SOME PROPERTIES AND RESULTS FOR CERTAIN SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS

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Abstract. In the present paper, we introduce and investigate some properties of two subclasses $\Lambda_n(\lambda, \beta)$ and $\Lambda^*_n(\lambda, \beta)$; meromorphic and starlike functions of order $\beta$. In particular, several inclusion relations, coefficient estimates, distortion theorems and covering theorems are proven here for each of these function classes.

1. Introduction

Let $A_n$ denotes the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N}),$$

which are analytic on the open unit disk

$$\mathbb{U} = \{z : |z| < 1\},$$

and suppose $A_1 = A$.

For $0 \leq \beta < 1$, we denote by $S^*(\beta)$ and $K(\beta)$ the subclasses of $A$ consisting of functions which are starlike of order $\beta$ and convex of order $\beta$ in $\mathbb{U}$, respectively, that is,

$$S^*(\beta) = \left\{ f \in A \cap S^* : \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{U} \right\},$$

$$K(\beta) = \left\{ f \in A \cap K : \Re \left( 1 + \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{U} \right\},$$

2010 Mathematics Subject Classification. 30C45, 30C80.
Key words and phrases. Analytic functions, Starlike functions, Convex functions, Coefficient estimates.
Received: 04 September 2016, Accepted: 06 February 2017.
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where $\mathcal{S}^*$ and $\mathcal{K}$ denote the starlike and convex functions, respectively. We set
\[ \mathcal{S}^* = \mathcal{S}^*(0), \quad \mathcal{K} = \mathcal{K}(0), \quad \mathcal{S}^*_n(\beta) = \mathcal{S}^*(\beta) \cap \mathcal{A}_n. \]
We say that $f(z) \in \mathcal{H}_n(\alpha, \beta)$ if only if $f(z)$ satisfies the following condition
\[ \Re \left( \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right) > \alpha \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n \alpha}{2}, \]
\[ (z \in \mathbb{U}, \alpha \geq 0, 0 \leq \beta < 1, f \in \mathcal{A}_n). \]
Obviously,
\[ \mathcal{H}_n(0, \beta) = \mathcal{S}^*(\beta) \quad (0 \leq \beta < 1). \]
In 1983 Li and Owa \[72\] proved the following theorem.

**Theorem 1.1.** Suppose that $\alpha \geq 0$ and $f \in \mathcal{A}$. If
\[ \Re \left( \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right) > -\frac{\alpha}{2} \quad (z \in \mathbb{U}), \]
then $f \in \mathcal{S}^*$.

Moreover, in 2012 Ravichandran et al \[11\] gave the following modification of Theorem 1.1.

**Theorem 1.2.** Let $\alpha \geq 0$ and $0 \leq \beta < 1$. Then
\[ \mathcal{H}_n(\alpha, \beta) \subset \mathcal{S}^*(\beta). \]

Recently Liu et al \[7\] investigated several other properties and characteristics of functions belonging to the subclasses $\mathcal{H}_n(\alpha, \beta)$. For more information about starlike functions, we refer the reader to \[10\]-\[4\] and the references therein.

In this paper, we introduce a new subclass of analytic starlike function and investigate some properties and results for certain classes.

## 2. Preliminaries

Let $\mathcal{P}_n$ denotes the class of functions $p(z)$ given by
\[ p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k, \quad (z \in \mathbb{U}), \tag{2.1} \]
which are analytic in $\mathbb{U}$ and let $\mathcal{P}_1 = \mathcal{P}$. For the proof of our main results in this paper, we need the following useful lemma, and we refer the reader to \[3\].

