

Non-Archimedean fuzzy metric spaces and best proximity point theorems

Mohadeseh paknazar

ABSTRACT. In this paper, we introduce some new classes of proximal contraction mappings and establish best proximity point theorems for such kinds of mappings in a non-Archimedean fuzzy metric space. As consequences of these results, we deduce certain new best proximity and fixed point theorems in partially ordered non-Archimedean fuzzy metric spaces. Moreover, we present an example to illustrate the usability of the obtained results.

1. INTRODUCTION AND PRELIMINARIES

Best approximation theorems furnish an approximate solution to the fixed point equation $Tx = x$, when the non-self mapping T has no fixed point. In particular, a well-known best approximation theorem, due to Fan [12], asserts that, if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \rightarrow E$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Ty) : y \in K\}$, where d is a metric on E .

Best proximity point evolves as a generalization of the concept of best approximation. Precisely, although a best approximation theorem guarantees the existence of an approximate solution, a best proximity point theorem is contemplated for solving the problem to find an approximate solution which is optimal. For any nonempty closed subsets A and B of E , when a non-self mapping $T : A \rightarrow B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. Now, in light of the fact that $d(x, Tx)$ is at least $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$, best proximity point theorems guarantee the existence of an element x^* such that $d(x^*, Tx^*) = d(A, B)$. This element

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Fuzzy metric space, Best proximity point, Proximal contraction.

Received: 21 October 2016, Accepted: 5 December 2016.

is called a best proximity point of T . Moreover, if the mapping under consideration is a self-mapping, we note that this best proximity theorem reduces to a fixed point. For some results in this direction, we refer to [3, 4, 9, 20, 22, 24, 25, 33, 34, 37].

On the other hand, the concept of fuzzy metric spaces was introduced in different ways by the authors in [8, 21]. Fixed point theory in such spaces has been intensively studied (see, for example, [6, 10, 11, 16, 36]), but the best proximity point theory is not considered yet. Here, we recall the notion of fuzzy metric space, which was introduced by Kramosil and Michalek [21] and later modified by George and Veeramani [13, 14]. Recently, Mihet [23] enlarged the class of fuzzy contractive mappings of Gregori and Sapena [16] and proved a fuzzy Banach contraction result in a complete non-Archimedean fuzzy metric space ([23], Theorem 3.16) (see also [15, 35]).

For the sake of completeness, we briefly recall some basic concepts used in the following.

Definition 1.1 (Schweizer and Sklar [31]). A binary operation $\star : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if it satisfies the following assertions:

- (T1) \star is commutative and associative;
- (T2) \star is continuous;
- (T3) $a \star 1 = a$ for all $a \in [0, 1]$;
- (T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

Definition 1.2 (George and Veeramani [13]). A fuzzy metric space is an ordered triple (X, M, \star) such that X is a nonempty set, \star is a continuous t -norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

- (F1) $M(x, y, t) > 0$;
- (F2) $M(x, y, t) = 1$ if and only if $x = y$;
- (F3) $M(x, y, t) = M(y, x, t)$;
- (F4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$;
- (F5) $M(x, y, \cdot) : (0, +\infty) \rightarrow (0, 1]$ is left continuous.

If we replace (F4) by

- (F6) $M(x, y, t) \star M(y, z, s) \leq M(x, z, \max\{t, s\})$,

then the triple (X, M, \star) is called a non-Archimedean fuzzy metric space.

Note that, since (F6) implies (F4), each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 1.3 (George and Veeramani [13]). Let (X, M, \star) be a fuzzy metric space (or a non-Archimedean fuzzy metric space). Then

- (1) a sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if $\lim_{n \rightarrow +\infty} M(x_n, x, t) = 1$ for all $t > 0$;
- (2) a sequence $\{x_n\}$ in X is called a Cauchy sequence if, for all $\epsilon \in (0, 1)$ and $t > 0$, there exists a positive integer n_0 such that

$$M(x_n, x_m, t) > 1 - \epsilon,$$

for all $m, n \geq n_0$.

On the other hand, Samet et al. [30] defined the notion of α -admissible mappings as follows.

Definition 1.4. Let T be a self-mapping on X and $\alpha : X \times X \rightarrow [0, +\infty)$ be a function. We say that T is an α -admissible mapping if, for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Salimi et al. [28] generalized the notion of α -admissible mappings by the following ways.

Definition 1.5 ([28]). Let T be a self-mapping in X and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is an α -admissible mapping with respect to η if, for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \quad \Rightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that, if we take $\eta(x, y) = 1$, then this definition reduces to Definition 1.4. Also, if we take $\alpha(x, y) = 1$, then we say that T is an η -subadmissible mapping.

Definition 1.6 ([20]). A non-self mapping T is said to be α -proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{cases} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1,$$

for all $x_1, x_2, u_1, u_2 \in A$, where $\alpha : A \times A \rightarrow [0, \infty)$.

Clearly, if $A = B$, then the α -proximal admissibility of T implies that T is α -admissible.

Recently Hussain et al.[19] generalized the notion of α -proximal admissibility as follows:

Definition 1.7. Let $T : A \rightarrow B$ be a mapping and $\alpha, \eta : A \times A \rightarrow [0, \infty)$ be functions. Then T is said to be α -proximal admissible with respect to η if

$$\begin{cases} \alpha(x_1, x_2) \geq \eta(x_1, x_2), \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{cases} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2),$$

for all $x_1, x_2, u_1, u_2 \in A$.

Note that, if we take $\eta(x, y) = 1$ for all $x, y \in A$, then this definition reduces to Definition 1.6. In case, $\alpha(x, y) = 1$ for all $x, y \in A$ and then we say that T is an η -proximal subadmissible mapping.

Recently, Vetro and Salimi [37] considered the problem of finding a best proximity point which achieves the minimum distance between two nonempty sets in a non-Archimedean fuzzy metric space. First they introduced the following notions in fuzzy metric spaces and then proved their main results.

Let A and B be two nonempty subsets of a fuzzy metric space (X, M, \star) . We denote by $A_0(t)$ and $B_0(t)$ the following sets:

$$(1.1) \quad \begin{aligned} A_0(t) &= \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\}, \\ B_0(t) &= \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}, \end{aligned}$$

where $M(A, B, t) = \sup\{M(x, y, t) : x \in A, y \in B\}$.

2. BEST PROXIMITY POINT RESULTS IN FUZZY METRIC SPACES

First, we introduce the following notion.

