alpha(x, y) = (d(x, y))^\alpha$ defines a new metric d^α on X such that the generated topology on X by d^α coincides with the generated topology on X by d.

Let (X, d) be a compact metric space. We denote by $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ the set of all $\mathbb{K}$-valued Lipschitz functions on (X, d^α). Then $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is a subalgebra of $C_{\mathbb{K}}(X)$ over $\mathbb{K}$ containing 1_X and separates the points of X. Moreover, $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is a commutative unital Banach algebra over $\mathbb{K}$ under the d^α-Lipschitz norm

$$\|f\|_{\mathrm{Lip}_{\mathbb{K}}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f), \quad (f \in \mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)).$$

Note that the infiniteness of X implies that $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is infinite dimensional. For $\alpha \in (0, 1)$, the set of all supercontractive $\mathbb{K}$-valued functions on (X, d^α) is denoted by $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$. Clearly, $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ is a subalgebra of $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ over $\mathbb{K}$ containing 1_X and closed in $(\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}_{\mathbb{K}}(X,d^\alpha)})$. So $(\mathrm{lip}_{\mathbb{K}}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}_{\mathbb{K}}(X,d^\alpha)})$ is a unital commutative Banach algebra over $\mathbb{K}$. It is clear that $\mathrm{Lip}_{\mathbb{K}}(X, d^\beta) \subseteq \mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ where $0 < \alpha < \beta \leq 1$. We write $\mathrm{Lip}(X, d^\alpha)$ and $\mathrm{lip}(X, d^\alpha)$ instead of $\mathrm{Lip}_{\mathbb{C}}(X, d^\alpha)$ and $\mathrm{lip}_{\mathbb{C}}(X, d^\alpha)$, respectively.

The algebras $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1)$ are called Lipschitz algebra and little Lipschitz algebra of order α on

(X, d), respectively. These algebras were first introduced by Sherbert in [15, 16]. He showed [15, Theorem 5.1] that a linear map $T : \mathrm{Lip}(X, d^\alpha) \to \mathrm{Lip}(X, d^\alpha)$ is a unital endomorphism if and only if there exists a Lipschitz mapping $\varphi : X \to X$ on (X, d) such that T is induced by φ. One can show that this result holds for $\mathrm{lip}(X, d^\alpha)$ instead of $\mathrm{Lip}(X, d^\alpha)$.

Behrouzi studied quasicompact and Riesz unital endomorphisms of $\mathrm{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$ in [3].

Jiménez-Vargas, Lacrus and Villegas-Vallecillos in [8] studied the essential norm of composition operators on Banach spaces of Hölder functions on pointed compact metric spaces over $\mathbb{K}$ and got a formula for it.

Golbaharan and Mahyar in [7] generalized some results of [3], studied essential spectral radius of a unital endomorphism of these algebras and got a formula for it.

Sanatpour investigated quasicompact composition operators on certain classes of complex Lipschitz algebras and obtained certain properties of power-contractive self-maps of compact plane sets in [14].

Quasicompact and Riesz endomorphisms of certain subalgebras of Lipschitz algebras were studied in [11, 12, 13].

Let X be a topological space. A self-map $\tau : X \to X$ is called a topological involution on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Clearly, such τ is a homeomorphism from X onto X.

Let X be a compact Hausdorff space and τ be a topological involution on X. The map $\tau^* : C(X) \to C(X)$ defined by $\tau^*(f) = \bar{f} \circ \tau$ is an algebra involution on $C(X)$, which is called the algebra involution induced by τ on $C(X)$. We now define

$$C(X, \tau) = \{f \in C(X) : \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a unital self-adjoint uniformly closed real subalgebra of $C(X)$ that separates the points of X, $i1_X \notin C(X, \tau)$, $C(X) = C(X, \tau) \oplus iC(X, \tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2 \max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C(X, \tau)$. Real Banach algebra $C(X, \tau)$ was defined explicitly by Kulkarni and Limaye in [9]. For further general facts about $C(X, \tau)$ and certain real subalgebras, we refer to [10].