**Lemma 2.1.** If the function $p \in \mathcal{P}_n$ in given by (2.1) and satisfies the $\Re(p(z)) > 0$, then $|p_k| \leq 2$ ($k \leq n$).
Lemma 2.2. If the function $f \in A_n$ given by \( (1.1) \), then

$$\frac{zf'(z)}{f(z)} \in P_n.$$ 

Let $0 \leq \lambda$, $0 \leq \beta < 1$ and $A_n(\lambda, \beta)$ denotes the class of functions $p(z) \in P_n$ satisfies the condition

$$\Re \left( \frac{\lambda z^2 p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} \right) > \lambda \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2}. \tag{2.3}$$

Finally, let $A_n^+(\lambda, \beta)$ denotes the subset of $A_n(\lambda, \beta)$ such that all functions $p \in A_n(\lambda, \beta)$ have the following form:

$$p(z) = 1 - \sum_{k=n}^{\infty} p_k z^k, \quad (p_k \geq 0; k \geq n).$$

Theorem 2.3. Let $0 \leq \lambda < \frac{1}{2}$, $0 \leq \beta < 1$ and $p \in A_n(\lambda, \beta)$. Then

$$zp(z) \in H_n \left( \frac{\lambda}{1 - 2\lambda}, \beta \right).$$

Proof. Let $p(z) \in A_n(\lambda, \beta)$, then

$$\Re \left( \frac{\lambda z^2 p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} \right) > \lambda \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2} \quad \text{and} \quad \beta - \frac{n\lambda}{2}.$$ 

Since $0 \leq \lambda < \frac{1}{2}$, then

$$\Re \left( \frac{\lambda z^2 p''(z)}{p(z)} + \frac{1}{1 - 2\lambda} \frac{zp'(z)}{p(z)} + 1 \right) > \frac{\lambda}{1 - 2\lambda} \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2} \frac{\lambda}{2}.$$ 

(2.3)

Obviously, $f(z) = zp(z) \in A_n$. Hence from (2.3)

$$\Re \left( \frac{\lambda z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right) = \Re \left( \frac{\lambda z^2 p''(z)}{p(z)} + \frac{1}{1 - 2\lambda} \frac{zp'(z)}{p(z)} + 1 \right) \quad \text{and} \quad \beta - \frac{n\lambda}{2} \frac{\lambda}{2}.$$ 

that is,

$$f(z) \in H_n \left( \frac{\lambda}{1 - 2\lambda}, \beta \right),$$

and this completes the proof. \( \Box \)

Corollary 2.4. Let $0 \leq \lambda < \frac{1}{2}$, $0 \leq \beta < 1$ and $p(z) \in A_n(\lambda, \beta)$. Then $zp(z) \in S^*(\beta)$. 
In order to derive our main results, we need the following lemmas.

**Lemma 2.5** (7). Let \( f(z) \in A_n \) be given by (1.1). Then \( f(z) \in K \) if and only if
\[
\sum_{k=n+1}^{\infty} k^2|a_k| \leq 1.
\]

**Lemma 2.6.** Let \( 0 \leq \lambda \) and \( \gamma < 0 \). Suppose also that the sequence \( \{A_k\}_{k=1}^{\infty} \) is defined by
\[
A_1 = -2\gamma, \quad \text{and} \quad A_{k+1} = \frac{-2\gamma}{(k+1)[\lambda k + 1]} \left( 1 + \sum_{l=1}^{k} A_l \right), \quad k \geq 1.
\]
Then
\[
A_k = -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j-1) + 1] - 2\gamma}{(j+1)[\lambda j + 1]}, \quad (k \in \mathbb{N} - \{1\}).
\]

**Proof.** From (2.3), we have
\[
(k + 1)[\lambda k + 1]A_{k+1} = -2\gamma \left( 1 + \sum_{l=1}^{k} A_l \right),
\]
and
\[
k[\lambda(k-1) + 1]A_k = -2\gamma \left( 1 + \sum_{l=1}^{k-1} A_l \right).
\]
So we obtain that
\[
\frac{A_{k+1}}{A_k} = \frac{k[\lambda(k-1) + 1] - 2\gamma}{(k+1)[\lambda k + 1]}.
\]
Thus, for \( k \geq 2 \), we have
\[
A_k = \frac{A_k}{A_{k-1}} \cdot \frac{A_{k-1}}{A_{k-2}} \cdots \frac{A_2}{A_1} \cdot A_1 = -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j-1) + 1]}{(j+1)[\lambda j + 1]},
\]
and this completes the proof. \( \square \)

