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# SOME PROPERTIES AND RESULTS FOR CERTAIN SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS

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Abstract. In the present paper, we introduce and investigate some properties of two subclasses $\Lambda_n(\lambda,\beta)$ and $\Lambda_n^+(\lambda,\beta)$; meromorphic and starlike functions of order $\beta$. In particular, several inclusion relations, coefficient estimates, distortion theorems and covering theorems are proven here for each of these function classes.

## 1. Introduction

Let $\mathcal{A}_n$ denotes the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N}), \tag{1.1}$$

which are analytic on the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\},$$

and suppose $\mathcal{A}_1 = \mathcal{A}$.
For $0 \leq \beta < 1$, we denote by $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ the subclasses of $\mathcal{A}$ consisting of functions which are starlike of order $\beta$ and convex of order $\beta$ in $\mathbb{U}$, respectively, that is,

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} \cap \mathcal{S}^* : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \ , \ z \in \mathbb{U} \right\},$$
$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} \cap \mathcal{K} : \Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \beta \ , \ z \in \mathbb{U} \right\},$$

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where $\mathcal{S}^*$ and $\mathcal{K}$ denote the starlike and convex functions, respectively. We set

$$\mathcal{S}^* = \mathcal{S}^*(0), \qquad \mathcal{K} = \mathcal{K}(0), \qquad \mathcal{S}_n^*(\beta) = \mathcal{S}^*(\beta) \cap \mathcal{A}_n.$$

We say that $f(z) \in \mathcal{H}_n(\alpha, \beta)$ if only and if $f(z)$ satisfies the following condition

$$\Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \alpha\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\alpha}{2},$$

$$(z \in \mathbb{U}, \alpha \geq 0, 0 \leq \beta < 1, f \in \mathcal{A}_n).$$

Obviously,

$$\mathcal{H}_n(0, \beta) = \mathcal{S}^*(\beta) \quad (0 \leq \beta < 1).$$

In 1983 Li and Owa [6] proved the following theorem.

**Theorem 1.1.** *Suppose that $\alpha \geq 0$ and $f \in \mathcal{A}$. If*

$$\Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > -\frac{\alpha}{2} \quad (z \in \mathbb{U}),$$

*then $f \in \mathcal{S}^*$.*

Moreover, in 2012 Ravichandran et al [11] gave the following modification of Theorem 1.1.

**Theorem 1.2.** *Let $\alpha \geq 0$ and $0 \leq \beta < 1$. Then*

$$\mathcal{H}_n(\alpha, \beta) \subset \mathcal{S}^*(\beta).$$

Recently Liu et al [7] investigated several other properties and characteristics of functions belonging to the subclasses $\mathcal{H}_n(\alpha, \beta)$. For more information about starlike functions, we refer the reader to [10]-[4] and the references therein.

In this paper, we introduce a new subclass of analytic starlike function and investigate some properties and results for certain classes.

## 2. Preliminaries

Let $\mathcal{P}_n$ denotes the class of functions $p(z)$ given by

$$p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k, \quad (z \in \mathbb{U}), \tag{2.1}$$

which are analytic in $\mathbb{U}$ and let $\mathcal{P}_1 = \mathcal{P}$. For the proof of our main results in this paper, we need the following useful lemma, and we refer the reader to [3].

**Lemma 2.1.** *If the function $p \in \mathcal{P}_n$ in given by (2.1) and satisfies the $\Re(p(z)) > 0$, then $|p_k| \leq 2$ $(k \leq n)$.*

**Lemma 2.2.** *If the function $f \in \mathcal{A}_n$ in given by (1.1), then*

$$\frac{zf^{'}(z)}{f(z)} \in \mathcal{P}_n.$$

*Let $0 \leq \lambda$, $0 \leqslant \beta < 1$ and $\Lambda_n(\lambda, \beta)$ denotes the class of functions $p(z) \in \mathcal{P}_n$ satisfies the condition*

$$\Re\left(\lambda\frac{z^2p^{''}(z)}{p(z)} + \frac{zp'(z)}{p(z)}\right) > \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}. \tag{2.2}$$

