

on the stability of the Pexiderized cubic functional equation in multi-normed spaces

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ABSTRACT. In this paper, we investigate the Hyers-Ulam stability of the orthogonally cubic equation and Pexiderized cubic equation

$$f(kx + y) + f(kx - y) = g(x + y) + g(x - y) + \frac{2}{k}g(kx) - 2g(x),$$

in multi-normed spaces by the direct method and the fixed point method. Moreover, we prove the Hyers-Ulam stability of the 2-variables cubic equation

$$\begin{aligned} f(2x + y, 2z + t) + f(2x - y, 2z - t) \\ = 2f(x + y, z + t) + 2f(x - y, z - t) + 12f(x, z), \end{aligned}$$

and orthogonally cubic type and k -cubic equation in multi-normed spaces. A counter example for non stability of the cubic equation is also discussed.

1. INTRODUCTION

The stability problem of functional equations originated from a question of S.M. Ulam [14] concerning the stability of group homomorphisms: Given a group G_1 , a metric group (G_2, d) and a positive number ε , does there exist a number $\delta > 0$ such that if a mapping $f : G_1 \rightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$ then there exists a homomorphism $T : G_1 \rightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$?

D.H. Hyers [5] gave a first affirmative answer to the question of Ulam for Banach spaces:

Let X be a normed space and Y a Banach space. Suppose that for some

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$\varepsilon > 0$, the mapping $f : X \rightarrow Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.

Jun and Kim [6] introduced the following cubic functional equation

$$(1.1) \quad f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x),$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.1).

Every solution of the cubic functional equation is said to be a cubic mapping.

T.Z. Xu et al. [15] proved the generalized Hyers-Ulam stability of the general mixed additive-cubic functional equation

$$(1.2) \quad f(kx+y) + f(kx-y) = kf(x+y) + kf(x-y) + 2f(kx) - 2kf(x),$$

in quasi-Banach spaces.

In [9], S. Ostadbashi and M. Soleimaninia considered the Pexiderized cubic functional equation

$$(1.3) \quad f(kx+y) + f(kx-y) = g(x+y) + g(x-y) + \frac{2}{k}g(kx) - 2g(x),$$

and they investigated the stability of the functional equation (1.3) in topological vector spaces.

The functional equation

$$(1.4) \quad \begin{aligned} f(2x+y, 2z+t) + f(2x-y, 2z-t) \\ = 2f(x+y, z+t) + 2f(x-y, z-t) + 12f(x, z), \end{aligned}$$

is called the 2-variables cubic functional equation.

It is easy to see that the function $f(x, y) = x^3 + y^3$ is a solution of equation (1.4).

Every solution of the 2-variables cubic functional equation is said to be a 2-variables cubic mapping.

Chang, Jun and Jung [1] introduced a cubic type functional equation as follows:

$$(1.5) \quad \begin{aligned} f(x_1+x_2+2x_3) + f(x_1+x_2-2x_3) + f(2x_1) + f(2x_2) + 7[f(x_1) + f(-x_1)] \\ = 2f(x_1+x_2) + 4[f(x_1+x_3) + f(x_1-x_3) + f(x_2+x_3) + f(x_2-x_3)], \end{aligned}$$

and they investigated the modified Hyers-Ulam-Rassias stability of this equation by using the fixed point method. Moreover I.S. Chang and Y.S. Jung [2] established the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.5).

It is easy to see that the function $f(x) = x^3 + C$, where $C \in \mathbb{R}$ is a constant is a solution of the equation (1.5).

K. Ravi, M.J. Rassias, P. Narasimman, R.K. Kumar [13] introduced the k -cubic functional equation

$$(1.6) \quad kf(x+ky) - f(kx+y) = \frac{k(k^2-1)}{2}[f(x+y) + f(x-y)] \\ + (k^4-1)f(y) - 2k(k^2-1)f(x),$$

where $k \geq 2$ and they established the generalized Hyers-Ulam stability problem for the functional equation (1.6).

Note that the function $f(x) = x^3$ is a solution of the functional equation (1.6).

Every solution of the k -cubic functional equation is said to be a k -cubic mapping.

During the last decades, several stability problems of functional equations have been investigated. The reader is referred to [3, 8, 11] and references therein for detailed information on stability of functional equations.

Pinsker [10] characterized orthogonal additive functional equation on an inner product space. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad x \perp y,$$

in which \perp is an orthogonality relation, is first investigated by Gudder and Strawther [4]. In 1985, Ratz [12] introduced a new definition of orthogonality by using more restrictive axioms than Gudder and Strawther. Moreover, he investigated the structure of the orthogonally additive mappings.

There are several orthogonality notations on a real normed space. But here, we present the orthogonal concept introduced by Ratz. This is given in the following definition.

Definition 1.1. Suppose that X is a vector space (algebraic module) with $\dim X \geq 2$, and \perp is a binary relation on X with the following properties:

- (i) totality of \perp for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
- (ii) independence: if $x, y \in X - \{0\}$ and $x \perp y$, then x and y are linearly independent;
- (iii) homogeneity: if $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (iv) Thalesian propriety: if P is a 2-dimensional subspace of X , $x \in P$ and $\lambda \geq 0$ then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair (X, \perp) is called an orthogonal space (resp., module). By an orthogonal normed space (normed module), we mean an orthogonal space (resp., module) having a normed (resp., normed module) structure.

Let $(X, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by \mathfrak{G}_k the group of permutations on k symbols.

Definition 1.2. A multi-norm on $\{X^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N}),$$

such that $\|\cdot\|_k$ is a norm on X^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in X$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(MN1)

$$\|x_{\sigma(1)}, \dots, x_{\sigma(k)}\|_k = \|x_1, \dots, x_k\|_k, \quad (\sigma \in \mathfrak{G}_k, x_1, \dots, x_k \in X);$$

(MN2)

$$\|\alpha_1 x_1, \dots, \alpha_k x_k\|_k \leq \left(\max_{i \in \mathbb{N}_k} |\alpha_i|\right) \|x_1, \dots, x_k\|_k,$$

$$(\alpha_1, \dots, \alpha_k \in \mathbb{C}, x_1, \dots, x_k \in X);$$

(MN3)

$$\|x_1, \dots, x_{k-1}, 0\|_k = \|x_1, \dots, x_{k-1}\|_{k-1}, \quad (x_1, \dots, x_{k-1} \in X);$$

(MN4)

$$\|x_1, \dots, x_{k-1}, x_{k-1}\|_k = \|x_1, \dots, x_{k-1}\|_{k-1}, \quad (x_1, \dots, x_{k-1} \in X).$$

In this case, we say that $((X^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Example 1.3. Let $(X, \|\cdot\|)$ be a Banach lattice, and define

$$\|x_1, \dots, x_k\|_k := \||x_1| \vee \dots \vee |x_k|\|, \quad (x_1, \dots, x_k \in X).$$

Then $((X^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Lemma 1.4. Let (X, d) be a complete generalized metric space and $J : X \rightarrow X$ be a strictly contractive mapping, that is,

$$d(Jx, Jy) \leq \alpha d(x, y), \quad \forall x, y \in X,$$

for some $\alpha < 1$. Then for each element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad (n \geq 0),$$

or there exists $n_0 \geq 0$ such that:

- (i) $d(J^n x, J^{n+1} x) < \infty, (n \geq n_0)$;
- (ii) the sequence $(J^n x)$ converges to a fixed point y^* of J ;
- (iii) y^* is the unique fixed point of J in the set

$$U = \{y \in X : d(J^{n_0} x, y) < \infty\};$$

- (iv) $d(y, y^*) \leq d(y, Jy) / (1 - \alpha), (y \in U)$.

In this paper, we investigate the Hyers-Ulam stability of the orthogonally cubic equation and Pexiderized cubic equation

$$f(kx + y) + f(kx - y) = g(x + y) + g(x - y) + \frac{2}{k}g(kx) - 2g(x),$$

in multi-normed spaces by the direct method and the fixed point method. Moreover we prove the Hyers-Ulam stability of a 2-variables cubic equation

$$f(2x + y, 2z + t) + f(2x - y, 2z - t) = 2f(x + y, z + t) + 2f(x - y, z - t) + 12f(x, z),$$

and orthogonally cubic type and k -cubic equation in multi-normed spaces. Throughout this paper, assume that X is an orthogonal space (except in Section 3, 4) and that Y is a Banach space. Let $((Y^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space.

2. STABILITY OF THE CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the orthogonally cubic functional equation in multi-normed spaces. For convenience, we use the following abbreviation for a given mapping $f : X \rightarrow Y$,

$$Df(x, y) = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x),$$

for all $x, y \in X$ with $x \perp y$.

Definition 2.1. A mapping $f : X \rightarrow Y$ is called an orthogonally cubic mapping if

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),$$

for all $x, y \in X$ with $x \perp y$.

Theorem 2.2. Let $\alpha \geq 0$ and $f : X \rightarrow Y$ be a mapping satisfying

$$(2.1) \quad \sup_{k \in \mathbb{N}} \|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \alpha,$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$ with $x_i \perp y_i (i = 1, \dots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$(2.2) \quad \sup_{k \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \frac{\alpha}{14},$$

for all $x_1, \dots, x_k \in X$.

Proof. Let $x_1, \dots, x_k \in X$. Letting $y_1 = \dots = y_k = 0$ in (2.1), we get

$$(2.3) \quad \sup_{k \in \mathbb{N}} \|f(2x_1) - 8f(x_1), \dots, f(2x_k) - 8f(x_k)\| \leq \frac{\alpha}{2},$$

for all $x_1, \dots, x_k \in X$, since $0 \perp x_i$, ($i = 1, \dots, k$). Replacing x_1, \dots, x_k by $2^n x_1, \dots, 2^n x_k$ in relation (2.3) and dividing by 8^{n+1} , from relation (2.3) one gets

$$(2.4) \quad \sup_{k \in \mathbb{N}} \left\| \frac{f(2^{n+1}x_1)}{8^{n+1}} - \frac{f(2^n x_1)}{8^n}, \dots, \frac{f(2^{n+1}x_k)}{8^{n+1}} - \frac{f(2^n x_k)}{8^n} \right\|_k \leq \frac{\alpha}{2} \left(\frac{1}{8^{n+1}} \right).$$

It follows from the last inequality that

$$(2.5) \quad \sup_{k \in \mathbb{N}} \left\| \frac{f(2^{n+m}x_1)}{8^{n+m}} - \frac{f(2^n x_1)}{8^n}, \dots, \frac{f(2^{n+m}x_k)}{8^{n+m}} - \frac{f(2^n x_k)}{8^n} \right\|_k \leq \frac{\alpha}{2} \left(\frac{1}{8^{n+1}} + \dots + \frac{1}{8^{n+m}} \right),$$

for $n, m \in \mathbb{N}, m \geq 1$.