Definition 2.1. Let A and B be two nonempty subsets of a fuzzy metric space (X, M, \star) . Let $T : A \rightarrow B$ and $\alpha^* : A \times A \times (0, \infty) \rightarrow [0, \infty)$. We say that T is triangular α^* -proximal admissible if the following conditions are satisfied:

[T1] T is α^* -proximal admissible, that is, for all $x_1, x_2, u_1, u_2 \in A$ and $t > 0$,

$$\begin{cases} \alpha^*(x_1, x_2, t) \geq 1, \\ M(u_1, Tx_1, t) = M(A, B, t), \\ M(u_2, Tx_2, t) = M(A, B, t), \end{cases} \quad \Rightarrow \quad \alpha^*(u_1, u_2, t) \geq 1;$$

[T2] for all $x, y, z \in A$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1, \quad \alpha^*(y, z, t) \geq 1 \quad \Rightarrow \quad \alpha^*(x, z, t) \geq 1.$$

Example 2.2. Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$. Consider $M(x, y, t) = \left(\frac{t}{t+1}\right)^{d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Moreover, consider $A = (-\infty, -1]$, $B = [1, +\infty)$ and define $T : A \rightarrow B$

by

$$Tx = \begin{cases} 2, & \text{if } x \in (-\infty, -14); \\ 6, & \text{if } x \in [-14, -12); \\ 5, & \text{if } x \in [-12, -10); \\ 7, & \text{if } x \in [-10, -8); \\ 9, & \text{if } x \in [-8, -6); \\ 3, & \text{if } x \in [-6, -4); \\ 8, & \text{if } x \in [-4, -2); \\ 1, & \text{if } x \in [-2, -1], \end{cases}$$

Also, define $\alpha^* : X \times X \times (0, \infty) \rightarrow [0, +\infty)$ by

$$\alpha^*(x, y, t) = \begin{cases} t + 1, & \text{if } x \in x, y \in [-2, -1]; \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

Clearly, $M(A, B, t) = \sup\{M(x, y, t) \mid x \in A, y \in B\} = \left(\frac{t}{t+1}\right)^2$. Suppose

$$\begin{cases} \alpha^*(x, y, t) \geq 1, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

then

$$\begin{cases} x, y \in [-2, -1], \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Hence, $u = v = -1$, that is, $\alpha^*(u, v, t) \geq 1$. Also, assume, $\alpha^*(x, y, t) \geq 1$, $\alpha^*(y, z, t) \geq 1$. Then, $x, y, z \in [-2, -1]$. So, $\alpha^*(x, z, t) \geq 1$. Therefore T is a triangular α -proximal admissible mapping.

Forward, Φ denotes the set of all functions $\varphi : [0, 1] \rightarrow [0, 1]$ with the following properties:

- (φ 1) φ is decreasing and continuous;
- (φ 2) $\varphi(s) = 0$ if and only if $s = 1$.

Definition 2.3. Let A and B be nonempty subsets of a fuzzy metric space (X, M, \star) . Let $T : A \rightarrow B$ be a non-self mapping and $\alpha^* : A \times A \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say T is a modified α^* - φ - ω -proximal contractive mapping if, for all $x, y, u, v \in A$ and $t > 0$,

$$(2.1) \quad \left. \begin{array}{l} \alpha^*(x, y, t) \geq 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \Rightarrow \varphi(M(u, v, t)) \leq \omega(t)\varphi(M(x, y, t)),$$

where $\varphi \in \Phi$ and $\omega : (0, +\infty) \rightarrow (0, 1)$ is a function.

Definition 2.4. Let A and B be nonempty subsets of a fuzzy metric space (X, M, \star) . Let $T : A \rightarrow B$ be a non-self mapping and $\alpha^* : A \times A \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say T is a modified α^* - ϕ -proximal contractive mapping if, for all $x, y, u, v \in A$ and $t > 0$,

$$(2.2) \quad \left. \begin{array}{l} \alpha^*(x, y, t) \geq 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \Rightarrow M(u, v, t) \geq M(x, y, t) + \phi(M(x, y, t)),$$

where $\phi : [0, 1] \rightarrow [0, 1]$ is continuous and $\phi(s) > 0$ for all $s \in (0, 1)$.

Definition 2.5. Let A and B be nonempty subsets of a fuzzy metric space (X, M, \star) . Let $T : A \rightarrow B$ be a non-self mapping and $\alpha^* : A \times A \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say T is a modified α^* - β -proximal contractive mapping if there exists a function $\beta : [0, 1] \rightarrow [1, +\infty)$ such that, for any sequence $\{s_n\} \subseteq [0, 1]$ of positive reals, $\beta(s_n) \rightarrow 1$ implies $s_n \rightarrow 1$ and, for all $x, y, u, v \in A$ and $t > 0$,

$$(2.3) \quad \left. \begin{array}{l} \alpha^*(x, y, t) \geq 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \Rightarrow M(u, v, t) \geq \beta(M(x, y, t))M(x, y, t).$$

Now, we are ready to prove our first main result.

Theorem 2.6. *Let A and B be nonempty closed subsets of a complete non-Archimedean fuzzy metric space (X, M, \star) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is a triangular α^* -proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is a modified α^* - φ - ω -proximal contractive mapping;
- (iii) If $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) There exist elements x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1$$

for all $t > 0$;

- (v) If $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$, that is, T has a best proximity point $x^* \in A$.

Moreover, if $M(x, Tx, t) = M(A, B, t)$ and $M(y, Ty, t) = M(A, B, t)$ imply that $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique best proximity point.

Proof. By (iv), there exist elements x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1,$$

for all $t > 0$. On the other hand, since $T(A_0(t)) \subseteq B_0(t)$, there exists $x_2 \in A_0(t)$ such that

$$M(x_2, Tx_1, t) = M(A, B, t).$$

Now, since T is an α^* -proximal admissible mapping, then $\alpha^*(x_1, x_2, t) \geq 1$, that is,

$$M(x_2, Tx_1, t) = M(A, B, t), \quad \alpha^*(x_1, x_2, t) \geq 1.$$

Again, since $T(A_0(t)) \subseteq B_0(t)$, there exists $x_3 \in A_0(t)$ such that

$$M(x_3, Tx_2, t) = M(A, B, t).$$

Thus we have

$$\begin{aligned} M(x_2, Tx_1, t) &= M(A, B, t), \\ M(x_3, Tx_2, t) &= M(A, B, t), \\ \alpha^*(x_1, x_2, t) &\geq 1. \end{aligned}$$

Again, since T is an α^* -proximal admissible mapping, then $\alpha^*(x_2, x_3, t) \geq 1$. Hence it follows that

$$M(x_3, Tx_2, t) = M(A, B, t), \quad \alpha^*(x_2, x_3, t) \geq 1.$$

Continuing this process, we get

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \alpha^*(x_n, x_{n+1}, t) \geq 1,$$

for any $n \geq 0$ and $t > 0$. Now, applying [T2] on $\alpha^*(x_n, x_{n+1}, t) \geq 1$ and $\alpha^*(x_{n+1}, x_{n+2}, t) \geq 1$, we get

$$\alpha^*(x_n, x_{n+2}, t) \geq 1.$$

Again, applying [T2] on $\alpha^*(x_n, x_{n+2}, t) \geq 1$ and $\alpha^*(x_{n+2}, x_{n+3}, t) \geq 1$, we get

$$\alpha^*(x_n, x_{n+3}, t) \geq 1.$$

Continuing this process, we get

$$(2.4) \quad M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \alpha^*(x_n, x_m, t) \geq 1,$$

for all $m, n \geq 1$ with $n < m$. Also, define $\tau_n(t) = M(x_n, x_{n+1}, t)$ for all $n \geq 0$ and $t > 0$. From (2.1) and (2.4), it follows that

$$(2.5) \quad \begin{aligned} \varphi(\tau_n(t)) &= \varphi(M(x_n, x_{n+1}, t)) \\ &\leq \omega(t)\varphi(M(x_{n-1}, x_n, t)) \\ &< \varphi(\tau_{n-1}(t)). \end{aligned}$$

Since φ is decreasing, $\tau_{n-1}(t) < \tau_n(t)$, that is, the sequence $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Take $\lim_{n \rightarrow +\infty} \tau_n(t) = \tau(t)$.