Let (X, d) be a metric space. A self-map $\tau : X \to X$ is called a Lipschitz involution on (X, d) if $\tau(\tau(x)) = x$ for all $x \in X$ and τ is a Lipschitz mapping on (X, d).

Note that if τ is a Lipschitz involution on (X, d), then τ is a topological involution on (X, d) and $p(\tau) \geq 1$.

Let (X, d) be a compact metric space and τ be a topological involution on (X, d). Then for $\alpha \in (0, 1]$ we have $\tau^*(\mathrm{Lip}(X, d^\alpha)) =$

$\mathrm{Lip}(X, d^\alpha)$, $p_{(X,d^\alpha)}(\tau^*(f)) \leq (p(\tau))^\alpha p_{(X,d^\alpha)}(f)$ and $\|\tau^*(f)\|_{\mathrm{Lip}(X,d^\alpha)} \leq (p(\tau))^\alpha \|f\|_{\mathrm{Lip}(X,d^\alpha)}$ for all $f \in \mathrm{Lip}(X, d^\alpha)$. Moreover, $\tau^*(\mathrm{lip}(X, d^\alpha)) = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$. We now define

$$\mathrm{Lip}(X, d^\alpha, \tau) := \{f \in \mathrm{Lip}(X, d^\alpha) : \tau^*(f) = f\}, \quad (0 < \alpha \leq 1),$$
$$\mathrm{lip}(X, d^\alpha, \tau) := \{f \in \mathrm{lip}(X, d^\alpha) : \tau^*(f) = f\}, \quad (0 < \alpha < 1).$$

In fact, $\mathrm{Lip}(X, d^\alpha, \tau) = \mathrm{Lip}(X, d^\alpha) \cap C(X, \tau)$ and $\mathrm{lip}(X, d^\alpha, \tau) = \mathrm{lip}(X, d^\alpha) \cap C(X, \tau)$.

The following result is a modification of [1, Theorem 2.7].

Theorem 1.1. *Let (X, d) be a compact metric space and τ be a Lipschitz involution on (X, d). Suppose that $A = \mathrm{Lip}(X, d^\alpha, \tau)$ and $B = \mathrm{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$ ($A = \mathrm{lip}(X, d^\alpha, \tau)$ and $B = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$, respectively). Then the following statements hold.*

(i) *A is a self-adjoint real subalgebra of B, $1_X \in A$ and $i1_X \notin A$.*
(ii) *$B = A \oplus iA$ and A separates the points of X.*
(iii) *For all $f, g \in A$;*

$$\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} \leq (p(\tau))^\alpha \|f + ig\|_{\mathrm{Lip}(X,d^\alpha)}$$
$$\leq 2(p(\tau))^\alpha \max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\}.$$

(iv) *A is closed in $(B, \|.\|_{\mathrm{Lip}(X,d^\alpha)})$ and so $(A, \|.\|_{\mathrm{Lip}(X,d^\alpha)})$ is a real Banach algebra.*
(v) *$A = \mathrm{Lip}_{\mathbb{R}}(X, d^\alpha)$ ($A = \mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$, respectively), if and only if τ is the identity map on X.*

The algebras $\mathrm{Lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$ are called real Lipschitz algebra and real little Lipschitz algebra of complex-valued functions of order α on $((X, d), \tau)$, respectively. These algebras were first studied in [1]. In this paper we study some properties of quasicompact and Riesz unital endomorphisms of $\mathrm{Lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$, considering obtained results in [7]. Moreover, we show that the class of quasicompact (Riesz, respectively) unital endomorphisms of real Lipschitz algebras of complex-valued functions is larger than the class of quasicompact (Riesz, respectively) unital endomorphisms of complex Lipschitz algebras.

2. Results

Let E be a real linear space (real algebra, respectively). A complex linear space (complex algebra, respectively) $E_{\mathbb{C}}$ is called a complexification of E if there exists an injective real linear map (real algebra homomorphism, respectively) $J : E \to E_{\mathbb{C}}$ such that $E_{\mathbb{C}} = J(E) \oplus iJ(E)$. For example if A is a real algebra, then $A \times A$ with algebra operations

$$\begin{aligned}
&(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \quad (a_1, b_1, a_2, b_2 \in A),\\
&(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \beta a + \alpha b), \quad (\alpha, \beta \in \mathbb{R},\ a, b \in A),\\
&(a_1, b_1)(a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2), \quad (a_1, b_1, a_2, b_2 \in A),
\end{aligned}$$

is a complexification of A when $J : A \to A \times A$ is defined by $J(a) = (a, 0)$ $(a \in A)$.