It follows that $\left(\frac{f(2^n x)}{8^n} \right)$ is Cauchy and so is convergent in the complete multi-normed space Y .

Set

$$C(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{8^n}.$$

Hence for each $\varepsilon > 0$ there exists n_0 such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{f(2^n x)}{8^n} - C(x), \dots, \frac{f(2^{n+k-1}x)}{8^{n+k-1}} - C(x) \right\|_k < \varepsilon,$$

for all $n \geq n_0$. Therefore we get

$$(2.6) \quad \lim_{n \rightarrow \infty} \left\| \frac{f(2^n x)}{8^n} - C(x) \right\| = 0,$$

for all $x \in X$. Letting $n = 0$ in (2.5), we get

$$\sup_{k \in \mathbb{N}} \left\| \frac{f(2^m x_1)}{8^m} - f(x_1), \dots, \frac{f(2^m x_k)}{8^m} - f(x_k) \right\|_k \leq \frac{\alpha}{2} \sum_{i=1}^m \frac{1}{8^i}.$$

Letting m tend to infinity and using Lemma 1.4 and (2.6), we have

$$\sup_{k \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \frac{\alpha}{14}.$$

Let $x, y \in X$ and $x \perp y$. So we have $2^n x \perp 2^n y$. Letting $x_1 = \dots = x_k = 2^n x$ and $y_1 = \dots = y_k = 2^n y$ in (2.1) and divide both sides by 8^n to obtain

$$\frac{1}{8^n} \|Df(2^n x, 2^n y)\| \leq \frac{\alpha}{8^n}.$$

Taking the limit as $n \rightarrow \infty$, we get

$$DC(x, y) = 0.$$

Hence C is an orthogonally cubic mapping.

Let C' be another orthogonally cubic mapping satisfying (2.2). It is easy to see that every cubic mapping $f : X \rightarrow Y$ satisfies

$$f(kx) = k^3 f(x),$$

for all positive integer k and $x \in X$. So, we have

$$\begin{aligned} \|Cx - C'x\| &= \frac{1}{8^n} \|C(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n} \|C(2^n x) - f(2^n x)\| + \frac{1}{8^n} \|f(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n} \left(\frac{\alpha}{14} + \frac{\alpha}{14} \right). \end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. \square

We prove the generalized Hyers-Ulam stability of the cubic functional equation in multi-normed spaces by using the fixed point method.

Theorem 2.3. *Let $k \in \mathbb{N}$ and $\phi : X^{2k} \rightarrow [0, \infty)$ be a function such that*

$$\phi(2x_1, \dots, 2x_k, 2y_1, \dots, 2y_k) \leq a\phi(x_1, \dots, x_k, y_1, \dots, y_k),$$

for some $a > 0$ with $a < 8$ and all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$(2.7) \quad \|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \phi(x_1, \dots, x_k, y_1, \dots, y_k),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$ with $x_i \perp y_i$ ($i = 1, \dots, k$). Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$(2.8) \quad \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \frac{1}{16 - 2a} \phi(x_1, \dots, x_k, 0, \dots, 0),$$

for all $x_1, \dots, x_k \in X$.

Proof. Let $E = \{f : X \rightarrow Y\}$, and introduce the generalized metric d defined on E by

$$\begin{aligned} d(g, h) &= \inf \left\{ c \in (0, \infty) : \|g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)\|_k \right. \\ &\quad \left. \leq c\psi(x_1, \dots, x_k), \forall x_1, \dots, x_k \in X \right\}, \end{aligned}$$

where $\psi : X^k \rightarrow [0, \infty)$ is a mapping defined by

$$\psi(x_1, \dots, x_k) = \phi(x_1, \dots, x_k, 0, \dots, 0).$$

Then, it is easy to show that d is a complete generalized metric on E . We now define a mapping $J : E \rightarrow E$ by

$$Jg(x) = \frac{1}{8}g(2x), \quad \forall x \in X.$$

We claim that J is a strictly contractive mapping. Given $g, h \in E$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) < c$. It is easy to see that

$$\|g(x_1) - h(x_1), \dots, g(x_k) - h(x_k)\|_k \leq c\psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$. Therefore,

$$\begin{aligned} & \|Jg(x_1) - Jh(x_1), \dots, Jg(x_k) - Jh(x_k)\|_k \\ &= \left\| \frac{1}{8}g(2x_1) - \frac{1}{8}h(2x_1), \dots, \frac{1}{8}g(2x_k) - \frac{1}{8}h(2x_k) \right\|_k \\ &\leq \frac{1}{8}ca\psi(x_1, \dots, x_k), \end{aligned}$$

for all $x_1, \dots, x_k \in X$. So we have $d(Jg, Jh) \leq \frac{1}{8}ad(g, h)$ for all $g, h \in E$. Letting $y_1 = \dots = y_k = 0$ in (2.7), we get

$$(2.9) \quad \|f(2x_1) - 8f(x_1), \dots, f(2x_k) - 8f(x_k)\|_k \leq \frac{1}{2}\psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$. It follows from (2.9) that $d(Jf, f) \leq \frac{1}{16}$. According to Lemma 1.4, we deduce the existence of a fixed point of J , that is, the existence of a mapping $C : X \rightarrow Y$ such that $C(2x) = 8C(x)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \rightarrow 0$ that

$$\begin{aligned} C(x) &= \lim_{n \rightarrow \infty} (J^n f)(x) \\ &= \lim_{n \rightarrow \infty} \frac{f(2^n x)}{8^n}, \quad \forall x \in X. \end{aligned}$$

Moreover $d(f, C) \leq \frac{1}{1-\frac{1}{8}}d(Jf, f)$, implies the inequality

$$d(f, C) \leq \frac{1}{16 - 2a}.$$

So (2.8) is satisfied.

Let $x, y \in X$ and $x \perp y$. So $2^n x \perp 2^n y$. Letting $x_1 = \dots = x_k = 2^n x$ and $y_1 = \dots = y_k = 2^n y$ in (2.7) and dividing both sides by 8^n , we to get

$$\begin{aligned} \frac{1}{8^n} \|Df(2^n x, 2^n y)\| &\leq \frac{1}{8^n} \phi(2^n x, \dots, 2^n x, 2^n y, \dots, 2^n y) \\ &\leq \left(\frac{a}{8}\right)^n \phi(x, \dots, x, y, \dots, y). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we deduce that C is an orthogonally cubic mapping.

The uniqueness of C follows from the fact that C is the unique fixed point of J with the property that there exists $\lambda \in (0, \infty)$ such that

$$\|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \lambda\psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$. \square

Corollary 2.4. *Let $k \in \mathbb{N}$ and p, α be real numbers with $\alpha > 0$ and $p < 3$, and let $f : X \rightarrow Y$ be a mapping satisfying*

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \alpha(\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$ with $x_i \perp y_i (i = 1, \dots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \frac{\alpha}{16 - 2^{p+1}}(\|x_1\|^p + \dots + \|x_k\|^p),$$

where $x_1, \dots, x_k \in X$.

Proof. Let

$$\phi(x_1, \dots, x_k, y_1, \dots, y_k) = \alpha(\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$ and $a = 2^p$ in Theorem 2.3, we obtain the desired result. \square

The following example shows that the assumption $p < 3$ cannot be omitted in Corollary 2.4. We know from Example 1.3 that if

$$\|x_1, \dots, x_k\|_k = \sup\{|x_1|, \dots, |x_k|\},$$

then $(\mathbb{R}^k, \|\cdot\|_k) : k \in \mathbb{N}$ is a multi-normed space.

Example 2.5. Fix $k \in \mathbb{N}$. For $\epsilon > 0$ put $\lambda := \frac{7}{9216}\epsilon$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\phi(x) := \begin{cases} \lambda & x \in [1, \infty); \\ \lambda x^3 & x \in (-1, 1); \\ -\lambda & x \in (-\infty, -1]. \end{cases}$$

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ to be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{8^n}, \quad x \in \mathbb{R}.$$

Then f satisfies the following functional inequality (2.10)

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \epsilon(|x_1|^3 + \dots + |x_k|^3 + |y_1|^3 + \dots + |y_k|^3),$$

where $x_1, \dots, x_k, y_1, \dots, y_k \in X$.

Proof. We claim that

$$|Df(x, y)| \leq \epsilon(|x|^3 + |y|^3), \quad x, y \in \mathbb{R}.$$

We have

$$|f(x)| \leq \frac{8}{7}\lambda,$$

for all $x \in \mathbb{R}$. Therefore we see that f is bounded.

If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq \frac{1}{8}$, then

$$\begin{aligned} |Df(x, y)| &\leq \frac{144}{7}\lambda \\ &\leq \frac{144}{7}\lambda(8(|x|^3 + |y|^3)) \\ &\leq \varepsilon(|x|^3 + |y|^3), \end{aligned}$$

for all $x, y \in \mathbb{R}$. Now, suppose that $0 < |x|^3 + |y|^3 < \frac{1}{8}$. Then there exists a nonnegative integer k such that

$$\frac{1}{8^{k+2}} \leq |x|^3 + |y|^3 < \frac{1}{8^{k+1}}.$$

Hence,

$$2^k x < \frac{2}{3} \quad \text{and} \quad 2^k y < \frac{2}{3},$$

and

$$2^n(2x + y), 2^n(2x - y), 2^n(x + y), 2^n(x - y), 2^n x \in (-1, 1),$$

for all $n = 0, 1, \dots, k - 1$. As a result, we get

$$\begin{aligned} \frac{|Df(x, y)|}{|x|^3 + |y|^3} &\leq \sum_{n=k}^{\infty} \frac{18\lambda}{8^n(|x|^3 + |y|^3)} \\ &\leq \sum_{n=0}^{\infty} \frac{18\lambda}{8^{n+k}(|x|^3 + |y|^3)} \\ &\leq \sum_{n=0}^{\infty} \frac{18\lambda}{8^n 8^{k+2}(|x|^3 + |y|^3)} 64 \\ &\leq \sum_{n=0}^{\infty} \frac{1152\lambda}{8^n} \\ &= \frac{9216}{7}\lambda \\ &= \varepsilon. \end{aligned}$$

So

$$|Df(x, y)| \leq \varepsilon(|x|^3 + |y|^3), \quad x, y \in \mathbb{R}.$$

It is easy to see that f satisfies (2.10) for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}$. Now, we claim that the cubic functional equation in Corollary 2.4 is not stable for $p = 3$. Suppose on the contrary that there exists a cubic mapping $C : \mathbb{R} \rightarrow \mathbb{R}$ and $\beta > 0$ such that

$$\|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \beta(|x_1|^3 + \dots + |x_k|^3),$$

where $x_1, \dots, x_k \in \mathbb{R}$. So $|f(x) - C(x)| \leq \delta|x|^3$ for some constants $\delta > 0$ and all $x \in \mathbb{R}$. Then there exists a constant $\gamma \in \mathbb{R}$ for which $C(x) = \gamma x^3$ for all $x \in \mathbb{Q}$. So we get

$$\frac{|f(x)|}{x^3} \leq \delta + |\gamma|, \quad x \in \mathbb{Q}.$$

Let $M \in \mathbb{N}$ with $M\lambda > \delta + |\gamma|$. If x is a rational number in $(0, \frac{1}{2^{M-1}})$, then we have $2^n x \in (0, 1)$ for each $n = 0, 1, 2, \dots, M-1$. Consequently, for such an x we have

$$\begin{aligned} \frac{f(x)}{x^3} &= \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{8^n x^3} \\ &\geq \sum_{n=0}^{M-1} \frac{\lambda 8^n x^3}{8^n x^3} \\ &= M\lambda \\ &> \delta + |\gamma|, \end{aligned}$$

which yields a contradiction. \square

Next, we prove the generalized Hyers-Ulam stability of the orthogonally cubic functional equation in the closed balls of X^k . Assume that $N_r(X^k)$ is the closed ball in X^k of radius r around the origin and $((X^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.