### 3. Properties of \( \Lambda_n(\lambda, \beta) \)

In this section, we give some properties of \( \Lambda_n(\lambda, \beta) \). At first we prove the following inclusion result.
Theorem 3.1. Let $0 \leq \lambda_2 < \lambda_1 < \frac{1}{2}$, $0 \leq \beta_1 < \beta_2 < 1$ and $1 \leq n_1 < n_2$. Then
\[
\Lambda_{n_2}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_1, \beta_1).
\]

Proof. Obviously,
\[
\Lambda_{n_2}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_2, \beta_2), \quad (1 \leq n_1 < n_2).
\]

Now we prove that
\[
\Lambda_{n_1}(\lambda_2, \beta_2) \subset \Lambda_{n_1}(\lambda_1, \beta_1).
\]

Let $p \in \Lambda_{n_1}(\lambda_2, \beta_2)$. Then
\[
\Re \left( \frac{zp'(z)}{p(z)} + \lambda_2 \frac{z^2p''(z)}{p(z)} \right) > \lambda_2 \beta_2 \left( \beta_2 + \frac{n_1}{2} - 1 \right) + \beta_2 - \frac{n_1 \lambda_2}{2}
\]
\[
> \lambda_2 \beta_1 \left( \beta_1 + \frac{n_1}{2} - 1 \right) + \beta_1 - \frac{n_1 \lambda_2}{2},
\]
which implies that $p(z) \in \Lambda_{n_1}(\lambda_2, \beta_2)$. By Corollary 2.4, we get $zp(z) \in S^*(\beta_1)$. That is
\[
\Re \left( 1 + \frac{zp'(z)}{p(z)} \right) > \beta_1,
\]
or
\[
\Re \left( \frac{zp'(z)}{p(z)} - \beta_1 \right) > -1.
\]

Now, by setting $\lambda = \frac{\lambda_1}{\lambda_2}$, we have $\lambda > 1$. Therefore,
\[
\Re \left( \frac{zp'(z)}{p(z)} + \lambda \frac{z^2p''(z)}{p(z)} - \lambda_1 \beta_1 \left( \beta_1 + \frac{n_1}{2} - 1 \right) - \beta_1 + \frac{n_1 \lambda_1}{2} \right)
\]
\[
= \lambda \Re \left( \frac{zp'(z)}{p(z)} + \lambda_2 \frac{z^2p''(z)}{p(z)} - \lambda_2 \beta_1 \left( \beta_1 + \frac{n_1}{2} - 1 \right) - \beta_1 + \frac{n_1 \lambda_2}{2} \right)
\]
\[
+ (1 - \lambda) \Re \left( \frac{zp'(z)}{p(z)} - \beta_1 \right) > 0,
\]
and hence, $p(z) \in \Lambda_{n_1}(\lambda_1, \beta_1)$. \qed

Theorem 3.2. Let $0 \leq \lambda \leq \beta < \frac{1}{2}$ and $\gamma = \lambda \beta (\beta - \frac{1}{2}) + \beta - \lambda$. If $\gamma < 0$ and $p(z) \in \Lambda_1(\lambda, \beta)$, then
\[
|p_1| \leq -2\gamma,
\]
\[ |p_k| \leq -2\gamma \prod_{j=1}^{k-1} \frac{j[\lambda(j - 1) + 1] - 2\gamma}{(j + 1)[\lambda j + 1]}, \quad (k \geq 2). \]

Moreover each of these inequalities is sharp, with the extremal function given by

\[ p_0(z) = 1 - 2\gamma z - 2\gamma \sum_{k=2}^{\infty} \prod_{j=1}^{k-1} \frac{j[\lambda(j - 1) + 1] - 2\gamma}{(j + 1)[\lambda j + 1]} z^k. \]