Now, we show that $\tau(t) = 1$ for all $t > 0$. Suppose that $0 < \tau(t_0) < 1$ for some $t_0 > 0$. Since $\tau_n(t_0) \leq \tau(t_0)$ and φ is continuous, by taking limit as $n \rightarrow +\infty$ in (2.5) with $t = t_0$, we obtain

$$\varphi(\tau(t_0)) \leq \omega(t_0)\varphi(\tau(t_0)) < \varphi(\tau(t_0)),$$

which is a contradiction and thus $\tau(t) = 1$ for all $t > 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Assuming that this is not true, then there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that, for all $k \geq 1$, there are $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon.$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality, that is, equivalently,

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \epsilon,$$

and so, for all k ,

$$(2.6) \quad \begin{aligned} 1 - \epsilon &\geq M(x_{m(k)}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) \star M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &> \tau_{m(k)}(t_0) \star (1 - \epsilon). \end{aligned}$$

Passing to limit as $n \rightarrow +\infty$ in (2.6), we deduce

$$\lim_{k \rightarrow +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon.$$

From

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) &\geq \\ M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)}, x_{n(k)}, t_0) \star M(x_{n(k)}, x_{n(k)+1}, t_0), \end{aligned}$$

and

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, t_0) &\geq \\ M(x_{m(k)}, x_{m(k)+1}, t_0) \star M(x_{m(k)+1}, x_{n(k)+1}, t_0) \star M(x_{n(k)+1}, x_{n(k)}, t_0), \end{aligned}$$

it follows that

$$\lim_{n \rightarrow +\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

From (2.4), we know that

$$(2.7) \quad \begin{cases} \alpha^*(x_{n(k)}, x_{m(k)}, t) \geq 1, \\ M(x_{m(k)+1}, Tx_{m(k)}, t_0) = M(A, B, t_0), \\ M(x_{n(k)+1}, Tx_{n(k)}, t_0) = M(A, B, t_0). \end{cases}$$

Hence, by (2.1) and (2.7), we have

$$\varphi(M(x_{m(k)+1}, x_{n(k)+1}, t_0)) \leq \omega(t_0)\varphi(M(x_{m(k)}, x_{n(k)}, t_0)).$$

Using the continuity of the function φ , passing to limit as $k \rightarrow +\infty$ in the above inequality, we get

$$\varphi(1 - \epsilon) \leq \omega(t_0)\varphi(1 - \epsilon).$$

Now, if $\varphi(1 - \epsilon) = 0$, then, by ($\varphi 2$), we have $\epsilon = 0$, which is a contradiction. Otherwise, we assume that $\varphi(1 - \epsilon) > 0$, which implies $1 \leq \omega(t_0)$. This is a contradiction since $0 < \omega(t_0) < 1$. Thus it follows that $\{x_n\}$ is a Cauchy sequence. The completeness of (X, M, \star) ensures that the sequence $\{x_n\}$ converges to some $x^* \in X$, that is, $\lim_{n \rightarrow +\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Moreover, we have

$$\begin{aligned} M(A, B, t) &= M(x_{n+1}, Tx_n, t) \\ &\geq M(x_{n+1}, x^*, t) \star M(x^*, Tx_n, t) \\ &\geq M(x_{n+1}, x^*, t) \star M(x^*, x_{n+1}, t) \star M(x_{n+1}, Tx_n, t) \\ &= M(x_{n+1}, x^*, t) \star M(x^*, x_{n+1}, t) \star M(A, B, t), \end{aligned}$$

which implies

$$\begin{aligned} M(A, B, t) &\geq M(x_{n+1}, x^*, t) \star M(x^*, Tx_n, t) \\ &\geq M(x_{n+1}, x^*, t) \star M(x^*, x_{n+1}, t) \star M(A, B, t). \end{aligned}$$

Passing to limit as $n \rightarrow +\infty$ in the above inequalities, we get

$$\begin{aligned} M(A, B, t) &\geq 1 \star \lim_{n \rightarrow +\infty} M(x^*, Tx_n, t) \\ &\geq 1 \star 1 \star M(A, B, t), \end{aligned}$$

that is,

$$\lim_{n \rightarrow +\infty} M(x^*, Tx_n, t) = M(A, B, t),$$

and so, by the condition (iii), $x^* \in A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, there exists $z \in A_0(t)$ such that $M(z, Tx^*, t) = M(A, B, t)$. Combining this with (2.4) and (2.1), we obtain

$$\varphi(M(z, x_{n+1}, t)) \leq \omega(t)\varphi(M(x^*, x_n, t)) < \varphi(M(x^*, x_n, t)).$$

Passing to limit as $n \rightarrow +\infty$ in the above inequalities, we get

$$\lim_{n \rightarrow +\infty} \psi(M(z, x_{n+1}, t)) = 0,$$

which implies $\lim_{n \rightarrow +\infty} M(z, x_{n+1}, t) = 1$ for all $t > 0$, since φ is continuous. By the uniqueness of the limit, we conclude that $z = x^*$, that is, $M(x^*, Tx^*, t) = M(z, Tx^*, t) = M(A, B, t)$ for all $t > 0$. Thus x^* is a best proximity point of T .

Finally, we show that x^* is the unique best proximity point of T . Assume that $0 < M(x^*, w, t) < 1$ for all $t > 0$ and $w \neq x^*$ is another best proximity point of T , that is, $M(x^*, Tx^*, t) = M(A, B, t)$ and $M(w, Tw, t) = M(A, B, t)$. Now, if the condition (vi) holds, then it follows from (2.1) that

$$\varphi(M(x^*, w, t)) \leq \omega(t)\varphi(M(x^*, w, t)) < \varphi(M(x^*, w, t)),$$

which is a contradiction and hence $M(x^*, w, t) = 1$ for all $t > 0$, that is, $w = x^*$. This completes the proof. \square

Corollary 2.7. *Let A and B be nonempty closed subsets of a complete non-Archimedean fuzzy metric space (X, M, \star) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is an α^* -proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is a modified α^* - φ - ω -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1,$$

for all $t > 0$;

[T3] for all $x \in X$ and $t > 0$, $\alpha^*(x, x, t) \geq 1$ and, if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$, that is, T has a best proximity point $x^* \in A$.