Theorem 2.6. *Let $\phi_k : X^{2k} \rightarrow [0, \infty)$ ($k \in \mathbb{N}$) be a family of functions such that $\phi_k(\frac{\mathbf{x}}{2}, \frac{\mathbf{y}}{2}) \leq a\phi_k(\mathbf{x}, \mathbf{y})$ for some $a > 0$ with $a < \frac{1}{8}$ and all $\mathbf{x}, \mathbf{y} \in N_r(X^k)$ and $k \in \mathbb{N}$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying*

$$(2.11) \quad \|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \phi_k(\mathbf{x}, \mathbf{y}),$$

for all $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k) \in N_r(X^k)$ with $x_i \perp y_i (i = 1, \dots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$\sup_{k \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \frac{a \sup_{k \in \mathbb{N}} \phi_k(\mathbf{x}, 0)}{2(1 - 8a)},$$

where $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$.

Proof. Let $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$. Replacing \mathbf{x}, \mathbf{y} in (2.11) by $\frac{\mathbf{x}}{2}, 0$ respectively, we get

$$(2.12) \quad \left\| f(x_1) - 8f\left(\frac{x_1}{2}\right), \dots, f(x_k) - 8f\left(\frac{x_k}{2}\right) \right\|_k \leq \frac{1}{2}\phi_k\left(\frac{\mathbf{x}}{2}, 0\right) \\ \leq \frac{1}{2}a\phi_k(\mathbf{x}, 0),$$

since $0 \perp \frac{1}{2}x_i (i = 1, \dots, k)$. Letting \mathbf{x} by $\frac{\mathbf{x}}{2^n}$ and multiplying with 8^n in (2.12) we have

$$\begin{aligned} & \left\| 8^n f\left(\frac{x_1}{2^n}\right) - 8^{n+1} f\left(\frac{x_1}{2^{n+1}}\right), \dots, 8^n f\left(\frac{x_k}{2^n}\right) - 8^{n+1} f\left(\frac{x_k}{2^{n+1}}\right) \right\|_k \\ & \leq \frac{a}{2} (8a)^n \phi_k(\mathbf{x}, 0). \end{aligned}$$

Therefore the sequence $(8^n f(\frac{x}{2^n}))$ is Cauchy and so is convergent in the complete multi-norm space Y . Set

$$C'(x) = \lim_{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right), \quad (x \in N_r(X)).$$

It is easy to see that

$$\sup_{k \in \mathbb{N}} \|f(x_1) - C'(x_1), \dots, f(x_k) - C'(x_k)\|_k \leq \frac{a \sup_{k \in \mathbb{N}} \phi_k(\mathbf{x}, 0)}{2(1 - 8a)},$$

where $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$.

Let $x \in N_r(X)$. Then we have

$$8C'\left(\frac{x}{2}\right) = \lim_{n \rightarrow \infty} 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) = C'(x).$$

So for all $m \in \mathbb{N}$ we have $8^m C'\left(\frac{x}{2^m}\right) = C'(x)$. Therefore we can define the well-defined mapping $C : X \rightarrow Y$ by $C(x) := 8^n C'\left(\frac{x}{2^n}\right)$, where n is the least non-negative integer such that $\frac{x}{2^n} \in N_r(X)$, indeed $n = \log_2 \|x\|$. It is clear that

$$C(x) = \lim_{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right), \quad (x \in X).$$

Let $x, y \in X$ and $x \perp y$. So $\frac{x}{2^n} \perp \frac{y}{2^n}$. There exists a large enough n such that $\frac{x}{2^n}, \frac{y}{2^n} \in N_r(X)$. Letting $x_1 = \dots = x_k = \frac{x}{2^n}, y_1 = \dots = y_k = \frac{y}{2^n}$ in (2.11) and multiplying both sides with 8^n , we to get

$$\left\| 8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \leq (8a)^n \phi_k(x, \dots, x, y, \dots, y).$$

Taking the limit as $n \rightarrow \infty$, we conclude that C is an orthogonally cubic mapping.

The proof of the uniqueness assertion is similar to the ones of Theorem 2.2. \square

Corollary 2.7. *Let p, α be positive real numbers with $p > 3$, and let $f : X \rightarrow Y$ be a mapping satisfying*

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \alpha (\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p),$$

for all $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k) \in N_r(X^k)$ with $x_i \perp y_i (i = 1, \dots, k)$. Then there exists a unique orthogonally cubic

mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} & \sup_{k \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \\ & \leq \frac{\alpha}{2^{p+1} - 16} \sup_{k \in \mathbb{N}} (\|x_1\|^p + \dots + \|x_k\|^p), \end{aligned}$$

where $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$.

Proof. Letting $\phi_k(\mathbf{x}, \mathbf{y}) = \alpha(\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p)$ for all $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{y} = (y_1, \dots, y_k) \in X^k$ and $a = \frac{1}{2^p}$ in Theorem 2.6, we obtain the desired result. \square

3. STABILITY OF THE PEXIDERIZED CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the Pexiderized cubic functional equation (1.3) in multi-normed spaces. For given $f, g : X \rightarrow Y$, we set

$$Df(x, y) = f(kx+y) + f(kx-y) - kf(x+y) - kf(x-y) - 2f(kx) + 2kf(x),$$

and

$$D(f, g)(x, y) = f(kx+y) + f(kx-y) - g(x+y) - g(x-y) - \frac{2}{k}g(kx) + 2g(x),$$

for all $x, y \in X$. Suppose that $k > 1$. We need the following lemma. The proof is found in [15].

Lemma 3.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ satisfying (1.2). Then the mapping $G(x) := f(2x) - 8f(x)$ is additive and $H(x) := f(2x) - 2f(x)$ is cubic.*

Theorem 3.2. *Let $t \in \mathbb{N}$ and $\phi : X^{2t} \rightarrow [0, \infty)$ be a function such that*

$$\phi(2x_1, \dots, 2x_t, 2y_1, \dots, 2y_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < 2$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f : X \rightarrow Y$ is a mapping such that $f(0) = 0$ and

$$(3.1) \quad \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$(3.2) \quad \|f(2x_1) - 8f(x_1) - A(x_1), \dots, f(2x_t) - 8f(x_t) - A(x_t)\|_t \\ \leq \frac{1}{2-a}\psi(x_1, \dots, x_t),$$

where $\psi : X^t \rightarrow [0, \infty)$ is a mapping defined by

$$(3.3)$$

$$\psi(x_1, \dots, x_t) = \frac{1}{k^3 - k} [(1+k)\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t)$$

$$\begin{aligned}
& + \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t)] \\
& + \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t) \\
& + (1+8k^2)\phi(x_1, \dots, x_t, x_1, \dots, x_t) \\
& + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
& + \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
& + k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
& + \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\
& + 2\phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
& + 2\phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
& + 2\phi(2x_1, \dots, 2x_t, x_1, \dots, x_t) \\
& + 2\phi(2x_1, \dots, 2x_t, kx_1, \dots, kx_t) \\
& + 8\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{kx_1}{2}, \dots, \frac{kx_t}{2}\right) \\
& + 8k\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{(2k-1)x_1}{2}, \dots, \frac{(2k-1)x_t}{2}\right) \\
& + \frac{2}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) \\
& + 8\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{3kx_1}{2}, \dots, \frac{3kx_t}{2}\right) \\
& + \frac{k+1}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
& + 8k\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{(2k+1)x_1}{2}, \dots, \frac{(2k+1)x_t}{2}\right) \\
& + \frac{1+8k^2}{k-1}\phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t) \\
& + \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\
& + \frac{k^2}{k-1}\phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t) \\
& + \frac{k^2+k-1}{k-1}\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t) \\
& + \frac{8k}{k-1}\phi\left(0, \dots, 0, \frac{(3k-1)x_1}{2}, \dots, \frac{(3k-1)x_t}{2}\right) \\
& + \frac{8k}{k-1}\phi\left(0, \dots, 0, \frac{(k+1)x_1}{2}, \dots, \frac{(k+1)x_t}{2}\right)
\end{aligned}$$

$$+ \frac{8k^2 + 2k - 8}{k - 1} \phi(0, \dots, 0, kx_1, \dots, kx_t)$$

for all $x_1, \dots, x_t \in X$.