Applying the proof of [4, Proposition I.1.13], one can show that if $(E, \|\cdot\|)$ is a real Banach space, then there exists a norm $|||\cdot|||$ on $E \times E$ such that $|||(a, 0)||| = \|a\|$ for each $a \in E$ and $\max\{\|a\|, \|b\|\} \leq |||(a, b)||| \leq 2 \max\{\|a\|, \|b\|\}$ for all $a, b \in E$.

Definition 2.1. Let E be a real linear space and $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \to E_{\mathbb{C}}$. Suppose that $T : E \to E$ is a real linear operator on E and the map $T' : E_{\mathbb{C}} \to E_{\mathbb{C}}$ is defined by

$$T'(J(a) + iJ(b)) = J(Ta) + iJ(Tb), \quad (a, b \in E).$$

Clearly, T' is a complex linear operator on $E_{\mathbb{C}}$. We say that T' is the complex linear operator on $E_{\mathbb{C}}$ associated with T.

The following result is a modification of [2, Theorem 2.1].

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a real Banach space, $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \to E_{\mathbb{C}}$. Suppose that $|||\cdot|||$ is a norm on $E_{\mathbb{C}}$ with $|||J(a)||| = \|a\|$ for each $a \in E$ and there exist positive constants k_1 and k_2 such that*

$$\max\{\|a\|, \|b\|\} \leq k_1 |||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$. Let $T \in \mathcal{B}_{\mathbb{R}}(E)$ and $T' : E_{\mathbb{C}} \to E_{\mathbb{C}}$ be the complex linear operator on $E_{\mathbb{C}}$ associated with T. Then the following statements hold.

(i) *$T' \in \mathcal{B}_{\mathbb{C}}(E_{\mathbb{C}})$ and $\|T'\| \leq k_1 k_2 \|T\|$.*
(ii) *T is compact if and only if T' is compact.*
(iii) *T is injective if and only if T' is injective.*
(iv) *T is invertible if and only if T' is invertible.*
(v) *$\sigma_p(T) = \mathbb{R} \cap \sigma_p(T')$.*
(vi) *$\sigma(T) = \mathbb{R} \cap \sigma(T')$.*

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a real Banach space and $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \to E_{\mathbb{C}}$. Suppose that $|||\cdot|||$ is a norm on $E_{\mathbb{C}}$ with $|||J(a)||| = \|a\|$ for each $a \in E$ and there exist positive constants k_1 and k_2 such that*

$$\max\{\|a\|, \|b\|\} \leq k_1 |||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$. Let $T \in \mathcal{B}_{\mathbb{R}}(E)$ and $T' : E_{\mathbb{C}} \to E_{\mathbb{C}}$ be the complex linear operator on $E_{\mathbb{C}}$ associated with T. Then the following statements hold.

(i) $r_e(T') = r_e(T)$.

(ii) *T is quasicompact if and only if T' is quasicompact.*

(iii) *T is Riesz if and only if T' is Riesz.*

Proof. (i). Let $S \in \mathcal{K}_{\mathbb{R}}(E)$. Define the map $S' : E_{\mathbb{C}} \to E_{\mathbb{C}}$ by

$$S'(J(a) + iJ(b)) = J(Sa) + iJ(Sb) \quad (a, b \in E).$$

Then $S' \in \mathcal{K}_{\mathbb{C}}(E_{\mathbb{C}})$ and $\|S'\| \le k_1 k_2 \|S\|$ by parts (ii) and (i) of Theorem 2.2, respectively. Moreover,