Moreover, if $M(x, Tx, t) = M(A, B, t)$ and $M(y, Ty, t) = M(A, B, t)$ imply $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique best proximity point.

Proof. The condition [T3] implies property [T2] in the definition of the triangular α^* -admissible mapping. Indeed, if $\alpha^*(x, y, t) \geq 1$ and $\alpha^*(y, z, t) \geq 1$, then, applying [T3] to the sequence (x_n) defined by

$$x_1 := x, \quad x_2 := y, \quad x_n := z,$$

for all $n \geq 3$, we get $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $t > 0$. Thus $\alpha^*(x, z, t) \geq 1$ for all $t > 0$. Thus the required result follows from Theorem 2.6. \square

Corollary 2.8. *Let A and B be nonempty closed subsets of a complete non-Archimedean fuzzy metric space (X, M, \star) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is a triangular α^* -proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is a modified α^* - φ - ω -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1,$$

for all $t > 0$;

[T4] if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ with $\alpha^*(x_{k_n}, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$, that is, T has a best proximity point $x^* \in A$.

Moreover, if $M(x, Tx, t) = M(A, B, t)$ and $M(y, Ty, t) = M(A, B, t)$ implies $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique best proximity point.

Proof. Obviously, the conditions [T2] and [T4] imply the condition (v) in Theorem 2.6 and so the required result follows from Theorem 2.6. \square

Theorem 2.9. *Let A and B be nonempty closed subsets of a complete non-Archimedean fuzzy metric space (X, M, \star) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is a triangular α^* -proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is a modified α^* - ϕ -proximal contractive map;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

(iv) there exist x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1,$$

for all $t > 0$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Moreover, if $M(x, Tx, t) = M(A, B, t)$ and $M(y, Ty, t) = M(A, B, t)$ imply $\alpha^*(x, y, t) \geq 1$, then T has a unique best proximity point.

Proof. Following the same lines in the proof of Theorem 2.6, we can construct a sequence $\{x_n\}$ in $A_0(t)$ such that

$$(2.8) \quad M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \alpha^*(x_n, x_m, t) \geq 1,$$

for all $m, n \geq 1$ with $n < m$. In view of (2.8), it follows from (2.2) with $u = y = x_n$, $v = x_{n+1}$ and $x = x_{n-1}$ that

$$(2.9) \quad M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t) + \phi(M(x_{n-1}, x_n, t)),$$

which implies

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t),$$

and hence $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$. Then there exists $l(t) \in (0, 1]$ such that

$$\lim_{n \rightarrow +\infty} M(x_n, x_{n+1}, t) = l(t),$$

for all $t > 0$.

Now, we prove that $l(t) = 1$ for all $t > 0$. Suppose that there is $t_0 > 0$ such that $0 < l(t_0) < 1$. Passing to limit as $n \rightarrow +\infty$ in (2.9), we have

$$l(t_0) \geq l(t_0) + \phi(l(t_0)),$$

and, consequently, $\phi(l(t_0)) = 0$, which is a contradiction. This implies that $l(t) = 1$ for all $t > 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Again, assuming that this is not true and then proceeding as in the proof of Theorem 2.6, there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that, for all $k \geq 1$, there are $n(k), m(k) \geq 1$ with $m(k) > n(k) \geq k$ such that

$$\lim_{n \rightarrow +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon,$$

and

$$\lim_{n \rightarrow +\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

Applying (2.2) and (2.7), we obtain easily

$$M(x_{m(k)+1}, x_{n(k)+1}, t_0) \geq M(x_{m(k)}, x_{n(k)}, t_0) + \phi(M(x_{m(k)}, x_{n(k)}, t_0)).$$

Using the continuity of the function ϕ and passing to limit as $k \rightarrow +\infty$ in the above inequality, we get

$$1 - \epsilon \geq 1 - \epsilon + \phi(1 - \epsilon),$$

and so $\phi(1 - \epsilon) = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, M, \star) is a complete non-Archimedean fuzzy metric space, the sequence $\{x_n\}$ converges to a point $x^* \in X$, that is, $\lim_{n \rightarrow +\infty} M(x_n, x^*, t) = 1$ for all $t > 0$.

Again, proceeding as in the proof of Theorem 2.6, there exists $z \in A_0(t)$ such that $M(z, Tx^*, t) = M(A, B, t)$. Consequently, it follows from (v) and (2.2) with $u = x_{n+1}$, $v = z$, $x = x_n$ and $y = x^*$ that

$$M(x_{n+1}, z, t) \geq M(x_n, x^*, t) + \phi(M(x_n, x^*, t)).$$

Passing to limit as $n \rightarrow +\infty$ in the above inequality, we have

$$M(x^*, z, t) \geq 1 + \phi(1) \geq 1,$$

and hence $M(x^*, z, t) = 1$ for all $t > 0$, that is, $x^* = z$ and $M(x^*, Tx^*, t) = M(A, B, t)$.

Finally, we show that x^* is the unique best proximity point of T . Assume that $0 < M(x^*, w, t) < 1$ for all $t > 0$ and $w \neq x^*$ is another best proximity point of T , that is, $M(x^*, Tx^*, t) = M(A, B, t)$ and $M(w, Tw, t) = M(A, B, t)$. If (vi) holds, then it follows from (2.2) that

$$M(x^*, w, t) \geq M(x^*, w, t) + \phi(M(x^*, w, t)),$$

which implies $\phi(M(x^*, w, t)) = 0$. This is a contradiction. Therefore, $M(x^*, w, t) = 1$ for all $t > 0$ and so $x^* = w$. This completes the proof. \square

Theorem 2.10. *Let A and B be nonempty closed subsets of a complete non-Archimedean fuzzy metric space (X, M, \star) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is a triangular α^* -proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is a modified α^* - β -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

(iv) there exist x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1,$$

for all $t > 0$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Moreover, if $M(x, Tx, t) = M(A, B, t)$ and $M(y, Ty, t) = M(A, B, t)$ imply $\alpha^*(x, y, t) \geq 1$, then T has a unique best proximity point.

Proof. Following the same lines in the proof of Theorem 2.6, we can construct a sequence $\{x_n\}$ in $A_0(t)$ such that

$$(2.10) \quad M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \alpha^*(x_n, x_m, t) \geq 1,$$

for all $m, n \geq 1$ with $n < m$. In view of (2.10), it follows from (2.3) with $u = y = x_n$, $v = x_{n+1}$ and $x = x_{n-1}$ that

$$(2.11) \quad \begin{aligned} M(x_n, x_{n+1}, t) &\geq \beta(M(x_{n-1}, x_n, t))M(x_{n-1}, x_n, t) \\ &\geq M(x_{n-1}, x_n, t), \end{aligned}$$

and hence $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$. Thus there exists $l(t) \in (0, 1]$ such that $\lim_{n \rightarrow +\infty} M(x_n, x_{n+1}, t) = l(t)$ for all $t > 0$.