Proof. Let $x_1, \dots, x_t \in X$. Letting $x_1 = \dots = x_t = 0$ in (3.1), we obtain

(3.4)

$$\|f(y_1) + f(-y_1), \dots, f(y_t) + f(-y_t)\|_t \leq \frac{1}{k - 1} \phi(0, \dots, 0, y_1, \dots, y_t),$$

for all $y_1, \dots, y_t \in X$. Letting $y_1 = x_1, \dots, y_t = x_t$ in (3.1), we get

$$(3.5) \quad \begin{aligned} & \|f((k + 1)x_1) + f((k - 1)x_1) - kf(2x_1) - 2f(kx_1) \\ & \quad + 2kf(x_1), \dots, f((k + 1)x_t) + f((k - 1)x_t) \\ & \quad - kf(2x_t) - 2f(kx_t) + 2kf(x_t)\|_t \\ & \leq \phi(x_1, \dots, x_t, x_1, \dots, x_t). \end{aligned}$$

Hence

$$(3.6) \quad \begin{aligned} & \|f(2(k + 1)x_1) + f(2(k - 1)x_1) - kf(4x_1) - 2f(2kx_1) \\ & \quad + 2kf(2x_1), \dots, f(2(k + 1)x_t) + f(2(k - 1)x_t) \\ & \quad - kf(4x_t) - 2f(2kx_t) + 2kf(2x_t)\|_t \\ & \leq \phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t). \end{aligned}$$

Letting $y_1 = kx_1, \dots, y_t = kx_t$ in (3.1), we get

$$(3.7) \quad \begin{aligned} & \|f(2kx_1) - kf((k + 1)x_1) - kf(-(k - 1)x_1) - 2f(kx_1) \\ & \quad + 2kf(x_1), \dots, f(2kx_t) - kf((k + 1)x_t) \\ & \quad - kf(-(k - 1)x_t) - 2f(kx_t) + 2kf(x_t)\|_t \\ & \leq \phi(x_1, \dots, x_t, kx_1, \dots, kx_t). \end{aligned}$$

Letting $y_1 = (k + 1)x_1, \dots, y_t = (k + 1)x_t$ in (3.1), we have

$$(3.8) \quad \begin{aligned} & \|f((2k + 1)x_1) + f(-x_1) - kf((k + 2)x_1) - kf(-kx_1) \\ & \quad - 2f(kx_1) + 2kf(x_1), \dots, f((2k + 1)x_t) + f(-x_t) \\ & \quad - kf((k + 2)x_t) - kf(-kx_t) - 2f(kx_t) + 2kf(x_t)\|_t \\ & \leq \phi(x_1, \dots, x_t, (k + 1)x_1, \dots, (k + 1)x_t). \end{aligned}$$

Letting $y_1 = (k - 1)x_1, \dots, y_t = (k - 1)x_t$ in (3.1), we have

$$(3.9) \quad \begin{aligned} & \|f((2k - 1)x_1 - (k + 2)f(kx_1) - kf(-(k - 2)x_1) \\ & \quad + (2k + 1)f(x_1), \dots, f((2k - 1)x_t - (k + 2)f(kx_t) \end{aligned}$$

$$\begin{aligned} & -kf(-(k-2)x_t) + (2k+1)f(x_t)\|_t \\ & \leq \phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t). \end{aligned}$$

Replacing $x_1 = 2x_1, \dots, x_t = 2x_t, y_1 = x_1, \dots, y_t = x_t$ in (3.1), we have

$$\begin{aligned} (3.10) \quad & \|f((2k+1)x_1) + f((2k-1)x_1) - 2f(2kx_1) - kf(3x_1) \\ & + 2kf(2x_1) - kf(x_1), \dots, f((2k+1)x_t) + f((2k-1)x_t) \\ & - 2f(2kx_t) - kf(3x_t) + 2kf(2x_t) - kf(x_t)\|_t \\ & \leq \phi(2x_1, \dots, 2x_t, x_1, \dots, x_t). \end{aligned}$$

Letting $x_1 = 3x_1, \dots, x_t = 3x_t, y_1 = x_1, \dots, y_t = x_t$ in (3.1), we obtain

$$\begin{aligned} (3.11) \quad & \|f((3k+1)x_1) + f((3k-1)x_1) - 2f(3kx_1) - kf(4x_1) \\ & - kf(2x_1) + 2kf(3x_1), \dots, f((3k+1)x_t) + f((3k-1)x_t) \\ & - 2f(3kx_t) - kf(4x_t) - kf(2x_t) + 2kf(3x_t)\|_t \\ & \leq \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t). \end{aligned}$$

Setting $x_1 = 2x_1, \dots, x_t = 2x_t, y_1 = kx_1, \dots, y_t = kx_t$ in (3.1), we obtain

$$\begin{aligned} (3.12) \quad & \|f(3kx_1) + f(kx_1) - kf((k+2)x_1) - kf(-(k-2)x_1) \\ & - 2f(2kx_1) + 2kf(2x_1), \dots, f(3kx_t) + f(kx_t) - kf((k+2)x_t) \\ & - kf(-(k-2)x_t) - 2f(2kx_t) + 2kf(2x_t)\|_t \\ & \leq \phi(2x_1, \dots, 2x_t, kx_1, \dots, kx_t). \end{aligned}$$

Letting $y_1 = (2k+1)x_1, \dots, y_t = (2k+1)x_t$ in (3.1), we have

$$\begin{aligned} (3.13) \quad & \|f((3k+1)x_1) + f(-(k+1)x_1) - kf(2(k+1)x_1) \\ & - kf(-2kx_1) - 2f(kx_1) + 2kf(x_1), \dots, f((3k+1)x_t) \\ & + f(-(k+1)x_t) - kf(2(k+1)x_t) \\ & - kf(-2kx_t) - 2f(kx_t) + 2kf(x_t)\|_t \\ & \leq \phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t). \end{aligned}$$

Letting $y_1 = (2k-1)x_1, \dots, y_t = (2k-1)x_t$ in (3.1), we have

$$\begin{aligned} (3.14) \quad & \|f((3k-1)x_1) + f(-(k-1)x_1) - kf(-2(k-1)x_1) \\ & - kf(2kx_1) - 2f(kx_1) + 2kf(x_1), \dots, f((3k-1)x_t) \end{aligned}$$

$$\begin{aligned}
 & + f(-(k-1)x_t) - kf(-2(k-1)x_t) \\
 & - kf(2kx_t) - 2f(kx_t) + 2kf(x_t) \|_t \\
 & \leq \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t).
 \end{aligned}$$

Putting $y_1 = 3kx_1, \dots, y_t = 3kx_t$ in (3.1), we have

$$\begin{aligned}
 (3.15) \quad & \|f(4kx_1) + f(-2kx_1) - kf((3k+1)x_1) - kf(-(3k-1)x_1) \\
 & - 2f(kx_1) + 2kf(x_1), \dots, f(4kx_t) + f(-2kx_t) \\
 & - kf((3k+1)x_t) - kf(-(3k-1)x_t) \\
 & - 2f(kx_t) + 2kf(x_t) \|_t \\
 & \leq \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t).
 \end{aligned}$$

It follows from (3.4), (3.5), (3.11), (3.13) and (3.14) that

$$\begin{aligned}
 (3.16) \quad & \|kf(2(k+1)x_1) + kf(-2(k-1)x_1) + 6f(kx_1) - 2f(3kx_1) \\
 & - kf(4x_1) + 2kf(3x_1) - 6kf(x_1), \dots, kf(2(k+1)x_t) \\
 & + kf(-2(k-1)x_t) + 6f(kx_t) - 2f(3kx_t) \\
 & - kf(4x_t) + 2kf(3x_t) - 6kf(x_t) \|_t \\
 & \leq \phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\
 & + \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t) \\
 & + \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t) \phi(x_1, \dots, x_t, x_1, \dots, x_t) \\
 & + \frac{1}{k-1} \phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
 & \times \frac{1}{k-1} \phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t) \\
 & + \frac{k}{k-1} \phi(0, \dots, 0, 2kx_1, \dots, 2kx_t).
 \end{aligned}$$

By using (3.4), (3.8) and (3.9) we deduce that

$$\begin{aligned}
 (3.17) \quad & \|f((2k+1)x_1) + f((2k-1)x_1) - kf((k+2)x_1) - kf(-(k-2)x_1) \\
 & - 4f(kx_1) + 4kf(x_1), \dots, f((2k+1)x_t) + f((2k-1)x_t) \\
 & - kf((k+2)x_t) - kf(-(k-2)x_t) - 4f(kx_t) + 4kf(x_t) \|_t \\
 & \leq \phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
 & + \phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
 & + \frac{1}{k-1} \phi(0, \dots, 0, x_1, \dots, x_t) + \frac{k}{k-1} \phi(0, \dots, 0, kx_1, \dots, kx_t).
 \end{aligned}$$

It follows from (3.10) and (3.17) that

$$\begin{aligned}
(3.18) \quad & \|kf((k+2)x_1) + kf(-(k-2)x_1) - 2f(2kx_1) + 4f(kx_1) \\
& - kf(3x_1) + 2kf(2x_1) - 5kf(x_1), \dots, kf((k+2)x_t) \\
& + kf(-(k-2)x_t) - 2f(2kx_t) + 4f(kx_t) \\
& - kf(3x_t) + 2kf(2x_t) - 5kf(x_t)\|_t \\
& \leq \phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
& + \phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
& + \phi(2x_1, \dots, 2x_t, x_1, \dots, x_t) + \frac{1}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) \\
& + \frac{k}{k-1}\phi(0, \dots, 0, kx_1, \dots, kx_t).
\end{aligned}$$

From (3.12) and (3.18) we have

$$\begin{aligned}
(3.19) \quad & \|f(3kx_1) - 4f(2kx_1) + 5f(kx_1) - kf(3x_1) + 4kf(2x_1) \\
& - 5kf(x_1), \dots, f(3kx_t) - 4f(2kx_t) + 5f(kx_t) \\
& - kf(3x_t) + 4kf(2x_t) - 5kf(x_t)\|_t \\
& \leq \phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
& + \phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
& + \phi(2x_1, \dots, 2x_t, x_1, \dots, x_t) + \phi(2x_1, \dots, 2x_t, kx_1, \dots, kx_t) \\
& + \frac{1}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) + \frac{k}{k-1}\phi(0, \dots, 0, kx_1, \dots, kx_t).
\end{aligned}$$

It follows from (3.4), (3.13), (3.14) and (3.15) that

$$\begin{aligned}
(3.20) \quad & \|kf(-(k+1)x_1) - kf(-(k-1)x_1) - k^2f(2(k+1)x_1) \\
& + k^2f(-2(k-1)x_1) + k^2f(2kx_1) - (k^2-1)f(-2kx_1) \\
& + f(4kx_1) - 2f(kx_1) + 2kf(x_1), \dots, kf(-(k+1)x_t) \\
& - kf(-(k-1)x_t) - k^2f(2(k+1)x_t) + k^2f(-2(k-1)x_t) \\
& + k^2f(2kx_t) - (k^2-1)f(-2kx_t) + f(4kx_t) \\
& - 2f(kx_t) + 2kf(x_t)\|_t \\
& \leq k\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\
& + k\phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t) \\
& + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
& + \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t).
\end{aligned}$$