**Proof.** Let

\[ q(z) = \frac{zp'(z)}{p(z)} + \lambda \frac{z^2p''(z)}{p(z)} - \lambda \beta \left( \beta - \frac{1}{2} \right) - \beta + \frac{\lambda}{2}. \]

Then, from \( p \in \Lambda_1(\lambda, \beta) \), it is easy to see that \( p(z) \) is analytic in \( U \), \( q(0) = -\gamma > 0 \) and \( \Re[q(z)] > 0 \). Therefore,

\[ h(z) = \frac{q(z)}{-\gamma} \in \mathcal{P}, \]

If we put

\[ q(z) = q_0 + \sum_{k=1}^{\infty} q_k z^k, \quad (q_0 = -\gamma), \]

then by Lemma 2.1, we get

\[ |q_k| \leq -2\gamma, \quad (k \in \mathbb{N}). \]

Also

\[ q(z)p(z) = zp'(z) + \lambda z^2 p''(z) - \gamma p(z), \]

and so

\[ \left( q_0 + \sum_{k=1}^{\infty} q_k z^k \right) \left( 1 + \sum_{k=1}^{\infty} p_k z^k \right) = \lambda \sum_{k=1}^{\infty} k(k - 1)p_k z^k + \sum_{k=1}^{\infty} kp_k z^k - \gamma \left( 1 + \sum_{k=1}^{\infty} p_k z^k \right). \]

Thus, noting that \( q_0 = -\gamma \), \( p_1 = q_1 \) and

\[ p_{k+1} = \frac{1}{(k + 1)[\lambda k + 1]} \left( q_{k+1} + \sum_{t=1}^{k} p_t q_{k+1-t} \right). \]

Therefore

\[ |p_1| \leq -2\gamma, \]
and
\[ |p_{k+1}| \leq \frac{-2\gamma}{(k + 1)(\lambda k + 1)} \left( 1 + \sum_{l=1}^{k} |p_l| \right), \quad (k \in \mathbb{N}). \]

Next, we define the sequence \( \{A_k\} \) as follows,
\[ A_1 = -2\gamma, \]
\[ A_{k+1} = \frac{-2\gamma}{(k + 1)(\lambda k + 1)} \left( 1 + \sum_{l=1}^{k} |A_l| \right), \quad (k \in \mathbb{N}). \]

Hence, by the principle of mathematical induction,
\[ |p_k| \leq A_k, \quad (k \in \mathbb{N}). \]

Now by using Lemma 2.6, we conclude that the conditions (2.5) hold.

Finally, in order to verify that the inequalities are sharp, we set
\[ q_0(z) = -\gamma \frac{1 + z}{1 - z}, \quad (z \in \mathbb{U}). \]

Obviously, \( p_0 \in \mathcal{P} \). By using (3.3) and (3.4) and by some simple computations, we obtain that
\[ q_0(z)p_0(z) = \lambda z^2 p''_0(z) + zp'_0(z) - \gamma p_0(z), \quad (z \in \mathbb{U}). \]

So, it follows that
\[ p_0(z) \in \Lambda_1(\lambda, \beta), \]
which completes the proof. \( \square \)

**Corollary 3.3.** Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \) and \( \gamma = \lambda \beta \left( \beta - \frac{1}{2} \right) + \beta - \frac{\lambda}{2} \). If \( p(z) \in \Lambda_1(\lambda, \beta) \), then
\[ |p(z)| \leq 1 - 2\gamma \left( r + \sum_{k=2}^{r-1} \prod_{j=1}^{k-1} \frac{j[\lambda(j-1) + 1] - 2\gamma}{(j+1)[\lambda j + 1]} \right), \quad (|z| = r < 1, \) 

and
\[ |p'(z)| \leq -2\gamma \left( 1 + \sum_{k=2}^{\infty} \prod_{j=1}^{k-1} \frac{j[\lambda(j-1) + 1] - 2\gamma}{(j+1)[\lambda j + 1]} \right), \quad (|z| = r < 1). \]

Moreover, each of those inequalities is sharp, with the extremal function given by (3.3).