Now, we prove that $l(t) = 1$ for all $t > 0$. By (2.11), we deduce

$$\frac{M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)} \geq \beta(M(x_{n-1}, x_n, t)) \geq 1,$$

which implies $\lim_{n \rightarrow +\infty} \beta(M(x_{n-1}, x_n, t)) = 1$. In view of the property of the function β , we conclude that

$$\lim_{n \rightarrow +\infty} M(x_n, x_{n+1}, t) = 1.$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Proceeding as in the proof of Theorem 2.6, there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that, for all $k \geq 1$ there are $n(k), m(k) \geq 1$ with $m(k) > n(k) \geq k$ such that

$$\lim_{n \rightarrow +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon,$$

and

$$\lim_{n \rightarrow +\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

Applying opportunely (2.3) and (2.7), we have easily

$$M(x_{m(k)+1}, x_{n(k)+1}, t_0) \geq \beta(M(x_{m(k)}, x_{n(k)}, t_0))M(x_{m(k)}, x_{n(k)}, t_0),$$

which implies

$$\frac{M(x_{m(k)+1}, x_{n(k)+1}, t_0)}{M(x_{m(k)}, x_{m(k)}, t_0)} \geq \beta(M(x_{m(k)}, x_{m(k)}, t_0)) \geq 1.$$

Passing to the limit as $k \rightarrow +\infty$ in the above inequality, we get

$$\lim_{k \rightarrow +\infty} \beta(M(x_{m(k)}, x_{n(k)}, t_0)) = 1.$$

It follows that

$$1 - \epsilon = \lim_{k \rightarrow +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1,$$

and so $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, M, \star) is a complete non-Archimedean fuzzy metric space, the sequence $\{x_n\}$ converges to a point $x^* \in X$, that is, $\lim_{n \rightarrow +\infty} M(x_n, x^*, t) = 1$ for all $t > 0$.

Again, proceeding as in the proof of Theorem 2.6, there exists $z \in A_0(t)$ such that $M(z, Tx^*, t) = M(A, B, t)$. Consequently, it follows from (v) and (2.3) with $u = x_{n+1}$, $v = z$, $x = x_n$ and $y = x^*$ that

$$M(x_{n+1}, z, t) \geq \beta(M(x_n, x^*, t))M(x_n, x^*, t) \geq M(x_n, x^*, t).$$

Passing to the limit as $n \rightarrow +\infty$ in the above inequality, we have $M(x^*, z, t) = 1$ for all $t > 0$, that is, $x^* = z$ and $M(x^*, Tx^*, t) = M(A, B, t)$.

Finally, we show that x^* is the unique best proximity point of T . Assume that $0 < M(x^*, w, t) < 1$ for all $t > 0$ and $w \neq x^*$ is another best proximity point of T , that is, $M(x^*, Tx^*, t) = M(A, B, t)$ and $M(w, Tw, t) = M(A, B, t)$. Then, from (vi) and (2.3), we have

$$1 = \frac{M(x^*, w, t_0)}{M(x^*, w, t_0)} \geq \beta(M(x^*, w, t_0)) \geq 1,$$

which implies $M(x^*, w, t_0) = 1$. This is a contradiction. Therefore, $M(x^*, w, t_0) = 1$ for all $t > 0$ and so $x^* = w$. This completes the proof. \square

Remark 2.11. We may obtain some results similar to Corollaries 2.7 and 2.8 as an immediate consequence of Theorems 2.9 and 2.10.

Example 2.12. Let X be \mathbb{R}^2 endowed with the non-Archimedean fuzzy metric $M : X \times X \times (0, +\infty) \rightarrow (0, 1]$ given by

$$M(x, y, t) = \left(\frac{t}{t+1} \right)^{d(x,y)},$$

where $d : X \times X \rightarrow [0, +\infty)$ is the metric defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. Thus (X, M, \star) is complete with $a \star b \leq ab$ for all $a, b \in [0, 1]$. Define the sets as follows:

$$A = \{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}, \quad B = \{(1, x) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$

so that $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$ for all $t > 0$. Clearly, A and B are nonempty closed subsets of X . Define a mapping $T : A \rightarrow B$ by

$$T(x_1, x_2) = \begin{cases} (1, 2\pi), & \text{if } (x_1, x_2) \in A \setminus V; \\ (1, \frac{1}{2n}), & \text{if } (x_1, x_2) = (0, \frac{1}{n}) \text{ for all } n \geq 1; \\ (1, 0), & \text{if } (x_1, x_2) = (0, 0), \end{cases}$$

where

$$V = \left\{ \left(0, \frac{1}{n}\right) : n \geq 1 \right\} \cup \{(0, 0)\}.$$

Notice that $A_0(t) = A$, $B_0(t) = B$, $T(A_0(t)) \subseteq B_0(t)$ and the hypothesis (iii) of Theorem 2.10 holds true. Also, define a function $\alpha^* : A \times A \times (0, \infty) \rightarrow [0, \infty)$ by

$$\alpha^*((0, x), (0, y), t) = \begin{cases} 2, & \text{if } (0, x), (0, y) \in V, \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Let $\alpha^*((0, x), (0, y), t) \geq 1$ and $\alpha^*((0, y), (0, z), t) \geq 1$. Then

$$(0, x), (0, y), (0, z) \in V,$$

that is, $\alpha^*((0, x), (0, z), t) \geq 1$. Also, assume that

$$\begin{cases} \alpha^*(x, y, t) \geq 1, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

and so

$$\begin{cases} (x, y) \in V, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Then we have

$$(u, x), (v, y) \in \left\{ ((0, 0), (0, 0)), \left(\left(0, \frac{1}{2n}\right), \left(0, \frac{1}{n}\right) \right) : n \geq 1 \right\}.$$

First, we conclude that $\alpha^*(u, v, t) \geq 1$, that is, T is a triangular α^* -proximal admissible mapping.

Next, we distinguish the following cases:

- (i) If $(u, x) = ((0, \frac{1}{2n}), (0, \frac{1}{n}))$ and $(v, y) = ((0, \frac{1}{2m}), (0, \frac{1}{m}))$ for all $n, m \geq 1$, then we have

$$\begin{aligned}
 M(u, v, t) &= \left(\frac{t}{t+1} \right)^{d(u,v)} \\
 &= \left(\frac{t}{t+1} \right)^{\left| \frac{1}{2n} - \frac{1}{2m} \right|} \\
 &\geq \beta \left(\left(\frac{t}{t+1} \right)^{\left| \frac{1}{n} - \frac{1}{m} \right|} \right) \left(\frac{t}{t+1} \right)^{\left| \frac{1}{n} - \frac{1}{m} \right|} \\
 &= \beta \left(\left(\frac{t}{t+1} \right)^{d(x,y)} \right) \left(\frac{t}{t+1} \right)^{d(x,y)} \\
 &= \beta(M(x, y, t))M(x, y, t).
 \end{aligned}$$