By (3.4), (3.6), (3.7) and (3.20), we get

$$\begin{aligned}
 (3.21) \quad & \|f(4kx_1) - 2f(2kx_1) - k^3f(4x_1) + 2k^3f(2x_1), \dots, f(4kx_t) \\
 & - 2f(2kx_t) - k^3f(4x_t) + 2k^3f(2x_t)\|_t \\
 & \leq k\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\
 & \quad + k\phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t) \\
 & \quad + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
 & \quad + \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
 & \quad + k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
 & \quad + \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\
 & \quad + \frac{k}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
 & \quad + \frac{k^2}{k-1}\phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t) \\
 & \quad + (k+1)\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t).
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 (3.22) \quad & \|f(2kx_1) - 2f(kx_1) - k^3f(2x_1) + 2k^3f(x_1), \dots, f(2kx_t) \\
 & - 2f(kx_t) - k^3f(2x_t) + 2k^3f(x_t)\|_t \\
 & \leq k\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{(2k+1)x_1}{2}, \dots, \frac{(2k+1)x_t}{2}\right) \\
 & \quad + k\phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{(2k-1)x_1}{2}, \dots, \frac{(2k-1)x_t}{2}\right) \\
 & \quad + \phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{3kx_1}{2}, \dots, \frac{3kx_t}{2}\right) \\
 & \quad + \phi\left(\frac{x_1}{2}, \dots, \frac{x_t}{2}, \frac{kx_1}{2}, \dots, \frac{kx_t}{2}\right) \\
 & \quad + k^2\phi(x_1, \dots, x_t, x_1, \dots, x_t) \\
 & \quad + \frac{k}{k-1}\phi\left(0, \dots, 0, \frac{(3k-1)x_1}{2}, \dots, \frac{(3k-1)x_t}{2}\right) \\
 & \quad + \frac{k}{k-1}\phi\left(0, \dots, 0, \frac{(k+1)x_1}{2}, \dots, \frac{(k+1)x_t}{2}\right) \\
 & \quad + \frac{k^2}{k-1}\phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t)
 \end{aligned}$$

$$+ (k+1)\phi(0, \dots, 0, kx_1, \dots, kx_t).$$

By using (3.7), we obtain

$$(3.23) \quad \begin{aligned} & \|f(4kx_1) - kf(2(k+1)x_1) - kf(-2(k-1)x_1) \\ & \quad - 2f(2kx_1) + 2kf(2x_1), \dots, f(4kx_t) - kf(2(k+1)x_t) \\ & \quad - kf(-2(k-1)x_t) - 2f(2kx_t) + 2kf(2x_t)\|_t \\ & \leq \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t). \end{aligned}$$

It follows from (3.21) and (3.23) that

$$(3.24) \quad \begin{aligned} & \|kf(2(k+1)x_1) + kf(-2(k-1)x_1) - k^3f(4x_1) \\ & \quad + (2k^3 - 2k)f(2x_1), \dots, kf(2(k+1)x_t) \\ & \quad + kf(-2(k-1)x_t) - k^3f(4x_t) + (2k^3 - 2k)f(2x_t)\|_t \\ & \leq k\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\ & \quad + k\phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t) \\ & \quad + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\ & \quad + \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\ & \quad + k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\ & \quad + \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\ & \quad + \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\ & \quad + \frac{k}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\ & \quad + \frac{k^2}{k-1}\phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t) \\ & \quad + (k+1)\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t). \end{aligned}$$

Using (3.16) and (3.24), we deduce that

$$(3.25) \quad \begin{aligned} & \|2f(3kx_1) - 6f(kx_1) + (k - k^3)f(4x_1) - 2kf(3x_1) \\ & \quad + (2k^3 - 2k)f(2x_1) + 6kf(x_1), \dots, 2f(3kx_t) \\ & \quad - 6f(kx_t) + (k - k^3)f(4x_t) - 2kf(3x_t) \\ & \quad + (2k^3 - 2k)f(2x_t) + 6kf(x_t)\|_t \\ & \leq (1+k)[\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\ & \quad + \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t)] \\ & \quad + \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t) + \phi(x_1, \dots, x_t, x_1, \dots, x_t) \end{aligned}$$

$$\begin{aligned}
& + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
& + \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
& + k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
& + \frac{1}{k-1}\phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t) \\
& + \frac{k+1}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
& + \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\
& + \frac{(k^2+k-1)}{k-1}\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t) \\
& + \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\
& + \frac{k^2}{k-1}\phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t).
\end{aligned}$$

Moreover from (3.19) and (3.25), we have

$$\begin{aligned}
(3.26) \quad & \|8f(2kx_1) - 16f(kx_1) + (k-k^3)f(4x_1) + (2k^3-10k)f(2x_1) \\
& + 16kf(x_1), \dots, f(2kx_t) - 16f(kx_t) + (k-k^3)f(4x_t) \\
& + (2k^3-10k)f(2x_t) + 16kf(x_t)\|_t \\
& \leq (1+k)[\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\
& + \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t)] \\
& + \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t) \\
& + \phi(x_1, \dots, x_t, x_1, \dots, x_t) \\
& + \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
& + \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
& + k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
& + \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\
& + 2\phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
& + 2\phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
& + 2\phi(2x_1, \dots, 2x_t, x_1, \dots, x_t) \\
& + 2\phi(2x_1, \dots, 2x_t, kx_1, \dots, kx_t) \\
& + \frac{2}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) \\
& + \frac{2k}{k-1}\phi(0, \dots, 0, kx_1, \dots, kx_t)
\end{aligned}$$

$$\begin{aligned}
& + \frac{k+1}{k-1} \phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
& + \frac{1}{k-1} \phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t) \\
& + \frac{k^2+k-1}{k-1} \phi(0, \dots, 0, 2kx_1, \dots, 2kx_t) \\
& + \frac{k}{k-1} \phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\
& + \frac{k^2}{k-1} \phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t).
\end{aligned}$$

So by (3.22) and (3.26), we get

$$\begin{aligned}
& \|f(4x_1) - 10f(2x_1) + 16f(x_1), \dots, f(4x_t) - 10f(2x_t) + 16f(x_t)\|_t \\
& \leq \psi(x_1, \dots, x_t),
\end{aligned}$$

for all $x_1, \dots, x_t \in X$. Let $g : X \rightarrow Y$ be the mapping defined by $g(x) := f(2x) - 8f(x)$. Therefore we have

$$\|g(2x_1) - 2g(x_1), \dots, g(2x_t) - 2g(x_t)\|_t \leq \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. It is easy to see that the mapping $A : X \rightarrow Y$ given by

$$A(x) := \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^n},$$

satisfies

$$\|g(x_1) - A(x_1), \dots, g(x_t) - A(x_t)\|_t \leq \frac{1}{2-a} \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. So (3.2), is satisfied. Moreover the mapping A satisfies (1.2) and $A(2x) = 2A(x)$ for all $x \in X$. By Lemma 3.1 the mapping A is additive. The uniqueness assertion is easy and we omit its proof. \square

Theorem 3.3. *Let $t \in \mathbb{N}$ and $\phi : X^{2t} \rightarrow [0, \infty)$ be a function such that*

$$\phi(2x_1, \dots, 2x_t, 2y_1, \dots, 2y_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < 8$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f : X \rightarrow Y$ is a mapping such that $f(0) = 0$ and

$$\|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned}
(3.27) \quad & \|f(2x_1) - 2f(x_1) - C(x_1), \dots, f(2x_t) - 2f(x_t) - C(x_t)\|_t \\
& \leq \frac{1}{8-a} \psi(x_1, \dots, x_t),
\end{aligned}$$

where $\psi : X^t \rightarrow [0, \infty)$ is a mapping defined by (3.3) for all $x_1, \dots, x_t \in X$.

Proof. We know from Theorem 3.2 that

$$\begin{aligned} & \|f(4x_1) - 10f(2x_1) + 16f(x_1), \dots, f(4x_t) - 10f(2x_t) + 16f(x_t)\|_t \\ & \leq \psi(x_1, \dots, x_t), \end{aligned}$$

for all $x_1, \dots, x_t \in X$. Let $h : X \rightarrow Y$ be the mapping defined by $h(x) := f(2x) - 2f(x)$. Therefore, we have

$$\|h(2x_1) - 8h(x_1), \dots, h(2x_t) - 8h(x_t)\|_t \leq \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. It is easy to see that the mapping $C : X \rightarrow Y$ given by

$$C(x) := \lim_{n \rightarrow \infty} \frac{h(2^n x)}{8^n},$$

satisfies

$$\|h(x_1) - C(x_1), \dots, h(x_t) - C(x_t)\|_t \leq \frac{1}{8-a} \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. So (3.27), is satisfied. Moreover the mapping C satisfies (1.2) and $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 3.1 the mapping C is cubic. The uniqueness assertion is easy and we omit its proof. \square

Theorem 3.4. Let $t \in \mathbb{N}$ and $\phi : X^{2t} \rightarrow [0, \infty)$ be a function such that

$$\phi(2x_1, \dots, 2x_t, 2y_1, \dots, 2y_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < 2$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f : X \rightarrow Y$ is a mapping such that $f(0) = 0$ and

$$\|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} (3.28) \quad & \|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \\ & \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi(x_1, \dots, x_t), \end{aligned}$$

where $\psi : X^t \rightarrow [0, \infty)$ is a mapping defined by (3.3) for all $x_1, \dots, x_t \in X$.

Proof. By Theorems 3.2 and 3.3, there exist a unique additive mapping $A_0 : X \rightarrow Y$ and a cubic mapping $C_0 : X \rightarrow Y$ such that

$$\begin{aligned} & \|f(2x_1) - 8f(x_1) - A_0(x_1), \dots, f(2x_t) - 8f(x_t) - A_0(x_t)\|_t \\ & \leq \frac{1}{2-a} \psi(x_1, \dots, x_t), \end{aligned}$$

and

$$\begin{aligned} & \|f(2x_1) - 2f(x_1) - C_0(x_1), \dots, f(2x_t) - 2f(x_t) - C_0(x_t)\|_t \\ & \leq \frac{1}{8-a} \psi(x_1, \dots, x_t), \end{aligned}$$

for all $x_1, \dots, x_t \in X$. Hence

$$\begin{aligned} & \|f(x_1) + \frac{1}{6}A_0(x_1) - \frac{1}{6}C_0(x_1), \dots, f(x_t) + \frac{1}{6}A_0(x_t) - \frac{1}{6}C_0(x_t)\|_t \\ & \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi(x_1, \dots, x_t), \end{aligned}$$

for all $x_1, \dots, x_t \in X$. Letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$, we obtain (3.28). For the uniqueness assertion, let A_1, C_1 be another additive and cubic mapping satisfying (3.28). Let $A' = A - A_1$ and $C' = C - C_1$. So

(3.29)

$$\begin{aligned} \|A'(x) + C'(x)\| & \leq \|f(x) - A(x) - C(x)\| + \|f(x) - A_1(x) - C_1(x)\| \\ & \leq \frac{1}{3} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi(x, \dots, x), \end{aligned}$$

for all $x \in X$. So

$$\frac{1}{8^n} \|A'(2^n x) + C'(2^n x)\| \leq \left(\frac{a}{8}\right)^n \left(\frac{1}{3}\right) \left(\frac{1}{2-a} + \frac{1}{8-a}\right) \psi(x, \dots, x),$$

for all $x \in X$. Hence $C' = 0$. It follows from (3.29) that

$$\|A'(x)\| \leq \frac{1}{3} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi(x, \dots, x),$$

for all $x \in X$. Hence $A' = 0$. \square

Following the same idea as in the proof of Theorems 3.2, 3.3 and 3.4, we get the following corollary.