$$\|(T')^n - S'\| = \|(T^n - S)'\| \le k_1 k_2 \|T^n - S\|,$$

for all $n \in \mathbb{N}$, by Theorem 2.2. This implies that

$$\|(T')^n\|_e \le k_1 k_2 \|T^n\|_e,$$

for all $n \in \mathbb{N}$. Therefore,

$$r_e(T') \le r_e(T). \tag{2.1}$$

Define the map $\Psi_1 : E \to E_{\mathbb{C}}$ by $\Psi_1(a) = J(a) + i0$, $(a \in E)$ and the map $P_1 : E_{\mathbb{C}} \to E$ by $P_1(J(a) + iJ(b)) = a$, $(a, b \in E)$. Clearly, Ψ_1 is a bounded real linear operator from E into $E_{\mathbb{C}}$, $\|\Psi_1\| \le 1$, P_1 is a bounded real linear operator from $E_{\mathbb{C}}$ into E and $\|P_1\| \le k_1$. Moreover, $P_1 \circ (T')^n \circ \Psi_1 = T^n$ for all $n \in \mathbb{N}$. Let $S \in \mathcal{K}_{\mathbb{C}}(E_{\mathbb{C}})$. Define the map $S_1 : E \to E$ by $S_1 = P_1 \circ S \circ \Psi_1$. Then $S_1 \in \mathcal{K}_{\mathbb{R}}(E)$ and

$$\begin{aligned}\|T^n - S_1\| &= \|P_1 \circ ((T')^n - S) \circ \Psi_1\| \\ &\le \|P_1\| \|(T')^n - S\| \|\Psi_1\| \\ &\le k_1 \|(T')^n - S\|,\end{aligned}$$

for all $n \in \mathbb{N}$. Hence,

$$\|T^n\|_e \le k_1 \|(T')^n\|_e,$$

for all $n \in \mathbb{N}$. Therefore,

$$r_e(T) \le r_e(T'). \tag{2.2}$$

From (2.1) and (2.2), we get $r_e(T') = r_e(T)$ and so (i) holds. Clearly, (ii) and (iii) are hold by (i). □

Theorem 2.4. *Let (X, d) be a compact metric space, $\tau : X \to X$ be a Lipschitz involution on (X, d) and $A = \mathrm{Lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1]$ or $A = \mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$.*

(i) *If $\varphi : X \to X$ is a Lipschitz mapping on (X, d) satisfying $\varphi \circ \tau = \tau \circ \varphi$, then $f \circ \varphi \in A$ for all $f \in A$.*

(ii) *If $T : A \to A$ is a unital endomorphism of A, then T is bounded and there exists a Lipschitz mapping φ on (X, d) satisfying $\varphi \circ \tau = \tau \circ \varphi$ such that T is induced by φ.*

Proof. Let $B = \mathrm{Lip}(X, d^\alpha)$ if $A = \mathrm{Lip}(X, d^\alpha, \tau)$ and $B = \mathrm{lip}(X, d^\alpha)$ if $A = \mathrm{lip}(X, d^\alpha, \tau)$. Clearly, B is the complexification of A under the injective real algebra homomorphism $J : A \to B$ defined by $J(f) = f$ $(f \in A)$. Moreover, by Theorem 1.1, we have

$$\begin{aligned}\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} &\le (p(\tau))^\alpha \|f + ig\|_{\mathrm{Lip}(X,d^\alpha)} \\ &\le 2(p(\tau))^\alpha \max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\},\end{aligned}$$

for all $f, g \in A$. Assume that $\varphi : X \to X$ is a Lipschitz mapping on (X, d) satisfying $\varphi \circ \tau = \tau \circ \varphi$ and $f \in A$. It is easy to see $f \circ \varphi \in B$. Since

$$\begin{aligned}(\tau^*(f \circ \varphi))(x) &= (\overline{f \circ \varphi} \circ \tau)(x) \\ &= \overline{(f \circ \varphi)(\tau(x))} \\ &= \overline{((f \circ \varphi) \circ \tau)(x)} \\ &= \overline{(f \circ (\varphi \circ \tau))(x)} \\ &= \overline{(f \circ (\tau \circ \varphi))(x)} \\ &= \overline{((f \circ \tau) \circ \varphi)(x)} \\ &= \overline{(f \circ \tau)(\varphi(x))} \\ &= (\overline{f} \circ \tau)(\varphi(x)) \\ &= (\tau^*(f))(\varphi(x)) \\ &= f(\varphi(x)) \\ &= (f \circ \varphi)(x),\end{aligned}$$

for each $x \in X$, we deduce that

$$\tau^*(f \circ \varphi) = f \circ \varphi.$$

Therefore, $f \circ \varphi \in A$. Hence, (i) holds.