*Finaly, let $\Lambda_n^+(\lambda, \beta)$ denotes the subset of $\Lambda_n(\lambda, \beta)$ such that all functions $p \in \Lambda_n(\lambda, \beta)$ have the following from:*

$$p(z) = 1 - \sum_{k=n}^{\infty} p_k z^k, \quad (p_k \geq 0; k \geq n).$$

**Theorem 2.3.** *Let $0 \leq \lambda < \frac{1}{2}$ , $0 \leq \beta < 1$ and $p \in \Lambda_n(\lambda, \beta)$. Then*

$$zp(z) \in \mathcal{H}_n\left(\frac{\lambda}{1-2\lambda}, \beta\right).$$

*Proof.* Let $p(z) \in \Lambda_n(\lambda, \beta)$, then

$$\begin{aligned}\Re\left(\lambda\frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)}\right) &> \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}\\ &> \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta(1-2\lambda) - \frac{n\lambda}{2}.\end{aligned}$$

Since $0 \leq \lambda < \frac{1}{2}$, then

$$\begin{aligned}\Re\left(\frac{\lambda}{1-2\lambda}\frac{z^2p''(z)}{p(z)} + \frac{1}{1-2\lambda}\frac{zp'(z)}{p(z)} + 1\right) >& \frac{\lambda}{1-2\lambda}\beta\left(\beta + \frac{n}{2} - 1\right)\\ &+ \beta - \frac{n}{2}\frac{\lambda}{1-2\lambda}.\end{aligned} \tag{2.3}$$

Obviously, $f(z) = zp(z) \in \mathcal{A}_n$. Hence from (2.3)

$$\begin{aligned}\Re\left(\frac{\lambda}{1-2\lambda}\frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) &= \Re\left(\frac{\lambda}{1-2\lambda}\frac{z^2p''(z)}{p(z)} + \frac{1}{1-2\lambda}\frac{zp'(z)}{p(z)} + 1\right)\\ &> \frac{1}{1-2\lambda}\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n}{2}\frac{\lambda}{1-2\lambda},\end{aligned}$$

that is,

$$f(z) \in \mathcal{H}_n\left(\frac{\lambda}{1-2\lambda}, \beta\right),$$

and this completes the proof. □

**Corollary 2.4.** *Let $0 \leq \lambda < \frac{1}{2}$ , $0 \leq \beta < 1$ and $p(z) \in \Lambda_n(\lambda, \beta)$. Then $zp(z) \in \mathcal{S}^*(\beta)$.*

In order to derive our main results, we need the following lemmas.

**Lemma 2.5** ([7])**.** *Let $f(z) \in \mathcal{A}_n$ be given by (1.1). Then $f(z) \in \mathcal{K}$ if and only if*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \leq 1.$$

**Lemma 2.6.** *Let $0 \leq \lambda$ and $\gamma < 0$. Suppose also that the sequence $\{A_k\}_{k=1}^{\infty}$ is defined by*

$$\begin{cases} A_1 = -2\gamma, & k = 1, \\ A_{k+1} = \dfrac{-2\gamma}{(k+1)[\lambda k + 1]}\left(1 + \displaystyle\sum_{l=1}^{k} A_l\right), & k \geq 1. \end{cases} \tag{2.4}$$

*Then*

$$A_k = -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j-1)+1] - 2\gamma}{(j+1)[\lambda j + 1]}, \quad (k \in \mathbb{N} - \{1\}). \tag{2.5}$$

*Proof.* From (2.4), we have

$$(k+1)[\lambda k + 1]A_{k+1} = -2\gamma\left(1 + \sum_{l=1}^{k} A_l\right),$$

and

$$k[\lambda(k-1)+1]A_k = -2\gamma\left(1 + \sum_{l=1}^{k-1} A_l\right).$$

So we obtain that

$$\frac{A_{k+1}}{A_k} = \frac{k[\lambda(k-1)+1] - 2\gamma}{(k+1)[\lambda k + 1]}.$$

Thus, for $k \geq 2$ , we have

$$A_k = \frac{A_k}{A_{k-1}} \cdot \frac{A_{k-1}}{A_{k-2}} \cdots \frac{A_k}{A_1} \cdot A_1 = -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j-1)+1]}{(j+1)[\lambda j + 1],}$$

and this completes the proof. □

## 3. Properties of $\Lambda_n(\lambda, \beta)$

In this section, we give some properties of $\Lambda_n(\lambda, \beta)$. At first we prove the following inclusion result.