**Theorem 3.4.** Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \) and \( \gamma_n = \lambda \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2}. \) If \( p(z) \in \Lambda_n(\lambda, \beta) \) then
\[ |p_k| \leq B_k(n), \]
(3.5)
where
\[
B_k(n) = \begin{cases} 
\frac{-2\gamma_n}{k[\lambda(k-1)+1]}, & n \leq k < 2n; \\
\frac{-2\gamma_n}{k[\lambda(k-1)+1]} \left( 1 + \sum_{i=n}^{k-1} B_i(n) \right), & k \geq 2n.
\end{cases}
\]

\[(3.6)\]

**Proof.** It is easy to see that \(\gamma_n < 0\) for all \(n\). If we set
\[
q(z) = \frac{zp'(z)}{p(z)} + \lambda \frac{z^2 p''(z)}{p(z)} - \gamma_n,
\]
then
\[
q(z) = -\gamma_n + n[\lambda(n-1)+1]p_n z^n + \cdots,
\]
is analytic in \(U\) and \(\Re[q(z)] > 0\) with \(q(0) = -\gamma_n > 0\). Hence,
\[
h(z) = \frac{q(z)}{\gamma_n} \in P_n.
\]
If
\[
q(z) = q_0 + \sum_{k=n}^{\infty} q_k z^k,
\]
then
\[
q(z)p(z) = zp'(z) + \lambda z^2 p''(z) - \gamma_n p(z),
\]
and so
\[
\left( q_0 + \sum_{k=n}^{\infty} q_k z^k \right) \left( 1 + \sum_{k=n}^{\infty} p_k z^k \right) = \sum_{k=n}^{\infty} k p_k z^k + \sum_{k=n}^{\infty} \lambda k(k-1)p_k z^k
\]
\[
- \gamma_n \left( 1 + \sum_{k=n}^{\infty} p_k z^k \right).
\]
Thus \(q_0 = -\gamma_n\),
\[
q_0 p_k + q_k = k[\lambda(k-1)+1]p_k - \gamma_n p_k, \quad (n \leq k < 2n),
\]
and
\[
q_0 p_k + q_k + \sum_{l=n}^{\infty} p_l q_{k+l} = k[\lambda(k-1)+1]p_k - \gamma_n p_k, \quad (k \geq 2n).
\]
Also, by Lemma 2.1
\[
|p_k| = \frac{|q_k|}{k[\lambda(k-1)+1]} \leq \frac{-2\gamma}{k[\lambda(k-1)+1]}, \quad (n \leq k < 2n),
\]
and

\[ |p_k| = \frac{1}{k[\lambda(k-1) + 1]} \left[ |q_k| + \sum_{l=n}^{k-n} |p_l||q_{k+l-n-l}| \right] \leq \frac{-2\gamma}{k[\lambda(k-1) + 1]} \left( 1 + \sum_{l=n}^{k-n} |p_l| \right), \quad (k \geq 2n). \]

Next, by applying the method of the proof of Theorem 3.2, we have,

\[ |p_k| \leq B_k(n), \quad (k \geq 2n), \]

which completes the proof. □

**Corollary 3.5.** Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \) and the sequence \( \{B_k(n)\}_{k=1}^{\infty} \) be defined by (3.4). If \( p(z) \in \Lambda_n(\lambda, \beta) \), then

\[ |p(z)| \leq 1 + \sum_{k=n}^{\infty} B_k(n)r^k, \quad (|z| = r < 1), \]

and

\[ |p'(z)| \leq \sum_{k=n}^{\infty} B_k(n)r^{k-1}, \quad (|z| = r < 1). \]

A covering theorem for the class \( \Lambda_n(\lambda, \beta) \) is provided by the following result.