To prove (ii), suppose that $T : A \to A$ is a unital endomorphism of A. Let $T' : B \to B$ be the complex linear operator on B associated with T. It is easy to see that T' is a unital complex endomorphism of B. This implies that there exists a Lipschitz mapping φ on (X, d) such that $T'h = h \circ \varphi$ for all $h \in B$. Since A is a subset of B and $Tf = T'f$ for all $f \in A$, we deduce that $Tf = f \circ \varphi$ for all $f \in A$. Therefore, T is induced by φ.

Let $x \in X$. For each $f \in A$, we have

$$\begin{aligned} f((\varphi \circ \tau)(x)) &= (f \circ (\varphi \circ \tau))(x) \\ &= ((f \circ \varphi) \circ \tau)(x) \\ &= (f \circ \varphi)(\tau(x)) \\ &= \overline{(f \circ \varphi)(x)} \\ &= \overline{f(\varphi(x))} \\ &= (f \circ \tau)(\varphi(x)) \\ &= ((f \circ \tau) \circ \varphi)(x) \\ &= (f \circ (\tau \circ \varphi))(x) \\ &= f((\tau \circ \varphi)(x)). \end{aligned}$$

This implies that $(\varphi \circ \tau)(x) = (\tau \circ \varphi)(x)$ since A separates the points of X. Hence, $\varphi \circ \tau = \tau \circ \varphi$. Therefore, (ii) holds. □

Theorem 2.5. *Let (X, d) be a compact metric space, $\tau : X \to X$ be a Lipschitz involution on (X, d) and $A = \mathrm{Lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1]$ or $A = \mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$. Suppose that $T : A \to A$ is a unital endomorphism of A induced by the Lipschitz mapping φ on (X, d). Then the following statements hold.*

(i) *If $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ and $\alpha \in (0, 1)$, then $r_e(T) = \lim_{n\to\infty} (p(\varphi_n))^{\frac{\alpha}{n}}$ and T is quasicompact.*

(ii) *If $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ and $\alpha = 1$, then $r_e(T) \leq \lim_{n\to\infty} (p(\varphi_n))^{\frac{\alpha}{n}}$ and T is quasicompact.*

(iii) *If (X, d) is connected and T is quasicompact, then $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$.*

(iv) *If $\lim_{n\to\infty} (p(\varphi_n))^{\frac{1}{n}} = 0$, then T is Riesz.*

(v) *If (X, d) is connected, $\alpha \in (0, 1)$ and T is Riesz, then $\lim_{n\to\infty} (p(\varphi_n))^{\frac{1}{n}} = 0$.*

(vi) *T is quasicompact if and only if there exists a decomposition of X into a finite number of mutually disjoint clopen subsets, say $X_1, X_2, \ldots, X_m$ such that, for each $i \in \{1, \ldots, m\}$, there exist $j \in \{1, \ldots, m\}$ and $n_i \in \mathbb{N}$ with $\tau(X_i) = X_j$, $\varphi_{n_i}(X_i) \subseteq X_i$ and $p(\varphi_{n_i}|_{X_i}) < 1$.*

(vii) *If $\alpha \in (0, 1)$ and T is quasicompact then*

$$\begin{aligned} \sigma_p(T) &\subseteq \{\lambda \in \mathbb{R} : |\lambda| \leq r_e(T)\} \cup \{-1, 1\}, \\ \sigma(T) &\subseteq \{\lambda \in \mathbb{R} : |\lambda| \leq r_e(T)\} \cup \{-1, 1\}. \end{aligned}$$

(viii) *If $\alpha \in (0, 1)$ and T is Riesz, then $\sigma(T) \subseteq \{0, -1, 1\}$.*

(ix) *If* $\alpha \in (0,1)$, (X,d) *is connected and* T *is Riesz, then* $\sigma(T) = \{0,1\}$.