- (ii) If $(u, x) = ((0, 0), (0, 0))$ and $(v, y) = ((0, \frac{1}{2m}), (0, \frac{1}{m}))$ for all $m \geq 1$, then we have

$$\begin{aligned}
 M(u, v, t) &= \left(\frac{t}{t+1} \right)^{d(u,v)} \\
 &= \left(\frac{t}{t+1} \right)^{\frac{1}{2m}} \\
 &\geq \beta \left(\left(\frac{t}{t+1} \right)^{\frac{1}{m}} \right) \left(\frac{t}{t+1} \right)^{\frac{1}{m}} \\
 &= \beta \left(\left(\frac{t}{t+1} \right)^{d(x,y)} \right) \left(\frac{t}{t+1} \right)^{d(x,y)} \\
 &= \beta(M(x, y, t))M(x, y, t).
 \end{aligned}$$

- (iii) If $(u, x) = (v, y) = ((0, 0), (0, 0))$, then we have

$$\begin{aligned}
 M(u, v, t) &= \left(\frac{t}{t+1} \right)^{d(u,v)} \\
 &= 1 \\
 &\geq \beta(1) \cdot 1 \\
 &= \beta \left(\left(\frac{t}{t+1} \right)^{d(x,y)} \right) \left(\frac{t}{t+1} \right)^{d(x,y)} \\
 &= \beta(M(x, y, t))M(x, y, t).
 \end{aligned}$$

The other possible cases can be easily covered using the symmetry of M and so we omit the details. We conclude that all the hypotheses of Theorem 2.10 are satisfied with $\beta : [0, 1] \rightarrow [1, +\infty)$ given by $\beta(s) = 1$ for all $s \in [0, 1]$ and so there exists a unique $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$. Here, $x^* = (0, 0)$ is the unique best proximity point of T .

Note that, if we choose $x = (0, \pi)$, $y = (0, 0)$, $u = (0, 2\pi)$ and $v = (0, 0)$, then we have

$$\begin{cases} M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

On the other hand, we have

$$\begin{aligned} M(u, v, t) &= \left(\frac{t}{t+1}\right)^{d(u,v)} \\ &= \left(\frac{t}{t+1}\right)^{2\pi} \\ &< \left(\frac{t}{t+1}\right)^{\pi} \\ &= \left(\frac{t}{t+1}\right)^{d(x,y)} \\ &= M(x, y, t) \\ &= \beta(M(x, y, t))M(x, y, t). \end{aligned}$$

That is, Theorem 4 of [37] can not be applied for this example.

3. APPLICATIONS TO FIXED POINT THEORY

First, we define the following notions.

Definition 3.1. Let $T : X \rightarrow X$ be a mapping and $\alpha^* : X \times X \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say that T is triangular α^* -admissible if

[T1] for all $x, y \in X$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1 \quad \Rightarrow \quad \alpha^*(Tx, Ty, t) \geq 1,$$

and

[T2] for all $x, y \in X$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1, \quad \alpha^*(y, z, t) \geq 1 \quad \Rightarrow \quad \alpha^*(x, z, t) \geq 1.$$

Definition 3.2. Let (X, M, \star) be a fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping and $\alpha^* : X \times X \times (0, \infty) \rightarrow [0, \infty)$ be a function. We

say T is a modified α^* - φ - ω - contractive mapping if, for all $x, y \in X$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1 \quad \Rightarrow \quad \varphi(M(Tx, Ty, t)) \leq \omega(t)\varphi(M(x, y, t)),$$

where $\varphi \in \Phi$ and $\omega : (0, +\infty) \rightarrow (0, 1)$ is a function.

Definition 3.3. Let (X, M, \star) be a fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping and $\alpha^* : X \times X \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say T is a modified α^* - ϕ - contractive mapping if, for all $x, y \in X$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1 \quad \Rightarrow \quad M(Tx, Ty, t) \geq M(x, y, t) + \phi(M(x, y, t)),$$

where $\phi : [0, 1] \rightarrow [0, 1]$ is continuous and $\phi(s) > 0$ for all $s \in (0, 1)$.

Definition 3.4. Let (X, M, \star) be a fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping and $\alpha^* : X \times X \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say T is a modified α^* - β - contractive mapping if there exists a function $\beta : [0, 1] \rightarrow [1, +\infty)$ such that, for any sequence $\{s_n\} \subseteq [0, 1]$ of positive reals, $\beta(s_n) \rightarrow 1$ implies $s_n \rightarrow 1$ and, for all $x, y \in X$ and $t > 0$,

$$\alpha^*(x, y, t) \geq 1 \quad \Rightarrow \quad M(Tx, Ty, t) \geq \beta(M(x, y, t))M(x, y, t).$$

As an immediate consequence of the best proximity results established above, we deduce the following fixed point results in a complete non-Archimedean fuzzy metric space.

Theorem 3.5. *Let (X, M, \star) be a complete non-Archimedean fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is a triangular α^* -admissible mapping;
- (ii) T is a modified α^* - φ - ω -contractive mapping;
- (iii) there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (iv) if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then T has a fixed point in X .

Moreover, if $x = Tx$ and $y = Ty$ imply $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique fixed point in X .

Corollary 3.6. *Let (X, M, \star) be a complete non-Archimedean fuzzy metric space and $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is an α^* -proximal admissible mapping;
- (ii) T is a modified α^* - φ - ω -proximal contractive mapping;
- (iii) there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$;

[T3] for all $x \in X$ and $t > 0$, $\alpha^*(x, x, t) \geq 1$ and, if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then T has a fixed point $x^* \in X$.

Moreover, if $x = Tx$ and $y = Ty$ imply $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique fixed point in X .

Corollary 3.7. Let (X, M, \star) be a complete non-Archimedean fuzzy metric space and $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:

- (i) T is a triangular α^* -proximal admissible mapping;
- (ii) T is a modified α^* - φ - ω -proximal contractive mapping;
- (iii) there exists $x_0 \in X$, such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$;

[T4] if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ with $\alpha^*(x_{k_n}, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then T has a fixed point in X .

Moreover, if $x = Tx$ and $y = Ty$ imply $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique fixed point in X .

Corollary 3.8 (Theorem 5 of [29]). Let (X, M, \star) be a complete non-Archimedean fuzzy metric space and f be a self-mapping on X . Assume that $k : (0, +\infty) \rightarrow (0, 1)$ is a function and $\varphi \in \Phi$. Also, suppose that

$$\varphi(M(fx, fy, t)) \leq k(t)\varphi(M(x, y, t)),$$

for all $x, y \in X$ with $x \neq y$ and $t > 0$. Then f has a unique fixed point in X .

Proof. By taking $\alpha^*(x, y, t) = 1$ and $\omega(t)^2 = k(t)$ in Theorem 3.5, we deduce this Corollary. \square

Theorem 3.9. Let (X, M, \star) be a complete non-Archimedean fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:

- (i) T is a triangular α^* -admissible mapping;
- (ii) T is a modified α^* - ϕ -contractive mapping;
- (iii) there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$;
- (v) if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.