**Theorem 3.1.** *Let* $0 \leq \lambda_2 < \lambda_1 < \dfrac{1}{2}$ , $0 \leq \beta_1 < \beta_2 < 1$ *and* $1 \leq n_1 < n_2$. *Then*

$$\Lambda_{n_2}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_1, \beta_1).$$

*Proof.* Obviously,

$$\Lambda_{n_2}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_2, \beta_2), \quad (1 \leq n_1 < n_2).$$

Now we prove that

$$\Lambda_{n_1}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_1, \beta_1).$$

Let $p \in \Lambda_{n_1}(\lambda_2, \beta_2)$. Then

$$\begin{aligned}\Re\left(\frac{zp'(z)}{p(z)} + \lambda_2 \frac{z^2 p''(z)}{p(z)}\right) &> \lambda_2\beta_2\left(\beta_2 + \frac{n_1}{2} - 1\right) + \beta_2 - \frac{n_1\lambda_2}{2} \\ &> \lambda_2\beta_1\left(\beta_1 + \frac{n_1}{2} - 1\right) + \beta_1 - \frac{n_1\lambda_2}{2},\end{aligned}$$

which implies that $p(z) \in \Lambda_{n_1}(\lambda_2, \beta_1)$. By Corollary 2.4, we get $zp(z) \in \mathcal{S}^*(\beta_1)$. That is

$$\Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \beta_1,$$

or

$$\Re\left(\frac{zp'(z)}{p(z)} - \beta_1\right) > -1.$$

Now, by setting $\lambda = \dfrac{\lambda_1}{\lambda_2}$, we have $\lambda > 1$. Therefore,

$$\begin{aligned}&\Re\left(\frac{zp'(z)}{p(z)} + \lambda\ \frac{z^2p''(z)}{p(z)} - \lambda_1\beta_1\left(\beta_1 + \frac{n_1}{2} - 1\right) - \beta_1 + \frac{n_1\lambda_1}{2}\right) \\ &\quad = \lambda\Re\left(\frac{zp'(z)}{p(z)} + \lambda_2\ \frac{z^2p''(z)}{p(z)} - \lambda_2\beta_1\left(\beta_1 + \frac{n_1}{2} - 1\right) - \beta_1 + \frac{n_1\lambda_2}{2}\right) \\ &\qquad + (1-\lambda)\Re\left(\frac{zp'(z)}{p(z)} - \beta_1\right) > 0,\end{aligned}$$

and hence, $p(z) \in \Lambda_{n_1}(\lambda_1, \beta_1)$. □

**Theorem 3.2.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$ *and* $\gamma = \lambda\beta(\beta - \frac{1}{2}) + \beta - \frac{\lambda}{2}$. *If* $\gamma < 0$ *and* $p(z) \in \Lambda_1(\lambda, \beta)$, *then*

$$|p_1| \leq -2\gamma, \tag{3.1}$$

*and*

$$|p_k| \le -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j-1)+1]-2\gamma}{(j+1)[\lambda j+1]}, \quad (k \ge 2). \tag{3.2}$$

*Moreover each of these inequalities is sharp, with the extremal function given by*

$$p_0(z) = 1 - 2\gamma z - 2\gamma \sum_{k=2}^{\infty} \prod_{j=1}^{k-1} \frac{j[\lambda(j-1)+1]-2\gamma}{(j+1)[\lambda j+1]} \; z^k. \tag{3.3}$$

*Proof.* Let

$$q(z) = \frac{zp'(z)}{p(z)} + \lambda \, \frac{z^2 p''(z)}{p(z)} - \lambda\beta\left(\beta - \frac{1}{2}\right) - \beta + \frac{\lambda}{2}.$$