Corollary 3.5. *Let $\alpha \geq 0$ and $f : X \rightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\sup_{t \in \mathbb{N}} \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exist a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} & \sup_{t \in \mathbb{N}} \|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \\ & \leq \frac{4}{21} \left(\frac{\alpha}{k^3 - k} \right) \left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k - 1} \right], \end{aligned}$$

for all $x_1, \dots, x_t \in X$.

Theorem 3.6. *Let $\alpha \geq 0$ and $f, g : X \rightarrow Y$ be two mappings such that $f(0) = g(0) = 0$ and*

$$\sup_{t \in \mathbb{N}} \|D(f, g)(x_1, y_1), \dots, D(f, g)(x_t, y_t)\|_t \leq \alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exist a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} & \sup_{t \in \mathbb{N}} \|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \\ & \leq \frac{4}{21} \left(\frac{2\alpha}{k(k-1)} \right) \left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1} \right], \end{aligned}$$

and

$$\begin{aligned} & \sup_{t \in \mathbb{N}} \|g(x_1) - kA(x_1) - kC(x_1), \dots, g(x_t) - kA(x_t) - kC(x_t)\|_t \\ & \leq \frac{4}{21} \left(\frac{2\alpha}{k^2 - 1} \right) \left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1} \right], \end{aligned}$$

where $x_1, \dots, x_t \in X$.

Proof. It is easy to see that

$$\begin{aligned} (3.30) \quad Df(x, y) &= D(f, g)(x, y) - D(f, g)(x, 0) - \frac{k}{2}D(f, g)\left(\frac{x+y}{k}, 0\right) \\ &\quad - \frac{k}{2}D(f, g)\left(\frac{x-y}{k}, 0\right) \\ &\quad + kD(f, g)\left(\frac{x}{k}, 0\right), \end{aligned}$$

for all $x, y \in X$. Using (3.30), we get

$$\sup_{t \in \mathbb{N}} \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq 2(k+1)\alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. It follows from Theorem 3.5 that there exist a unique additive mapping $A_1 : X \rightarrow Y$ and a unique cubic mapping $C_1 : X \rightarrow Y$ such that

$$\begin{aligned} & \sup_{t \in \mathbb{N}} \|f(x_1) - A_1(x_1) - C_1(x_1), \dots, f(x_t) - A_1(x_t) - C_1(x_t)\|_t \\ & \leq \frac{4}{21} \left(\frac{2\alpha}{k(k-1)} \right) \left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1} \right], \end{aligned}$$

for all $x_1, \dots, x_t \in X$. Moreover we see that

$$(3.31) \quad Dg(x, y) = kD(f, g)(x, y) - \frac{k}{2}D(f, g)\left(\frac{kx+y}{k}, 0\right)$$

$$-\frac{k}{2}D(f, g)\left(\frac{kx-y}{k}, 0\right),$$

for all $x, y \in X$. It follows from (3.31) that

$$\sup_{t \in \mathbb{N}} \|Dg(x_1, y_1), \dots, Dg(x_t, y_t)\|_t \leq 2k\alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Again using Theorem 3.5, we conclude that there exist a unique additive mapping $A_2 : X \rightarrow Y$ and a unique cubic mapping $C_2 : X \rightarrow Y$ such that

$$\begin{aligned} \sup_{t \in \mathbb{N}} \|g(x_1) - A_2(x_1) - C_2(x_1), \dots, g(x_t) - A_2(x_t) - C_2(x_t)\|_t \\ \leq \frac{4}{21} \left(\frac{2\alpha}{k^2 - 1} \right) \left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k - 1} \right], \end{aligned}$$

for all $x_1, \dots, x_t \in X$.

We show that $A_2 = kA_1$. To this end we have

$$\frac{k}{2}D(f, g)\left(\frac{x}{k}, 0\right) = kf(x) - g(x),$$

for all $x \in X$. Therefore, we get

$$\begin{aligned} \|kA_1(x) - A_2(x)\| &\leq \left(\frac{1}{6}\right) \left(\frac{1}{2^n}\right) \lim_{n \rightarrow \infty} \frac{k}{2} \left\| D(f, g)\left(\frac{2^{n+1}x}{k}, 0\right) \right\| \\ &\quad + 8\frac{k}{2} \left\| D(f, g)\left(\frac{2^n x}{k}, 0\right) \right\|. \end{aligned}$$

So $A_2 = kA_1$. Similarly $C_2 = kC_1$. The proof of the uniqueness assertion is similar to the ones of Theorem 3.4. \square

We prove the generalized Hyers-Ulam stability of the Pexiderized cubic functional equation in multi-normed spaces

Theorem 3.7. *Let $t \in \mathbb{N}$ and $\phi : X^{2t} \rightarrow [0, \infty)$ be a function such that*

$$\phi(2x_1, \dots, 2x_t, 2y_1, \dots, 2y_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < 2$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f, g : X \rightarrow Y$ are two mappings such that $f(0) = g(0) = 0$ and

$$\|D(f, g)(x_1, y_1), \dots, D(f, g)(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

and $\phi_1, \phi_2 : X^{2t} \rightarrow [0, \infty)$ are two mappings defined by

$$\begin{aligned} \phi_1(x_1, \dots, x_t, y_1, \dots, y_t) &:= \phi(x_1, \dots, x_t, y_1, \dots, y_t) \\ &\quad + \phi(x_1, \dots, x_t, 0, \dots, 0) \\ &\quad + \frac{k}{2}\phi\left(\frac{x_1 + y_1}{k}, \dots, \frac{x_t + y_t}{k}, 0, \dots, 0\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{k}{2} \phi \left(\frac{x_1 - y_1}{k}, \dots, \frac{x_t - y_t}{k}, 0, \dots, 0 \right) \\
& + k \phi \left(\frac{x_1}{k}, \dots, \frac{x_t}{k}, 0, \dots, 0 \right),
\end{aligned}$$

and

$$\begin{aligned}
\phi_2(x_1, \dots, x_t, y_1, \dots, y_t) & := k \phi(x_1, \dots, x_t, y_1, \dots, y_t) \\
& + \frac{k}{2} \phi \left(\frac{kx_1 + y_1}{k}, \dots, \frac{kx_t + y_t}{k}, 0, \dots, 0 \right) \\
& + \frac{k}{2} \phi \left(\frac{kx_1 - y_1}{k}, \dots, \frac{kx_t - y_t}{k}, 0, \dots, 0 \right),
\end{aligned}$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exist a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned}
& \|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \\
& \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi_1(x_1, \dots, x_t),
\end{aligned}$$

and

$$\begin{aligned}
& \|g(x_1) - kA(x_1) - kC(x_1), \dots, g(x_t) - kA(x_t) - kC(x_t)\|_t \\
& \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi_2(x_1, \dots, x_t),
\end{aligned}$$

for all $x_1, \dots, x_t \in X$ where $\psi_1, \psi_2 : X^t \rightarrow [0, \infty)$ are two mappings defined similar to the definition of ψ in Theorem 3.2 with this difference that ϕ (in the definition of ψ) is replaced by ϕ_1 and ϕ_2 anywhere respectively.

Proof. It follows from (3.30) that

$$\|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi_1(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Using Theorem 3.4, we conclude that there exist a unique additive mapping $A_1 : X \rightarrow Y$ and a unique cubic mapping $C_1 : X \rightarrow Y$ such that

$$\begin{aligned}
& \|f(x_1) - A_1(x_1) - C_1(x_1), \dots, f(x_t) - A_1(x_t) - C_1(x_t)\|_t \\
& \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi_1(x_1, \dots, x_t),
\end{aligned}$$

for all $x_1, \dots, x_t \in X$. Moreover, using (3.31), we get

$$\|Dg(x_1, y_1), \dots, Dg(x_t, y_t)\|_t \leq \phi_2(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Using Theorem 3.4 again, we deduce that there exist a unique additive mapping $A_2 : X \rightarrow Y$ and a unique cubic mapping $C_2 : X \rightarrow Y$ such that

$$\begin{aligned} & \|f(x_1) - A_2(x_1) - C_2(x_1), \dots, f(x_t) - A_2(x_t) - C_2(x_t)\|_t \\ & \leq \frac{1}{6} \left(\frac{1}{2-a} + \frac{1}{8-a} \right) \psi_2(x_1, \dots, x_t), \end{aligned}$$

for all $x_1, \dots, x_t \in X$.

Similar to the proof of Theorem 3.6, we have $A_2 = kA_1$ and $C_2 = kC_1$. The proof of the uniqueness assertion is similar to ones proof of Theorem 3.4. \square

4. STABILITY OF THE 2-VARIABLES CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the 2-variables, cubic functional equation

$$(4.1) \quad \begin{aligned} f(2x + y, 2z + t) + f(2x - y, 2z - t) &= 2f(x + y, z + t) \\ &+ 2f(x - y, z - t) + 12f(x, z), \end{aligned}$$

in multi-normed spaces. Throughout this section, assume that X is a vector space.

Lemma 4.1. *If $f : X^2 \rightarrow Y$ is a 2-variables cubic mapping, then $f(2^n x, 2^n y) = 8^n f(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$.*

Proof. Letting $y = t = 0$ in (4.1), we get $f(2x, 2z) = 8f(x, z)$ for all $x, z \in X$. So by induction we have $f(2^n x, 2^n y) = 8^n f(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$. \square

Given a mapping $f : X^2 \rightarrow Y$, we set

$$\begin{aligned} Df(x, y, z, t) &= f(2x + y, 2z + t) + f(2x - y, 2z - t) - 2f(x + y, z + t) \\ &\quad - 2f(x - y, z - t) - 12f(x, z), \end{aligned}$$

for all $x, y, z, t \in X$.

Theorem 4.2. *Let $\alpha \geq 0$ and $f : X \rightarrow Y$ be a mapping satisfying*

$$(4.2) \quad \sup_{k \in \mathbb{N}} \|Df(x_1, y_1, z_1, t_1), \dots, Df(x_k, y_k, z_k, t_k)\|_k \leq \alpha,$$

for all $x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, t_1, \dots, t_k \in X$. Then there exists a unique 2-variable cubic mapping $C : X \rightarrow Y$ such that

$$(4.3) \quad \sup_{k \in \mathbb{N}} \|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k \leq \frac{\alpha}{14},$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$.

Proof. Let $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Letting $z_1 = y_1, z_2 = y_2, \dots, z_k = y_k$ and $y_1 = \dots = y_k = t_1 = \dots = t_k = 0$ in (4.2), we get

$$\sup_{k \in \mathbb{N}} \|f(2x_1, 2y_1) - 8f(x_1, y_1), \dots, f(2x_k, 2y_k) - 8f(x_k, y_k)\|_k \leq \frac{\alpha}{2},$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$.