Proof. Suppose that $B = \mathrm{Lip}(X,d^\alpha)$ if $A = \mathrm{Lip}(X,d^\alpha,\tau)$ and $B = \mathrm{lip}(X,d^\alpha)$ if $A = \mathrm{lip}(X,d^\alpha,\tau)$. By Theorem 1.1, B is the complexification of A under the injective real algebra homomorphism $J : A \to B$ defined by $J(f) = f$ $(f \in A)$ and $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ is a norm on B satisfying

$$\begin{aligned}\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} &\leq (p(\tau))^\alpha \|f+ig\|_{\mathrm{Lip}(X,d^\alpha)}\\ &\leq 2(p(\tau))^\alpha \max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\},\end{aligned}$$

for all $f,g \in A$. Let $T' : B \to B$ be the complex linear operator on B associated with T. It is easy to see that T' is a unital endomorphism of B which is induced by the Lipschitz mapping φ on (X,d). Thus, by Theorem 2.2, we have

$$\sigma_p(T) = \mathbb{R} \cap \sigma_p(T'), \tag{2.3}$$

$$\sigma(T) = \mathbb{R} \cap \sigma(T'). \tag{2.4}$$

Moreover, we deduce that

$$r_e(T') = r_e(T), \tag{2.5}$$

T is quasicompact if and only if T' is quasicompact and T is Riesz if and only if T' is Riesz by Theorem 2.3.

Suppose that $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ and $\alpha \in (0,1)$. Then $r_e(T') = \lim\limits_{n\to\infty}(p(\varphi_n))^{\frac{\alpha}{n}}$ by [7, Theorem 2.2] and T' is quasicompact by part (i) of [7, Corollary 2.4]. Therefore, $r_e(T) = \lim\limits_{n\to\infty}(p(\varphi_n))^{\frac{\alpha}{n}}$ and T is quasicompact. Thus, (i) holds.

Let $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ and $\alpha = 1$. Then

$$r_e(T') \leq \lim_{n\to\infty}(p(\varphi_n))^{\frac{\alpha}{n}},$$

by [7, Theorem 2.3] and T' is quasicompact by part (i) of [7, Corollary 2.4]. Therefore,

$$r_e(T) \leq \lim_{n\to\infty}(p(\varphi_n))^{\frac{\alpha}{n}},$$

and T is quasicompact. Thus, (ii) holds.

Suppose that (X,d) is connected and T is quasicompact. Then T' is quasicompact. Hence, $p(\varphi_{n_0}) < 1$ for some $n_0 \in \mathbb{N}$ by part (ii) of [7, Corollary 2.4]. Thus (iii) holds.

Let $\lim\limits_{n\to\infty}(p(\varphi_n))^{\frac{1}{n}} = 0$. By part (i) of [7, Corollary 2.5], T' is Riesz. Therefore, T is Riesz and so (iv) holds.

Suppose that (X,d) is connected, $\alpha \in (0,1)$ and T is Riesz. Then T' is Riesz. Hence, $\lim\limits_{n\to\infty}(p(\varphi_n))^{\frac{1}{n}} = 0$ by part (ii) of [7, Corollary 2.5] and so (v) holds.

Since $\tau : X \to X$ is a homeomorphism, (vi) holds by [7, Theorem 2.10].

Suppose that $\alpha \in (0,1)$ and T is quasicompact. Then T' is quasicompact. Hence,

$$\sigma_p(T') \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r_e(T')\} \cup \{\lambda \in \mathbb{C} : \lambda^n = 1\},$$
$$\sigma(T') \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r_e(T')\} \cup \{\lambda \in \mathbb{C} : \lambda^n = 1\},$$

for some $n \in \mathbb{N}$ by [7, Theorem 2.12]. Thus, by (2.3), (2.4) and (2.5) we get

$$\sigma_p(T) \subseteq \{\lambda \in \mathbb{R} : |\lambda| \leq r_e(T)\} \cup \{-1, 1\},$$
$$\sigma(T) \subseteq \{\lambda \in \mathbb{R} : |\lambda| \leq r_e(T)\} \cup \{-1, 1\}.$$

Therefore, (vii) holds.