Then, from $p \in \Lambda_1(\lambda, \beta)$, it is easy to see that $p(z)$ is analytic in $\mathbb{U}$ , $q(0) = -\gamma > 0$ and $\Re[q(z)] > 0$. Therefore,

$$h(z) = \frac{q(z)}{-\gamma} \in \mathcal{P},$$

If we put

$$q(z) = q_0 + \sum_{k=1}^{\infty} q_k z^k, \quad (q_0 = -\gamma),$$

then by Lemma 2.1, we get

$$|q_k| \le -2\gamma, \quad (k \in \mathbb{N}).$$

Also

$$q(z)p(z) = zp'(z) + \lambda z^2 p''(z) - \gamma p(z),$$

and so

$$\left(q_0 + \sum_{k=1}^{\infty} q_k z^k\right)\left(1 + \sum_{k=1}^{\infty} p_k z^k\right) = \lambda \sum_{k=1}^{\infty} k(k-1) p_k z^k + \sum_{k=1}^{\infty} k p_k z^k - \gamma\left(1 + \sum_{k=1}^{\infty} p_k z^k\right).$$

Thus, noting that $q_0 = -\gamma$ , $p_1 = q_1$ and

$$p_{k+1} = \frac{1}{(k+1)(\lambda k+1)}\left(q_{k+1} + \sum_{l=1}^{k} p_l q_{k+1-l}\right).$$

Therefore

$$|p_1| \le -2\gamma,$$

and

$$|p_{k+1}| \le \frac{-2\gamma}{(k+1)(\lambda k+1)}\left(1+\sum_{l=1}^{k}|p_l|\right), \quad (k \in N).$$

Next, we define the sequence $\{A_k\}$ as follows,

$$A_1 = -2\gamma,$$

$$A_{k+1} = \frac{-2\gamma}{(k+1)(\lambda k+1)}\left(1+\sum_{l=1}^{k}|A_l|\right), \quad (k \in \mathbb{N}). \tag{3.4}$$

Hence, by the principle of mathematical induction,

$$|p_k| \le A_k, \quad (k \in \mathbb{N}).$$

Now by using Lemma 2.6, we conclude that the conditions (2.5) hold.

Finally, in order to verify that the inequalities are sharp, we set

$$q_0(z) = -\gamma\,\frac{1+z}{1-z}, \quad (z \in \mathbb{U}).$$

Obviously, $p_0 \in \mathcal{P}$. By using (3.3) and (3.4) and by some simple computations, we obtain that

$$q_0(z)p_0(z) = \lambda z^2 p_0^{''}(z) + z p_0^{'}(z) - \gamma p_0(z), \quad (z \in \mathbb{U}).$$

So, it follows that

$$p_0(z) \in \Lambda_1(\lambda, \beta),$$

which completes the proof. □

**Corollary 3.3.** *Let* $0 \le \lambda \le \beta < \frac{1}{2}$ *and* $\gamma = \lambda\beta\left(\beta - \frac{1}{2}\right) + \beta - \frac{\lambda}{2}$. *If* $p(z) \in \Lambda_1(\lambda, \beta)$, *then*

$$|p(z)| \le 1 - 2\gamma\left(r + \sum_{k=2}^{\infty}\prod_{j=1}^{k-1}\frac{j[\lambda(j-1)+1]-2\gamma}{(j+1)[\lambda j+1]}r^k\right), \quad (|z| = r < 1,)$$

*and*

$$|p^{'}(z)| \le -2\gamma\left(1 + \sum_{k=2}^{\infty}k\prod_{j=1}^{k-1}\frac{j[\lambda(j-1)+1]-2\gamma}{(j+1)[\lambda j+1]}r^{k-1}\right), \quad (|z| = r < 1).$$

*Moreover, each of those inequalities is sharp, with the extremal function given by* (3.3).