Replacing $x_1, \dots, x_k, y_1, \dots, y_k$ by $2^n x_1, \dots, 2^n x_k, 2^n y_1, \dots, 2^n y_k$ and dividing the last inequality by 8^{n+1} , one gets

$$\begin{aligned} \sup_{k \in \mathbb{N}} \left\| \frac{f(2^{n+1}x_1, 2^{n+1}y_1)}{8^{n+1}} - \frac{f(2^n x_1, 2^n y_1)}{8^n}, \dots, \frac{f(2^{n+1}x_k, 2^{n+1}y_k)}{8^{n+1}} \right. \\ \left. - \frac{f(2^n x_k, 2^n y_k)}{8^n} \right\|_k \leq \frac{\alpha}{2} \left(\frac{1}{8^{n+1}} \right). \end{aligned}$$

It follows that $\left(\frac{f(2^n x, 2^n y)}{8^n} \right)$ is Cauchy and so is convergent in the complete multi-normed space Y . Set

$$C(x, y) := \lim_{n \rightarrow \infty} \frac{f(2^n x, 2^n y)}{8^n}.$$

It is easy to see that

$$\sup_{k \in \mathbb{N}} \|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k \leq \frac{\alpha}{14}.$$

Let $x, y, z, t \in X$. Replacing $x_1 = \dots = x_k = 2^n x, y_1 = \dots = y_k = 2^n y, z_1 = \dots = z_k = 2^n z, t_1 = \dots = t_k = 2^n t$ in (4.2) and dividing the both sides by 8^n , we obtain

$$\frac{1}{8^n} \|Df(2^n x, 2^n y, 2^n z, 2^n t)\| \leq \frac{\alpha}{8^n}.$$

Taking the limit as $n \rightarrow \infty$, we get

$$DC(x, y, z, t) = 0.$$

Hence C is a 2-variables cubic mapping. Let C' be another 2-variables cubic mapping satisfying (4.3). So by Lemma 4.1 we have

$$\begin{aligned} \|C(x, y) - C'(x, y)\| &= \frac{1}{8^n} \|C(2^n x, 2^n y) - C'(2^n x, 2^n y)\| \\ &\leq \frac{1}{8^n} \|C(2^n x, 2^n y) - f(2^n x, 2^n y)\| \\ &\quad + \frac{1}{8^n} \|f(2^n x, 2^n y) - C'(2^n x, 2^n y)\| \\ &\leq \frac{1}{8^n} \left(\frac{\alpha}{14} + \frac{\alpha}{14} \right). \end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. \square

We prove the generalized Hyers-Ulam stability of the 2-variables cubic functional equation in multi-normed spaces by using the fixed point method.

Theorem 4.3. *Let $k \in \mathbb{N}$ and $\phi : X^{4k} \rightarrow [0, \infty)$ be a function such that*

$$\begin{aligned} &\phi(2x_1, \dots, 2x_k, 2y_1, \dots, 2y_k, 2z_1, \dots, 2z_k, 2t_1, \dots, 2t_k) \\ &\leq a\phi(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, t_1, \dots, t_k), \end{aligned}$$

for some $a > 0$ with $a < 8$ and all

$$x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, t_1, \dots, t_k \in X.$$

Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$(4.4) \quad \begin{aligned} &\|Df(x_1, y_1, z_1, t_1), \dots, Df(x_k, y_k, z_k, t_k)\|_k \\ &\leq \phi(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, t_1, \dots, t_k), \end{aligned}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique 2-variable cubic mapping $C : X \rightarrow Y$ such that

$$(4.5) \quad \begin{aligned} &\|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k \\ &\leq \frac{1}{16 - 2a} \phi(x_1, \dots, x_k, 0, \dots, 0, y_1, \dots, y_k, 0, \dots, 0), \end{aligned}$$

for all $x_1, \dots, x_k \in X$.

Proof. Let $E = \{f : X^2 \rightarrow Y\}$, and consider the generalized metric d defined on E by

$$\begin{aligned} d(g, h) &= \inf \left\{ c > 0, \|g(x_1, y_1) - h(x_1, y_1), \dots, g(x_k, y_k) - h(x_k, y_k)\|_k \right. \\ &\quad \left. \leq c\psi(x_1, \dots, x_k, y_1, \dots, y_k), \forall x_1, \dots, x_k, y_1, \dots, y_k \in X \right\}, \end{aligned}$$

where $\psi : X^{2k} \rightarrow [0, \infty)$ is a mapping defined by

$$\psi(x_1, \dots, x_k, y_1, \dots, y_k) = \phi(x_1, \dots, x_k, 0, \dots, 0, y_1, \dots, y_k, 0, \dots, 0),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then, it is easy to show that d is a complete generalized metric on E . We now define a mapping $J : E \rightarrow E$ by

$$Jg(x, y) = \frac{1}{8}g(2x, 2y), \quad \forall x, y \in X.$$

We claim that J is a strictly contractive mapping. Given $g, h \in E$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) < c$. It is easy to see that

$$\begin{aligned} &\|g(x_1, y_1) - h(x_1, y_1), \dots, g(x_k, y_k) - h(x_k, y_k)\|_k \\ &\leq c\psi(x_1, \dots, x_k, y_1, \dots, y_k), \end{aligned}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Therefore,

$$\begin{aligned} & \|Jg(x_1, y_1) - Jh(x_1, y_1), \dots, Jg(x_k, y_k) - Jh(x_k, y_k)\|_k \\ &= \left\| \frac{1}{8}g(2x_1, 2y_1) - \frac{1}{8}h(2x_1, 2y_1), \dots, \frac{1}{8}g(2x_k, 2y_k) - \frac{1}{8}h(2x_k, 2y_k) \right\|_k \\ &\leq \frac{1}{8}c\psi(2x_1, \dots, 2x_k, 2y_1, \dots, 2y_k) \leq \frac{1}{8}ac\psi(x_1, \dots, x_k, y_1, \dots, y_k), \end{aligned}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. So $d(Jg, Jh) \leq \frac{a}{8}d(g, h)$. Letting $z_1 = y_1, z_2 = y_2, \dots, z_k = y_k$ and $y_1 = \dots = y_k = t_1 = \dots = t_k = 0$ in (4.4), we get

$$(4.6) \quad \begin{aligned} & \|f(2x_1, 2y_1) - 8f(x_1, y_1), \dots, f(2x_k, 2y_k) - 8f(x_k, y_k)\|_k \\ &\leq \frac{1}{2}\psi(x_1, \dots, x_k, y_1, \dots, y_k), \end{aligned}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. It follows from (4.6) that $d(Jf, f) \leq \frac{1}{16}$. According to Lemma 1.4, we deduce the existence of a fixed point of J , that is, the existence of a mapping $C : X \rightarrow Y$ such that $C(2x, 2y) = 8C(x, y)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \rightarrow 0$ that

$$\begin{aligned} C(x, y) &= \lim_{n \rightarrow \infty} (J^n f)(x, y) \\ &= \lim_{n \rightarrow \infty} \frac{f(2^n x, 2^n y)}{8^n}, \quad \forall x \in X. \end{aligned}$$

Moreover $d(f, C) \leq \frac{1}{1-\frac{1}{8}}d(Jf, f)$, implies the inequality

$$d(f, C) \leq \frac{1}{16 - 2a}.$$

So (4.5) is satisfied.

Let $x, y, z, t \in X$. Letting $x_1 = \dots = x_k = 2^n x, y_1 = \dots = y_k = 2^n y, z_1 = \dots = z_k = 2^n z, t_1 = \dots = t_k = 2^n t$ in (4.4) and dividing the both sides by 8^n , we obtain

$$\begin{aligned} \frac{1}{8^n} \|Df(2^n x, 2^n y, 2^n z, 2^n t)\| &\leq \frac{1}{8^n} \phi(2^n x, \dots, 2^n x, 2^n y, \dots, 2^n y, \\ &\quad 2^n z, \dots, 2^n z, 2^n t, \dots, 2^n t) \\ &\leq \left(\frac{a}{8}\right)^n \phi(x, \dots, x, y, \dots, y, z, \dots, z, t, \dots, t). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we get

$$DC(x, y, z, t) = 0.$$

Hence C is a 2-variables cubic mapping.

The uniqueness of C follows from the fact that C is the unique fixed point of J with the property that there exists $\lambda \in (0, \infty)$ such that

$$\|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k$$

$$\leq \lambda\psi(x_1, \dots, x_k, y_1, \dots, y_k),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. \square

5. STABILITY OF THE CUBIC TYPE FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the orthogonally cubic type functional equation in multi-normed spaces. Given a mapping $f : X \rightarrow Y$, we set

$$\begin{aligned} Df(x_1, x_2, x_3) &= f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) \\ &\quad + f(2x_1) + f(2x_2) + 7[f(x_1) + f(-x_1)] - 2f(x_1 + x_2) \\ &\quad - 4[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)], \end{aligned}$$

for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ ($i, j = 1, 2, 3$).

We need the following lemma. The proof is found in [2].

Lemma 5.1. *Let X and Y be an orthogonal space and a real vector space, respectively. If a mapping $f : X \rightarrow Y$ satisfies the functional equation (1.5) for all $x_1, x_2, x_3 \in X$ with $x_i \perp x_j$ ($i, j = 1, 2, 3$), then C is orthogonally cubic, where $C : X \rightarrow Y$ is a mapping defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Theorem 5.2. *Let $\alpha \geq 0$ and $f : X \rightarrow Y$ be a mapping satisfying*

$$(5.1) \quad \sup_{k \in \mathbb{N}} \|Df(x_{11}, x_{21}, x_{31}), \dots, Df(x_{1k}, x_{2k}, x_{3k})\|_k \leq \alpha,$$

for all $x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k} \in X$ with $x_{in} \perp x_{jn}$ ($i, j = 1, 2, 3, n = 1, \dots, k$). Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$(5.2) \quad \sup_{k \in \mathbb{N}} \|f(x_{11}) - C(x_{11}) - f(0), \dots, f(x_{1k}) - C(x_{1k}) - f(0)\|_k \leq \frac{\alpha}{7},$$

for all $x_{11}, \dots, x_{1k} \in X$.