Suppose that $\alpha \in (0,1)$ and T is Riesz. Then T' is Riesz and so $\sigma(T') \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^n = 1\}$ for some $n \in \mathbb{N}$ by part (i) of [7, Corollary 2.13]. Hence, $\sigma(T) \subseteq \{0, -1, 1\}$ by (2.4) and so (viii) holds.

Suppose that $\alpha \in (0,1)$, (X,d) is connected and T is Riesz. Then T' is Riesz. Hence, $\sigma(T') = \{0,1\}$ by considered remark of the end paragraph in [7]. Thus $\sigma(T) = \{0,1\}$ by (2.4) and so (ix) holds. $\square$

The following result is a modification of [2, Theorem 2.4] and [2, Theorem 2.9] and shows that the class of real Lipschitz algebras of complex-valued functions is larger than the class of complex Lipschitz algebras regarded as real Banach algebras.

Theorem 2.6. *Let (X,d) be a compact metric space, $Y = X \times \{0,1\}$ and ρ be the metric on Y defined by $\rho((x_1,j_1),(x_2,j_2)) = \max\{d(x_1,x_2), |j_1 - j_2|\}$. Suppose that $\tau : Y \to Y$ be the self-map on Y defined by*

$$\tau(x,0) = (x,1), \qquad \tau(x,1) = (x,0), \quad (x \in X).$$

Then the following statements hold.

(i) *τ is a Lipschitz involution on the compact metric space (Y,ρ) and $p(\tau) = 1$.*

(ii) *For* $\alpha \in (0,1]$*, the map* $\Lambda : \mathrm{Lip}(X, d^\alpha) \to \mathrm{Lip}(X, \rho^\alpha, \tau)$ *defined by*

$$(\Lambda f)(x,0) = f(x), \qquad (f \in \mathrm{Lip}(X, d^\alpha), x \in X),$$
$$(\Lambda f)(x,1) = \overline{f(x)}, \qquad (f \in \mathrm{Lip}(X, d^\alpha), x \in X),$$

is an injective bounded real linear operator from $(\mathrm{Lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ *regarded as a real Banach algebra onto* $(\mathrm{Lip}(Y, \rho^\alpha, \tau), \|\cdot\|_{\mathrm{Lip}(Y,\rho^\alpha)})$ *satisfying*

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} \le \|\Lambda f\|_{\mathrm{Lip}(Y,\rho^\alpha)} \le 2\|f\|_{\mathrm{Lip}(X,d^\alpha)},$$

for all $f \in \mathrm{Lip}(X, d^\alpha)$.

(iii) *For* $\alpha \in (0,1)$*,* $\Lambda(\mathrm{lip}(X, d^\alpha)) \subseteq \mathrm{lip}(Y, \rho^\alpha, \tau)$ *and the map* $\Gamma = \Lambda|_{\mathrm{lip}(X,d^\alpha)} : \mathrm{lip}(X, d^\alpha) \to \mathrm{lip}(X, \rho^\alpha, \tau)$ *is an injective bounded real linear operator from* $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ *regarded as a real Banach algebra onto* $(\mathrm{lip}(Y, \rho^\alpha, \tau), \|\cdot\|_{\mathrm{Lip}(Y,\rho^\alpha)})$ *satisfying*

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} \le \|\Gamma f\|_{\mathrm{Lip}(Y,\rho^\alpha)} \le 2\|f\|_{\mathrm{Lip}(X,d^\alpha)},$$

for all $f \in \mathrm{lip}(X, d^\alpha)$.

We now show that the class of quasicompact (Riesz, respectively) unital endomorphisms of real Lipschitz algebras of complex-valued functions on compact metric spaces with Lipschitz involutions is larger than the class of quasicompact (Riesz, respectively) unital complex endomorphisms of complex Lipschitz algebras.