**Theorem 3.4.** *Let* $0 \le \lambda \le \beta < \frac{1}{2}$ *and* $\gamma_n = \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}$. *If* $p(z) \in \Lambda_n(\lambda, \beta)$ *then*

$$|p_k| \le B_k(n), \tag{3.5}$$

*where*

$$\left\{\begin{array}{ll} B_k(n)=\dfrac{-2\gamma_n}{k[\lambda(k-1)+1]}, & n\leq k<2n; \\ \\ B_k(n)=\dfrac{-2\gamma_n}{k[\lambda(k-1)+1]}\left(1+\displaystyle\sum_{l=n}^{k-1}B_l(n)\right), & k\geq 2n. \end{array}\right. \tag{3.6}$$

*Proof.* It is easy to see that $\gamma_n<0$ for all $n$. If we set

$$q(z)=\frac{zp^{'}(z)}{p(z)}+\lambda\frac{z^2p^{''}(z)}{p(z)}-\gamma_n,$$

then

$$q(z)=-\gamma_n+n[\lambda(n-1)+1]p_nz^n+\cdots,$$

is analytic in $\mathbb{U}$ and $\Re[q(z)]>0$ whit $q(0)=-\gamma_n>0$. Hence,

$$h(z)=\frac{q(z)}{\gamma_n}\in P_n.$$

If

$$q(z)=q_0+\sum_{k=n}^{\infty}q_kz^k,$$

then

$$q(z)p(z)=zp^{'}(z)+\lambda z^2p^{''}(z)-\gamma_np(z),$$

and so

$$\begin{aligned}\left(q_0+\sum_{k=n}^{\infty}q_kz^k\right)\left(1+\sum_{k=n}^{\infty}p_kz^k\right)&=\sum_{k=n}^{\infty}kp_kz^k+\sum_{k=n}^{\infty}\lambda k(k-1)p_kz^k\\&\quad-\gamma_n\left(1+\sum_{k=n}^{\infty}p_kz^k\right).\end{aligned}$$

Thus $q_0=-\gamma_n$,

$$q_0p_k+q_k=k[\lambda(k-1)+1]p_k-\gamma_np_k,\quad (n\leq k<2n),$$

and

$$q_0p_k+q_k+\sum_{l=n}^{\infty}p_lq_{k+n-l}=k[\lambda(k-1)+1]p_k-\gamma_np_k,\quad (k\geq 2n).$$

Also, by Lemma 2.1

$$|p_k|=\frac{|q_k|}{k[\lambda(k-1)+1]}\leq\frac{-2\gamma}{k[\lambda(k-1)+1]},\quad (n\leq k<2n),$$

and

$$|p_k| = \frac{1}{k[\lambda(k-1)+1]}\left[|q_k| + \sum_{l=n}^{k-n}|p_l||q_{k+n-l}|\right]$$
$$\leq \frac{-2\gamma}{k[\lambda(k-1)+1]}\left(1+\sum_{l=n}^{k-n}|p_l|\right), \quad (k \geq 2n).$$

Next, by applying the method of the proof of Theorem 3.2, we have,

$$|p_k| \leq B_k(n), \quad (k \geq 2n),$$

which completes the proof. □

**Corollary 3.5.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$ *and the sequence* $\{B_k(n)\}_{k=1}^{\infty}$ *be defined by* (3.6)*. If* $p(z) \in \Lambda_n(\lambda, \beta)$*, then*

$$|p(z)| \leq 1 + \sum_{k=n}^{\infty} B_k(n) r^k, \quad (|z| = r < 1),$$

*and*

$$|p^{'}(z)| \leq \sum_{k=n}^{\infty} B_k(n) r^{k-1}, \quad (|z| = r < 1).$$

A covering theorem for the class $\Lambda_n(\lambda, \beta)$ is provided by the following result.