**Theorem 3.6.** Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \). If \( p(z) \in \Lambda_n(\lambda, \beta) \), then the unit disk \( \mathbb{U} \) is mapped by onto a domain that contains the disk \( |w| < r_0 \), where

\[ r_0 = \max \left\{ n^{-\frac{1}{n+1}}, \left( \frac{n|\lambda(n-1) + 1|}{n(n+1)|\lambda(n-1) + 1| - 2\lambda\beta(2\beta + n - 2) - \beta + n\lambda} \right)^{\frac{1}{n}} \right\}. \]

**Proof.** Let \( w_0 \) be any complex number such that

\[ p(z) \neq w_0 \ (z \in \mathbb{U}). \]

Then \( w_0 \neq 0 \) and, by Theorem 1.2, the function

\[ \frac{w_0zp(z)}{w_0 - zp(z)} = z + \frac{1}{w_0} z^2 + \cdots + \frac{1}{w_0^{n-1}} z^n + \left( p_n + \frac{1}{w_0^n} \right) z^{n+1} + \cdots, \]

is univalent with

\[ \left| \frac{1}{w_0} \right| \leq 2, \quad \left| \frac{1}{w_0^{n-1}} \right| \leq n \text{ and } \left| p_n + \frac{1}{w_0^n} \right| \leq n + 1. \]

Therefore, according to Theorem 5.1, we find that

\[ |w_0| \geq n^{-\frac{1}{n-1}}, \quad (n \in \mathbb{N} - \{1\}), \]
and

\[ |w_0| \geq \left[ \frac{n \lambda (n-1) + 1}{n(n+1)[\lambda (n-1) + 1] - 2 \gamma_n} \right]^{\frac{1}{n}}, \quad (n \in \mathbb{N}), \]

and this completes the proof. \(\square\)

**Theorem 3.7.** Let \(0 \leq \lambda \leq \beta < \frac{1}{2}\) and \(p(z) \in P_n\). If

\[
\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1) - \gamma_n]|p_k| \leq -\gamma_n,
\]

then \(p(z) \in \Lambda_n(\lambda, \beta)\).

**Proof.** Suppose that

\[ q(z) = \frac{zp'}{p(z)} + \lambda z^2p''(z) . \]

We prove that \( |q(z)| < -\gamma_n \quad (z \in \mathbb{U}) \).

Obviously,

\[
\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1) - \gamma_n]|p_k| \geq -\gamma_n \sum_{k=n}^{\infty} |p_k|.
\]

Also, by using (4.1), we obtain that

\[
\sum_{k=n}^{\infty} |p_k| < 1,
\]
and so

\[ |q(z)| = \left| \frac{\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1)]p_k z^k}{1 + \sum_{k=n}^{\infty} p_k z^k} \right|. \]

\[ \leq \frac{\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1)] |p_k|}{1 - \sum_{k=n}^{\infty} |p_k|} \]

\[ = \frac{\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1) - \gamma_n] |p_k| - (\gamma_n) \sum_{k=n}^{\infty} |p_k|}{1 - \sum_{k=n}^{\infty} |p_k|} \]

\[ \leq \frac{-\gamma_n - (\gamma_n) \sum_{k=n}^{\infty} |p_k|}{1 - \sum_{k=n}^{\infty} |p_k|} \]

\[ = -\gamma_n. \]

Hence, \( \Re(-q(z)) \leq |q(z)| < -\gamma_n. \) Thus \( \Re(q(z)) > \gamma_n. \) That is, \( p(z) \in \Lambda_n(\lambda, \beta). \) \( \square \)

4. Properties of \( \Lambda_n^+(\lambda, \beta) \)

From Theorem 3.1 and the definition of \( \Lambda_n^+(\lambda, \beta), \) we have the following inclusion result.