Then T has a fixed point in X .

Moreover, if $x = Tx$ and $y = Ty$ imply $\alpha^*(x, y, t) \geq 1$ for all $t > 0$, then T has a unique fixed point in X .

Theorem 3.10. *Let (X, M, \star) be a complete non-Archimedean fuzzy metric space. Let $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) *T is a triangular α^* -admissible mapping;*
- (ii) *T is a modified $\alpha^* - \beta$ - contractive mapping;*
- (iii) *there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$;*
- (v) *if $\{x_n\}$ is a sequence in X such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$ and $t > 0$.*

Then T has a fixed point in X .

Moreover, if $x = Tx$ and $y = Ty$ imply $\alpha^(x, y, t) \geq 1$ for all $t > 0$, then T has a unique fixed point in X .*

4. BEST PROXIMITY POINT RESULTS IN PARTIALLY ORDERED FUZZY METRIC SPACES

The study of the existence of fixed points in partially ordered sets has been initiated by Ran and Reurings [26] with applications to matrix equations. Agarwal et al. [1, 2], Ćirić et al. [7] and Hussain et al. [17, 18] presented some new results for nonlinear contractions in partially ordered metric spaces. Here, as an application of our results, we deduce some new best proximity and fixed point results in partially ordered fuzzy metric spaces.

Recall that, if (X, \preceq) is a partially ordered set and $T : X \rightarrow X$ is such that, for all $x, y \in X$ with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping T is said to be non-decreasing.

Definition 4.1. Let A and B be nonempty subsets of a partially ordered fuzzy metric space (X, M, \star, \preceq) . A mapping $T : A \rightarrow B$ is said to be proximally ordered-preserving if, for all $x_1, x_2, u_1, u_2 \in A$ and $t > 0$,

$$\begin{cases} x_1 \preceq x_2, \\ M(u_1, Tx_1, t) = M(A, B, t), \\ M(u_2, Tx_2, t) = M(A, B, t) \end{cases} \Rightarrow u_1 \preceq u_2.$$

Clearly, letting $A = B$, then, if $T : A \rightarrow A$ is proximally order-preserving, then T is a non-decreasing mapping.

Definition 4.2. Let A and B be nonempty subsets of a partially ordered fuzzy metric space (X, M, \star, \preceq) . Let $T : A \rightarrow B$ be a non-self mapping. We say that T is an ordered φ - ω -proximal contractive mapping if, for all $x, y, u, v \in A$ and $t > 0$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t), \end{cases} \Rightarrow \varphi(M(u, v, t)) \leq \omega(t)\varphi(M(x, y, t)),$$

where $\varphi \in \Phi$ and $\omega : (0, +\infty) \rightarrow (0, 1)$ is a function.

Definition 4.3. Let A and B be nonempty subsets of a partially ordered fuzzy metric space (X, M, \star, \preceq) . Let $T : A \rightarrow B$ be a non-self mapping. We say that T is an ordered ϕ -proximal contractive mapping if, for all $x, y, u, v \in A$ and $t > 0$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \Rightarrow M(u, v, t) \geq M(x, y, t) + \phi(M(x, y, t)), \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

where $\phi : [0, 1] \rightarrow [0, 1]$ is continuous and $\phi(s) > 0$ for all $s \in (0, 1)$.

Definition 4.4. Let A and B be nonempty subsets of a partially ordered fuzzy metric space (X, M, \star, \preceq) . Let $T : A \rightarrow B$ be a non-self mapping. We say that T is an ordered β -proximal contractive mapping if there exists a function $\beta : [0, 1] \rightarrow [1, +\infty)$ such that, for any sequence $\{s_n\} \subseteq [0, 1]$ of positive reals, $\beta(s_n) \rightarrow 1$ implies $s_n \rightarrow 1$ and, for all $x, y, u, v \in A$ and $t > 0$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \Rightarrow M(u, v, t) \geq \beta(M(x, y, t))M(x, y, t). \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

Theorem 4.5. Let A and B be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space (X, M, \star, \preceq) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:

- (i) T is proximally order-preserving and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is an ordered φ - ω -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ are such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist $x_0, x_1 \in A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad x_0 \preceq x_1,$$

for all $t > 0$;

- (v) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Proof. Define a function $\alpha : A \times A \times (0, \infty) \rightarrow [0, +\infty)$ by

$$\alpha^*(x, y, t) = \begin{cases} 2, & \text{if } x \preceq y; \\ 0, & \text{otherwise,} \end{cases}$$

for all $x, y \in A$ and $t > 0$.

First, we prove that T is an α^* -proximal admissible mapping. For this, assume that

$$\begin{cases} \alpha^*(x, y, t) \geq 1, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

and so

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Now, since T is proximally ordered-preserving and so $u \preceq v$, that is, $\alpha^*(u, v, t) \geq 1$ for all $t > 0$, which implies that T is α^* -proximal admissible. Since \preceq is partial order, $[T2]$ obviously holds. Thus T is a triangular α^* -proximal admissible mapping. The condition (ii) implies that T is a modified α^* - φ - ω -proximal contractive mapping. Further, by (iv), we have

$$M(x_1, Tx_0, t) = M(A, B, t), \quad \alpha^*(x_0, x_1, t) \geq 1.$$

Also, assume that $\alpha^*(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Then $x_n \preceq x_{n+1}$ for all $n \geq 1$. Hence, by (v), we get $x_n \preceq x$ for all $n \geq 1$ and so $\alpha^*(x_n, x, t) \geq 1$ for all $n \geq 1$, that is, all the conditions of Theorem 2.6 hold and T has a best proximity point. This completes the proof. \square

Corollary 4.6. *Let A and B be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space (X, M, \star, \preceq) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is proximally order-preserving and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is an ordered φ - ω -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist $x_0, x_1 \in A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad x_0 \preceq x_1,$$

for all $t > 0$;

- (v) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then there is a subsequence $\{x_{k_n}\}$ with $x_{k_n} \preceq x$ for all $n \geq 1$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Similarly, we have the following best proximity theorems.

Theorem 4.7. *Let A and B be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space (X, M, \star, \preceq) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is proximally ordered-preserving and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is an ordered ϕ -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ are such that

$$M(x, y_n, t) \rightarrow M(A, B, t),$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist $x_0, x_1 \in A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad x_0 \preceq x_1,$$

for all $t > 0$;

- (v) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Theorem 4.8. *Let A and B be nonempty closed subsets of a complete partially ordered non-Archimedean fuzzy metric space (X, M, \star, \preceq) such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \rightarrow B$ be a non-self mapping satisfying the following assertions:*

- (i) T is proximally ordered-preserving and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;
- (ii) T is an ordered β -proximal contractive mapping;
- (iii) if $\{y_n\}$ is a sequence in $B_0(t)$ and $x \in A$ is such that

$$M(x, y_n, t) \rightarrow M(A, B, t)$$

as $n \rightarrow +\infty$, then $x \in A_0(t)$ for all $t > 0$;

- (iv) there exist $x_0, x_1 \in A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t), \quad x_0 \preceq x_1,$$

for all $t > 0$;

- (v) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

Further, we can easily deduce the following new fixed point results from best proximity theorems proved above.