**Theorem 3.6.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$*. If* $p(z) \in \Lambda_n(\lambda, \beta)$*, then the unit disk* $\mathbb{U}$ *is mapped by onto a domain that contains the disk* $|w| < r_0$*, where*

$$r_0 = \max\left\{n^{-\frac{1}{n-1}}, \left(\frac{n[\lambda(n-1)+1]}{n(n+1)[\lambda(n-1)+1] - 2\lambda\beta(2\beta+n-2) - \beta + n\lambda}\right)^{\frac{1}{n}}\right\}.$$

*Proof.* Let $w_0$ be any complex number such that

$$p(z) \neq w_0 \ (z \in U).$$

Then $w_0 \neq 0$ and, by Theorem 1.2, the function

$$\frac{w_0 z p(z)}{w_0 - z p(z)} = z + \frac{1}{w_0} z^2 + \cdots + \frac{1}{w_0^{n-1}} z^n + \left(p_n + \frac{1}{w_0^n}\right) z^{n+1} + \cdots,$$

is univalent with

$$\left|\frac{1}{w_0}\right| \leq 2, \qquad \left|\frac{1}{w_0^{n-1}}\right| \leq n \text{ and } \left|p_n + \frac{1}{w_0^n}\right| \leq n+1.$$

Therefore, according to Theorem 3.4, we find that

$$|w_0| \geq n^{-\frac{1}{n-1}}, \quad (n \in \mathbb{N} - \{1\}),$$

and

$$|w_0| \geq \left[\frac{n[\lambda(n-1)+1]}{n(n+1)[\lambda(n-1)+1]-2\gamma_n}\right]^{\frac{1}{n}}, \quad (n \in \mathbb{N}),$$

and this completes the proof. □

**Theorem 3.7.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$ *and* $p(z) \in P_n$*. If*

$$\sum_{k=n}^{\infty}[k(\lambda(k-1)+1)-\gamma_n]|p_k| \leq -\gamma_n, \tag{3.7}$$

$$\left(\gamma_n = \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}\right),$$

*then* $p(z) \in \Lambda_n(\lambda, \beta)$.

*Proof.* Suppose that

$$q(z) = \frac{zp^{'}(z)}{p(z)} + \lambda\frac{z^2p^{''}(z)}{p(z)}.$$

We prove that $|q(z)| < -\gamma_n \quad (z \in \mathbb{U})$.
Obviously,

$$\sum_{k=n}^{\infty}[k(\lambda(k-1)+1)-\gamma_n]|p_k| \geq -\gamma_n \sum_{k=n}^{\infty}|p_k|.$$

Also, by using (3.7), we obtain that

$$\sum_{k=n}^{\infty}|p_k| < 1,$$

and so

$$|q(z)| = \left| \frac{\displaystyle\sum_{k=n}^{\infty} [k(\lambda(k-1)+1)] p_k z^k}{1 + \displaystyle\sum_{k=n}^{\infty} p_k z^k} \right|.$$

$$\leq \frac{\displaystyle\sum_{k=n}^{\infty} [k(\lambda(k-1)+1)] |p_k|}{1 - \displaystyle\sum_{k=n}^{\infty} |p_k|}$$

$$= \frac{\displaystyle\sum_{k=n}^{\infty} [k(\lambda(k-1)+1) - \gamma_n] |p_k| - (-\gamma_n) \sum_{k=n}^{\infty} |p_k|}{1 - \displaystyle\sum_{k=n}^{\infty} |p_k|}$$

$$\leq \frac{-\gamma_n - (-\gamma_n) \displaystyle\sum_{k=n}^{\infty} |p_k|}{1 - \displaystyle\sum_{k=n}^{\infty} |p_k|}$$

$$= -\gamma_n.$$

Hence, $\Re(-q(z)) \leq |-q(z)| < -\gamma_n$. Thus $\Re(q(z)) > \gamma_n$. That is, $p(z) \in \Lambda_n(\lambda, \beta)$. □

## 4. Properties of $\Lambda_n^+(\lambda, \beta)$

From Theorem 3.1 and the definition of $\Lambda_n^+(\lambda, \beta)$, we have the following inclusion result.