Theorem 2.7. *Let* (X, d) *be a compact metric space,* $B = \mathrm{Lip}(X, d^\alpha)$ *for* $\alpha \in (0,1]$ *or* $B = \mathrm{lip}(X, d^\alpha)$ *for* $\alpha \in (0,1)$ *and* $T : B \to B$ *be a unital complex endomorphism of* B *induced by the Lipschitz mapping* φ *on* (X, d)*. Let* $Y = X \times \{0,1\}$*,* ρ *be the metric on* Y *defined by* $\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}$ *and* τ *be the Lipschitz involution on* (Y, ρ) *defined by*

$$\tau(x,0) = (x,1), \qquad \tau(x,1) = (x,0) \quad (x \in X).$$

Suppose that $A = \mathrm{Lip}(Y, \rho^\alpha, \tau)$ *if* $B = \mathrm{Lip}(X, d^\alpha)$ *and* $A = \mathrm{lip}(Y, \rho^\alpha, \tau)$ *if* $B = \mathrm{lip}(X, d^\alpha)$*. Let* $\psi : Y \to Y$ *be the self-map of* Y *defined by*

$$\psi(x,0) = (\varphi(x), 0), \qquad \psi(x,1) = (\varphi(x), 1) \quad (x \in X).$$

Then the following statements hold.

(i) ψ *is a Lipschitz involution on* (Y, ρ) *and* $\psi \circ \tau = \tau \circ \psi$.

(ii) *If* $S : A \to A$ *is the composition endomorphism of* A *induced by* ψ*, then* $r_e(S) = r_e(T)$ *and so* S *is quasicompact (Riesz, respectively) if and only if* T *is quasicompact (Riesz, respectively).*

Proof. Clearly, (i) holds. We prove (ii) in the case $B = \mathrm{Lip}(X, d^\alpha)$ and $A = \mathrm{Lip}(Y, \rho^\alpha, \tau)$ for $\alpha \in (0, 1]$. Define the map $\Lambda : B \to A$ by

$$(\Lambda f)(x, 0) = f(x), \qquad (f \in B,\ x \in X),$$
$$(\Lambda f)(x, 1) = \overline{f(x)}, \qquad (f \in B,\ x \in X).$$

By Theorem 2.6, Λ is an injective bounded real linear operator from $(B, \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$, regarded as a real Banach algebra, onto $(A, \|\cdot\|_{\mathrm{Lip}(Y,\rho^\alpha)})$. We can easily show that

$$\Lambda \circ T^n \circ \Lambda^{-1} = S^n,$$

for all $n \in \mathbb{N}$. Let $K \in \mathcal{K}_{\mathbb{C}}(B)$. Clearly, $K \in \mathcal{K}_{\mathbb{R}}(B)$ where B is regarded as a real Banach algebra. By open mapping theorem, Λ^{-1} is a bounded real linear operator from $(A, \|\cdot\|_{\mathrm{Lip}(Y,\rho^\alpha)})$ into $(B, \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$. Hence, $\Lambda \circ K \circ \Lambda^{-1} \in \mathcal{K}_{\mathbb{R}}(A)$. Since

$$\begin{aligned}
\|S^n - \Lambda \circ K \circ \Lambda^{-1}\| &= \|\Lambda \circ T^n \circ \Lambda^{-1} - \Lambda \circ K \circ \Lambda^{-1}\| \\
&= \|\Lambda \circ (T^n - K) \circ \Lambda^{-1}\| \\
&\le \|\Lambda\| \|T^n - K\| \|\Lambda^{-1}\|,
\end{aligned}$$

for all $n \in \mathbb{N}$, we deduce that

$$\|S^n\|_e \le \|\Lambda\| \|T^n\|_e \|\Lambda^{-1}\|,$$

for all $n \in \mathbb{N}$. This implies that

$$r_e(S) \le r_e(T). \tag{2.6}$$

Similarly, we can show that the converse of the inequality (2.6) holds. Therefore, $r_e(S) = r_e(T)$ and so (ii) holds.

To prove (ii) in the case $B = \mathrm{lip}(X, d^\alpha)$ and $A = \mathrm{lip}(Y, \rho^\alpha, \tau)$ for $\alpha \in (0, 1)$, it is sufficient to apply $\Gamma = \Lambda|_{\mathrm{lip}(X,d^\alpha)}$ instead of Λ. □

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