Theorem 4.9. *Let (X, M, \star, \preceq) be a complete partially ordered non-Archimedean fuzzy metric space and $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is a non-decreasing mapping;
- (ii) T is an ordered φ - ω -contractive mapping;
- (iii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iv) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then T has a fixed point in X .

Theorem 4.10. *Let (X, M, \star, \preceq) be a complete partially ordered non-Archimedean fuzzy metric space and $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is a non-decreasing mapping;
- (ii) T is an ordered ϕ -contractive mapping;
- (iii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iv) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then T has a fixed point in X .

Theorem 4.11. *Let (X, M, \star, \preceq) be a complete partially ordered non-Archimedean fuzzy metric space and $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is a non-decreasing mapping;
- (ii) T is an ordered β -contractive mapping;
- (iii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iv) if $\{x_n\}$ is an increasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $x_n \preceq x$ for all $n \geq 1$.

Then T has a fixed point in X .

REFERENCES

1. R.P. Agarwal, M.A. El-Gebeily, and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., 87 (2008), pp. 109-116.
2. R.P. Agarwal, N. Hussain, and M.A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal., (2012), Article ID 245872, 15 pp.
3. S.S. Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., 54 (2012), pp. 511-517.

4. S.S. Basha, *Common best proximity points: global minimization of multi-objective functions*, J. Global Optim., 54 (2012), pp. 367-373.
5. S. Cang and D. Partridge, *Feature ranking and best feature subset using mutual information*, Neural Computing Appl., 13 (2004), pp. 175-184.
6. A. Celikyilmaz and I.B. Türksen, *Fuzzy functions with support vector machines*, Inform. Sci., 177 (2007), pp. 51-63.
7. L. Ćirić, M. Abbas, R. Saadati, and N. Hussain, *Common fixed points of almost generalized contractive mappings in ordered metric spaces*, Appl. Math. Comput., 217 (2011), pp. 5784-5789.
8. Z.K. Deng, *Fuzzy pseudometric spaces*, J. Math. Anal. Appl., 86 (1982), pp. 74-95.
9. C. Di Bari, T. Suzuki, and C. Vetro, *Best proximity points for cyclic Meir-Keeler contractions*, Nonlinear Anal., 69 (2008), pp. 3790-3794.
10. C. Di Bari and C. Vetro, *A fixed point theorem for a family of mappings in a fuzzy metric space*, Rend. del Circolo Mat. di Palermo., 52 (2003), pp. 315-321.
11. C. Di Bari and C. Vetro, *Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space*, J. Fuzzy Math., 13 (2005), pp. 973-982.
12. K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z., 112 (1969), pp. 234-240.
13. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., 64 (1994), pp. 395-399.
14. A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Syst., 90 (1997), pp. 365-368.
15. D. Gopal, M. Imdad, C. Vetro, and M. Hasan, *Fixed point theory for cyclic weak ϕ -contraction in fuzzy metric spaces*, J. Nonlinear Anal. Appl., 2012 (2012), pp. 1-11.
16. V. Gregori and A. Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets Syst., 125 (2002), pp. 245-252.
17. N. Hussain, S. Al-Mezel, and P. Salimi, *Fixed points for ψ -graphic contractions with application to integral equations*, Abstr. Appl. Anal., 2013 (2013), pp. 1-11.
18. N. Hussain, A.R. Khan, and R.P. Agarwal, *Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces*, J. Nonlinear Convex Anal., 11 (2010), pp. 475-489.
19. N. Hussain, M.A. Kutbi, and P. Salimi, *Best proximity point results for modified α - ψ -proximal rational contractions*, Abstr. Appl. Anal., (2013), Article ID 927457, 14 pp.

20. M. Jleli and B. Samet, *Best proximity points for $\alpha - \psi$ -proximal contractive type mappings and applications*, Bull. Sci. Math., 137 (2013), pp. 977-995.
21. I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika., 11 (1975), pp. 336-344.
22. E. Karapınar and I.M. Erhan, *Best proximity point on different type contractions*, Appl. Math. Inform. Sci., 5 (2011), pp. 558-569.
23. D. Mihet, *Fuzzy ψ -contractive mappings in non-Archimedean fuzzy metric spaces*, Fuzzy Sets Syst., 159 (2008), pp. 739-744.
24. C. Mongkolkeha, Y.J. Cho, and P. Kumam, *Best proximity points for generalized proximal C -contraction mappings in metric spaces with partial orders*, J. Inequal. Appl., 2013 (2013), 94 pp.
25. C. Mongkolkeha, Y.J. Cho, and P. Kumam, *Best proximity points for Geraghty's proximal contraction mappings*, Fixed Point Theory Appl., 2013 (2013), 180 pp.
26. A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132 (2003), pp. 1435-1443.
27. B.D. Ripley, *Pattern Recognition and Neural Networks*, Cambridge University Press, 1996.
28. P. Salimi, A. Latif, and N. Hussain, *Modified α - ψ -contractive mappings with applications*, Fixed Point Theory Appl., 2013 (2013), 151 pp.
29. P. Salimi, C. Vetro, and P. Vetro, *Some new fixed point results in non-Archimedean fuzzy metric spaces*, Nonlinear Anal. Model. Control., 18 (2013), pp. 344-358.
30. B. Samet, C. Vetro, and P. Vetro, *Fixed point theorem for $\alpha - \psi$ contractive type mappings*, Nonlinear Anal., 75 (2012), pp. 2154-2165.
31. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., 10 (1960), pp. 314-334.
32. B.W. Silverman, *Density Estimation for Statistics and Data Analysis*, Chapman & Hall, London, 1996.
33. T. Suzuki, M. Kikkawa, and C. Vetro, *The existence of best proximity points in metric spaces with the property (UC)*, Nonlinear Anal., 71 (2009), pp. 2918-2926.
34. C. Vetro, *Best proximity points: convergence and existence theorems for p -cyclic mappings*, Nonlinear Anal., 73 (2010), pp. 2283-2291.
35. C. Vetro, *Fixed points in weak non-Archimedean fuzzy metric spaces*, Fuzzy Sets Syst., 162 (2011), pp. 84-90.

36. C. Vetro and P. Vetro, *Common fixed points for discontinuous mappings in fuzzy metric spaces*, Rend. del Circolo Mat. di Palermo., 57 (2008), pp. 295-303.
37. C. Vetro and P. Salimi, *Best proximity point results in non-Archimedean fuzzy metric spaces*, Fuzzy Inf. Eng., 4 (2013), pp. 417-429.

DEPARTMENT OF MATHEMATICS, FARHANGIAN UNIVERSITY, IRAN.
E-mail address: `m.paknazar@cfu.ac.ir`