**Theorem 4.1.** *Let* $0 \leq \lambda_2 < \lambda_1 < \frac{1}{2}$, $0 \leq \beta_1 < \beta_2 < \frac{1}{2}$ *and* $1 \leq n_1 \leq n_2$. *Then*

$$\Lambda_{n_2}^+(\lambda_2, \beta_2) \subset \Lambda_{n_1}^+(\lambda_1, \beta_1).$$

In the following theorem we give a necessary and sufficient condition for an element belongs to $\Lambda_n^+(\lambda, \beta)$.

**Theorem 4.2.** *Let* $0 \leq \lambda < \beta < \frac{1}{2}$ *and* $p(z) \in \mathcal{P}_n$. *Then* $p \in \Lambda_n^+(\lambda, \beta)$ *if and only if*

$$\sum_{k=n}^{\infty} [k(\lambda(k-1)+1) - \gamma_n] p_k \leq -\gamma_n, \tag{4.1}$$

$$\left(\gamma_n = \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}\right).$$

*Proof.* Let $p(z) \in \mathcal{P}_n$. If $p$ satisfies (4.1) then by Theorem 3.7, we conclude that $p(z) \in \Lambda_n^+(\lambda, \beta)$.
Conversely, suppose that $p \in \Lambda_n^+(\lambda, \beta)$. Then

$$\Re\left(\frac{zp^{'}(z)}{p(z)} + \lambda\frac{z^2p^{''}(z)}{p(z)}\right) = \Re\left(-\frac{\sum_{k=n}^{\infty}[k(\lambda(k-1)+1)]p_kz^k}{1 - \sum_{k=n}^{\infty}p_kz^k}\right) > \gamma_n,$$

for $z = re^{i\theta}$ , $0 \le r < 1$ and $0 \le \theta < 2\pi$. Hence

$$-\gamma_n > \frac{\sum_{k=n}^{\infty}[k(\lambda(k-1)+1)]p_kr^k}{1 - \sum_{k=n}^{\infty}p_kr^k} \ge \frac{\sum_{k=n}^{\infty}[k(\lambda(k-1)+1)]p_kr^k}{1 + \sum_{k=n}^{\infty}p_kr^k},$$

and so

$$\sum_{k=n}^{\infty}[k(\lambda(k-1)+1) - \gamma_n]p_kr^k < -\gamma_n.$$

By letting $r \to 1$ in the above inequality, we get $p \in \Lambda_n^+(\lambda, \beta)$ and this completes the proof. □

**Corollary 4.3.** *If* $p(z) \in \Lambda_n^+(\lambda, \beta)$, *then*

$$p_k \le \frac{-\gamma_n}{k[\lambda(k-1)+1] - \gamma_n}, \quad \left(k \ge n,\ 0 \le \lambda < \beta < \frac{1}{2}\right).$$

*Proof.* This follows from Theorem 4.2, since in this case, the condition (4.1) is satisfied. □

**Theorem 4.4.** *Let* $0 \le \lambda \le \beta < 1$, $\gamma_n = \lambda\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}$ *and* $p(z) \in \Lambda_n^+(\lambda, \beta)$. *Then*

(4.2)
$$1 + \frac{\gamma_n}{n(\lambda(n-1)+1) - \gamma_n}r^n \le |p(z)| \le 1 - \frac{\gamma_n}{n(\lambda(n-1)+1) - \gamma_n}r^n, \quad (|z| = r < 1)$$

*and*

$$\frac{n\gamma_n}{n(\lambda(n-1)+1)n\gamma_n}r^{n-1} \le |p^{'}(z)| \le -\frac{n\gamma_n}{n(\lambda(n-1)+1)n\gamma_n}r^{n-1}, \quad (|z| = r < 1), \tag{4.3}$$

*Moreover, each of these inequalities is sharp, with the extremal function given by*

$$p_n(z) = 1 + \frac{\gamma_n}{n(\lambda(n-1)+1)-\gamma_n} z^n. \tag{4.4}$$

*Proof.* By using Theorem 4.2, we get

$$\sum_{k=n}^{\infty} p_k \leq \frac{-\gamma_n}{n(\lambda(n-1)+1)-\gamma_n}.$$

Therefore, the distortion inequalities in (4.2) follow from

$$1 - r^n \sum_{k=n}^{\infty} p_k \leq |p(z)| \leq 1 + r^n \sum_{k=n}^{\infty} p_k, \quad (|z| = r < 1).$$