**Theorem 4.1.** Let \( 0 \leq \lambda_2 < \lambda_1 < \frac{1}{2}, \) \( 0 \leq \beta_1 < \beta_2 < \frac{1}{2} \) and \( 1 \leq n_1 \leq n_2. \) Then

\[ \Lambda_{n_2}^+(\lambda_2, \beta_2) \subset \Lambda_{n_1}^+(\lambda_1, \beta_1). \]

In the following theorem we give a necessary and sufficient condition for an element belongs to \( \Lambda_n^+(\lambda, \beta). \)

**Theorem 4.2.** Let \( 0 \leq \lambda < \beta < \frac{1}{2} \) and \( p(z) \in \mathcal{P}_n. \) Then \( p \in \Lambda_n^+(\lambda, \beta) \) if and only if

\[ \sum_{k=n}^{\infty} [k(\lambda(k-1) + 1) - \gamma_n] p_k \leq -\gamma_n, \]
\[
\left( \gamma_n = \lambda \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2} \right).
\]

**Proof.** Let \( p(z) \in \mathcal{P}_n \). If \( p \) satisfies (4.1) then by Theorem 3.7, we conclude that \( p(z) \in \Lambda^+_n(\lambda, \beta) \).

Conversely, suppose that \( p \in \Lambda^+_n(\lambda, \beta) \). Then

\[
\Re \left( \frac{z p'(z)}{p(z)} + \lambda \frac{z^2 p''(z)}{p(z)} \right) = \Re \left( -\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1)]p_k z^k \right) > \gamma_n,
\]

for \( z = re^{i\theta} \), \( 0 \leq r < 1 \) and \( 0 \leq \theta < 2\pi \). Hence

\[
-\gamma_n > \sum_{k=n}^{\infty} [k(\lambda(k-1) + 1)]p_k r^k \geq \sum_{k=n}^{\infty} 1 - \sum_{k=n}^{\infty} p_k r^k \frac{1}{1 + \sum_{k=n}^{\infty} p_k r^k},
\]

and so

\[
\sum_{k=n}^{\infty} [k(\lambda(k-1) + 1) - \gamma_n]p_k r^k < -\gamma_n.
\]

By letting \( r \to 1 \) in the above inequality, we get \( p \in \Lambda^+_n(\lambda, \beta) \) and this completes the proof. \( \square \)

**Corollary 4.3.** If \( p(z) \in \Lambda^+_n(\lambda, \beta) \), then

\[
p_k \leq \frac{-\gamma_n}{k(\lambda(k-1) + 1) - \gamma_n}, \quad \left( k \geq n, \ 0 \leq \lambda < \beta < \frac{1}{2} \right).
\]

**Proof.** This follows from Theorem 4.2, since in this case, the condition (4.1) is satisfied. \( \square \)

**Theorem 4.4.** Let \( 0 \leq \lambda \leq \beta < 1 \), \( \gamma_n = \lambda \beta \left( \beta + \frac{n}{2} - 1 \right) + \beta - \frac{n\lambda}{2} \) and \( p(z) \in \Lambda^+_n(\lambda, \beta) \). Then

\[
1 + \frac{\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n} r^n \leq |p(z)| \leq 1 - \frac{\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n} r^n, \quad (|z| = r < 1)
\]

and

\[
\frac{n\gamma_n}{n(\lambda(n-1) + 1) n\gamma_n} r^{n-1} \leq |p'(z)| \leq -\frac{n\gamma_n}{n(\lambda(n-1) + 1) n\gamma_n} r^{n-1}, \quad (|z| = r < 1),
\]

\[
(4.2)
\]

and

\[
(4.3)
\]
Moreover, each of these inequalities is sharp, with the extremal function given by
\begin{equation}
  p_n(z) = 1 + \frac{\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n} z^n.
\end{equation}