Also, since

$$\sum_{k=n}^{\infty} k p_k \leq \frac{-n\gamma_n}{n(\lambda(n-1)+1)-\gamma_n},$$

then the distortion inequalities in (4.3) follow from

$$-r^n \sum_{k=n}^{\infty} k p_k \leq |p^{'}(z)| \leq r^n \sum_{k=n}^{\infty} k p_k, \quad (|z| = r < 1).$$

This completes the proof. □

**Corollary 4.5.** *Let $0 \leq \lambda \leq \beta < \frac{1}{2}$ and $p(z) \in \Lambda_n^+(\lambda, \beta)$. Then the unit disk $\mathbb{U}$ is mapped by $p(z)$ onto a domain that contains the disk $|w| < r_1$, where*

$$r_1 = \frac{n(\lambda(n-1)+1)-2\gamma_n}{n(\lambda(n-1)+1)-\gamma_n}.$$

*The result is sharp, with the extremal function $p_n(z)$ given by* (4.4).

**Corollary 4.6.** *Let $0 \leq \lambda \leq \beta < \frac{1}{2}$. Then $\Lambda_1^+(\lambda, \beta) \subset \mathcal{K}$.*

*Proof.* The proof follows from Theorem 4.2 and Lemma 2.5. □

**Corollary 4.7.** *Let $0 \leq \lambda \leq \beta < \dfrac{1}{2}$ and $\gamma_n = \lambda\beta\left(\beta + \dfrac{n}{2} - 1\right) + \beta - \dfrac{n\lambda}{2}$. Suppose that*

$$p_{n-1}(z) = 1, \qquad p_k(z) = 1 - \frac{-\gamma_n}{k(\lambda(k-1)+1)-\gamma_n} z^k, \quad (k \geq n). \tag{4.5}$$

*Then $p(z) \in \Lambda_n^+(\lambda, \beta)$ if and only if $p(z)$ can be expressed in the following form*

$$p(z) = \sum_{k=n-1}^{\infty} \mu_k p_k(z), \quad \left(\mu_k \geq 0\ ,\ k \geq n \text{ and } \sum_{k=n-1}^{\infty} \mu_k = 1\right).$$

*Proof.* Suppose that

$$\begin{aligned} p(z) &= \sum_{k=n-1}^{\infty} \mu_k p_k(z) \\ &= 1 - \sum_{k=n}^{\infty} \mu_k \frac{-\gamma_n}{k(\lambda(k-1)+1-\gamma_k} z^k. \end{aligned}$$

Then, by using Theorem 4.2, we can deduce that $p(z) \in \Lambda_n^+(\lambda, \beta)$.

Conversely, suppose that $p(z) \in \Lambda_n^+(\lambda, \beta)$. Then, from Corollary 4.3, we have

$$p_k \leq \frac{-\gamma_n}{k(\lambda(k-1)+1)-\gamma_n}, \quad (k \geq n).$$

Now, if we set

$$\mu_k = \frac{k(\lambda(k-1)+1)-\gamma_n}{-\gamma_n} p_k, \quad (k \geq n),$$

and

$$\mu_{n-1} = 1 - \sum_{k=1}^{\infty} \mu_k,$$

then

$$\sum_{k=n}^{\infty} \mu_k = 1, \quad \mu_k \geq 0, \quad (k \geq n),$$

and

$$p(z) = \sum_{k=n}^{\infty} \mu_k p_k(z).$$

□

**Corollary 4.8.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$. *Then the extreme points of the class* $\Lambda_n^+(\lambda, \beta)$ *are the functions* $p_k(z)$ *given by* (4.5).

Finally, by Lemma 2.5 and Theorem 4.2, we conclude the following inclusion relation.

**Theorem 4.9.** *Let* $0 \leq \lambda \leq \beta < \frac{1}{2}$. *Then* $\Lambda_1^+(\lambda, \beta) \subset \mathcal{K}$.

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