**Proof.** By using Theorem 4.2, we get
\[
  \sum_{k=n}^{\infty} p_k \leq \frac{-\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n}.
\]
Therefore, the distortion inequalities in (4.2) follow from
\[
  1 - r^n \sum_{k=n}^{\infty} p_k \leq |p(z)| \leq 1 + r^n \sum_{k=n}^{\infty} p_k, \quad (|z| = r < 1).
\]
Also, since
\[
  \sum_{k=n}^{\infty} k p_k \leq \frac{-n\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n},
\]
then the distortion inequalities in (4.3) follow from
\[
  -r^n \sum_{k=n}^{\infty} k p_k \leq |p'(z)| \leq r^n \sum_{k=n}^{\infty} k p_k, \quad (|z| = r < 1).
\]
This completes the proof. \(\square\)

**Corollary 4.5.** Let \(0 \leq \lambda \leq \beta < \frac{1}{2}\) and \(p(z) \in \Lambda_n^{+}(\lambda, \beta)\). Then the unit disk \(U\) is mapped by \(p(z)\) onto a domain that contains the disk \(|w| < r_1\), where
\[
  r_1 = \frac{n(\lambda(n-1) + 1) - 2\gamma_n}{n(\lambda(n-1) + 1) - \gamma_n}.
\]
The result is sharp, with the extremal function \(p_n(z)\) given by (4.4).

**Corollary 4.6.** Let \(0 \leq \lambda \leq \beta < \frac{1}{2}\). Then \(\Lambda_1^{+}(\lambda, \beta) \subset \mathcal{K}\).

**Proof.** The proof follows from Theorem 4.2 and Lemma 2.3. \(\square\)

**Corollary 4.7.** Let \(0 \leq \lambda \leq \beta < \frac{1}{2}\) and \(\gamma_n = \lambda \beta \left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{n\lambda}{2}\). Suppose that
\begin{equation}
  p_{n-1}(z) = 1, \quad p_k(z) = 1 - \frac{-\gamma_n}{k(\lambda(k-1) + 1) - \gamma_n} z^k, \quad (k \geq n).
\end{equation}
Then \(p(z) \in \Lambda_n^{+}(\lambda, \beta)\) if and only if \(p(z)\) can be expressed in the following form
\[
  p(z) = \sum_{k=n-1}^{\infty} \mu_k p_k(z), \quad \left(\mu_k \geq 0, \ k \geq n \text{ and } \sum_{k=n-1}^{\infty} \mu_k = 1\right).
\]
Proof. Suppose that
\[ p(z) = \sum_{k=1}^{\infty} \mu_k p_k(z) = 1 - \sum_{k=n}^{\infty} \mu_k \frac{-\gamma_n}{k(\lambda(k-1) + 1 - \gamma_k)} z^k. \]

Then, by using Theorem 4.2, we can deduce that \( p(z) \in \Lambda^+_{\mu}(\lambda, \beta) \).

Conversely, suppose that \( p(z) \in \Lambda^+_{\mu}(\lambda, \beta) \). Then, from Corollary 4.3, we have
\[ p_k \leq \frac{-\gamma_n}{k(\lambda(k-1) + 1) - \gamma_n}, \quad (k \geq n). \]

Now, if we set
\[ \mu_k = \frac{k(\lambda(k-1) + 1) - \gamma_n}{-\gamma_n} p_k, \quad (k \geq n), \]
and
\[ \mu_{n-1} = 1 - \sum_{k=1}^{\infty} \mu_k, \]
then
\[ \sum_{k=n}^{\infty} \mu_k = 1, \quad \mu_k \geq 0, \quad (k \geq n), \]
and
\[ p(z) = \sum_{k=n}^{\infty} \mu_k p_k(z). \]

\[ \square \]

Corollary 4.8. Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \). Then the extreme points of the class \( \Lambda^+_{\mu}(\lambda, \beta) \) are the functions \( p_k(z) \) given by (4.5).

Finally, by Lemma 2.3 and Theorem 4.2, we conclude the following inclusion relation.

Theorem 4.9. Let \( 0 \leq \lambda \leq \beta < \frac{1}{2} \). Then \( \Lambda^+_{\mu}(\lambda, \beta) \subset \mathcal{K} \).

References


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