Proof. Let $x_{11}, \dots, x_{1k} \in X$. Let F be a function on X defined by

$$F(x) = f(x) - f(0),$$

for all $x \in X$. Then we have $F(0) = 0$. Note that $x \perp 0$ for all $x \in X$. Letting $x_{11} = \dots = x_{1k} = x_{31} = \dots = x_{3k} = 0$ and $x_{21} = x_{11}, \dots, x_{2k} = x_{1k}$ in (5.1), we get

$$\sup_{k \in \mathbb{N}} \|F(2x_{11}) - 8F(x_{11}), \dots, F(2x_{1k}) - 8F(x_{1k})\|_k \leq \alpha.$$

Following the same approach as in the proof of Theorem 2.2, we can define a mapping $C : X \rightarrow Y$ by setting

$$C(x) := \lim_{n \rightarrow \infty} \frac{F(2^n x)}{8^n}.$$

Moreover, we have

$$\sup_{k \in \mathbb{N}} \|F(x_{11}) - C(x_{11}), \dots, F(x_{1k}) - C(x_{1k})\|_k \leq \frac{\alpha}{7},$$

which implies the inequality (5.2).

Let $x_1, x_2, x_3 \in X$ and $x_i \perp x_j$ ($i, j = 1, 2, 3$). So we have $2^n x_i \perp 2^n x_j$ ($i, j = 1, 2, 3$). Letting $x_{11} = \dots = x_{1k} = 2^n x_1$, $x_{21} = \dots = x_{2k} = 2^n x_2$ and $x_{31} = \dots = x_{3k} = 2^n x_3$ in (5.1) and dividing by 8^n , then it follows that

$$\frac{1}{8^n} \|Df(2^n x_1, 2^n x_2, 2^n x_3)\| \leq \frac{1}{8^n} \alpha.$$

Taking the limit as $n \rightarrow \infty$ we get $DC(x_1, x_2, x_3) = 0$. Since $C(0) = 0$, by Lemma 5.1 we conclude that C is an orthogonally cubic mapping. Let C' be another orthogonally cubic mapping satisfying (5.2). Then we have

$$\begin{aligned} \|C(x) - C'(x)\| &= \frac{1}{8^n} \|C(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n} \|C(2^n x) - f(2^n x) + f(0)\| \\ &\quad + \frac{1}{8^n} \|f(2^n x) - C'(2^n x) - f(0)\| \\ &\leq \frac{1}{8^n} \left(\frac{\alpha}{7} + \frac{\alpha}{7} \right). \end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. \square

Corollary 5.3. *Let $\alpha \geq 0$ and $f : X \rightarrow Y$ be a mapping satisfying (5.1) for all $x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k} \in X$ with $x_{in} \perp x_{jn}$ ($i, j = 1, 2, 3, n = 1, \dots, k$). Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that*

$$\sup_{k \in \mathbb{N}} \|f(x_{11}) - C(x_{11}), \dots, f(x_{1k}) - C(x_{1k})\|_k \leq \frac{\alpha + 7\|f(0)\|}{7}.$$

Proof. In view of Theorem 5.2, there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that (5.2) is satisfied. We have

$$\begin{aligned} &\sup_{k \in \mathbb{N}} \|f(x_{11}) - C(x_{11}), \dots, f(x_{1k}) - C(x_{1k})\|_k \\ &\leq \sup_{k \in \mathbb{N}} \|f(x_{11}) - C(x_{11}) - f(0), \dots, f(x_{1k}) \\ &\quad - C(x_{1k}) - f(0)\|_k + \|f(0), \dots, f(0)\|_k \\ &\leq \frac{\alpha + 7\|f(0)\|}{7}, \end{aligned}$$

for all $x_{11}, \dots, x_{1k} \in X$. \square

We prove the generalized Hyers-Ulam stability of the orthogonally cubic type functional equation in multi-normed spaces by using the fixed point method.

Theorem 5.4. *Let $k \in \mathbb{N}$ and $\phi : X^{3k} \rightarrow [0, \infty)$ be a function such that*

$$\begin{aligned} & \phi(2x_{11}, \dots, 2x_{1k}, 2x_{21}, \dots, 2x_{2k}, 2x_{31}, \dots, 2x_{3k}) \\ & \leq a\phi(x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k}), \end{aligned}$$

for some $a > 0$ with $a < 8$ and all $x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k} \in X$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$(5.3) \quad \begin{aligned} & \|Df(x_{11}, x_{21}, x_{31}), \dots, Df(x_{1k}, x_{2k}, x_{3k})\|_k \\ & \leq \phi(x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k}), \end{aligned}$$

for all $x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2k}, x_{31}, \dots, x_{3k} \in X$ with $x_{in} \perp x_{jn}$ ($i, j = 1, 2, 3, n = 1, \dots, k$). Then there exists a unique orthogonally cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} & \|f(x_{11}) - C(x_{11}) - f(0), \dots, f(x_{1k}) - C(x_{1k}) - f(0)\|_k \\ & \leq \frac{1}{8-a}\phi(0, \dots, 0, x_{11}, \dots, x_{1k}, 0, \dots, 0), \end{aligned}$$

for all $x_{11}, \dots, x_{1k} \in X$.

Proof. Let $E = \{f : X \rightarrow Y\}$, and consider the generalized metric d defined on E by

$$\begin{aligned} d(g, h) &= \inf \left\{ c > 0, \|g(x_{11}) - h(x_{11}), \dots, g(x_{1k}) - h(x_{1k})\|_k \right. \\ & \left. \leq c\psi(x_{11}, \dots, x_{1k}), \forall x_{11}, \dots, x_{1k} \in X \right\}, \end{aligned}$$

where $\psi : X^k \rightarrow [0, \infty)$ is a mapping defined by

$$\psi(x_{11}, \dots, x_{1k}) = \phi(0, \dots, 0, x_{11}, \dots, x_{1k}, 0, \dots, 0),$$

for all $x_{11}, \dots, x_{1k} \in X$. Then, it is easy to show that d is a complete generalized metric on E . We now define a mapping $J : E \rightarrow E$ by

$$Jg(x) = \frac{1}{8}g(2x), \quad \forall x \in X.$$

Let F be a function on X defined by

$$F(x) = f(x) - f(0),$$

for all $x \in X$. Letting $x_{11} = \dots = x_{1k} = x_{31} = \dots = x_{3k} = 0$ and $x_{21} = x_{11}, \dots, x_{2k} = x_{1k}$ in (5.3), we get

$$\|F(2x_{11}) - 8F(x_{11}), \dots, F(2x_{1k}) - 8F(x_{1k})\|_k \leq \psi(x_{11}, \dots, x_{1k}),$$

for all $x_{11}, \dots, x_{1k} \in X$. The rest of the proof is similar to the ones of Theorem 2.3. \square

6. STABILITY OF THE k -CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the k -cubic functional equation in multi-normed spaces. For a given mapping $f : X \rightarrow Y$, we set

$$Df(x, y) = kf(x + ky) - f(kx + y) - \frac{k(k^2 - 1)}{2}[f(x + y) + f(x - y)] - (k^4 - 1)f(y) + 2k(k^2 - 1)f(x),$$

for all $x, y \in X$ with $x \perp y$.

Definition 6.1. A mapping $f : X \rightarrow Y$ is called an orthogonally k -cubic mapping if

$$kf(x + ky) - f(kx + y) = \frac{k(k^2 - 1)}{2}[f(x + y) + f(x - y)] + (k^4 - 1)f(y) - 2k(k^2 - 1)f(x),$$

for all $x, y \in X$ with $x \perp y$.

We state the following lemma. The proof is obvious.

Lemma 6.2. *If $k \geq 2$ and $f : X \rightarrow Y$ is a k -cubic mapping then we have $f(kx) = k^3f(x)$ for all $x \in X$.*

Theorem 6.3. *Let $\alpha \geq 0$ and $k \geq 2$ and $f : X \rightarrow Y$ be a mapping satisfying*

$$(6.1) \quad \sup_{t \in \mathbb{N}} \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$ with $x_i \perp y_i (i = 1, \dots, t)$. If

$$\lambda = \frac{(k+1)(k^2+1)}{k^2+k+1},$$

then there exists a unique orthogonally k -cubic mapping $C : X \rightarrow Y$ such that

$$(6.2) \quad \sup_{t \in \mathbb{N}} \|f(x_1) - C(x_1) - \lambda f(0), \dots, f(x_t) - C(x_t) - \lambda f(0)\|_t \leq \frac{\alpha}{k^3 - 1},$$

for all $x_1, \dots, x_t \in X$.

Proof. Let $x_1, \dots, x_t \in X$ and $\beta = k^4 - 1$. Letting $y_1 = \dots = y_t = 0$ in (6.1), we obtain

$$\sup_{t \in \mathbb{N}} \|f(kx_1) - k^3f(x_1) + \beta f(0), \dots, f(kx_t) - k^3f(x_t) + \beta f(0)\|_t \leq \alpha,$$

since $0 \perp x_i (i = 1, \dots, t)$. So the sequence $\left(\frac{f(k^n x)}{k^{3n}}\right)$ is Cauchy and so is convergent in the complete multi-normed space Y . Set

$$C(x) := \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}.$$

Similar to the proof of Theorem 2.2, we can conclude the inequality (6.2).

Let $x, y \in X$ and $x \perp y$. Letting $x_1 = \dots = x_t = k^n x$, $y_1 = \dots = y_t = k^n y$ in (6.1) and dividing the both sides by k^{3n} , we obtain

$$\frac{1}{k^{3n}} \|Df(k^n x, k^n y)\| \leq \frac{\alpha}{k^{3n}}.$$

Taking the limit as $n \rightarrow \infty$, we get

$$DC(x, y) = 0.$$

So C is an orthogonally k -cubic mapping.

Let C' be another orthogonally k -cubic mapping satisfying (6.2). So by Lemma 6.2 and induction, we have

$$C(k^n x) = k^{3n} C(x), \quad C'(k^n x) = k^{3n} C'(x),$$

for all $x \in X$ and $n \in \mathbb{N}$. So we have

$$\begin{aligned} \|C(x) - C'(x)\| &= \frac{1}{k^{3n}} \|C(k^n x) - C'(k^n x)\| \\ &\leq \frac{1}{k^{3n}} \|C(k^n x) - f(k^n x) + \lambda f(0)\| \\ &\quad + \frac{1}{k^{3n}} \|C'(k^n x) - f(k^n x) + \lambda f(0)\| \\ &\leq \frac{1}{k^{3n}} \left(\frac{\alpha}{k^3 - 1} + \frac{\alpha}{k^3 - 1} \right). \end{aligned}$$

Hence $C = C'$. This completes the proof. \square

Corollary 6.4. *Let $\alpha \geq 0$ and $k \geq 2$ and $f : X \rightarrow Y$ be a mapping satisfying (6.1) for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$ with $x_i \perp y_i (i = 1, \dots, t)$. Then there exists a unique orthogonally k -cubic mapping $C : X \rightarrow Y$ such that*

$$\sup_{t \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_t) - C(x_t)\|_t \leq \frac{\alpha + (k^4 - 1)\|f(0)\|}{k^3 - 1}.$$

Proof. By Theorem 6.3, we conclude the existence of an orthogonally k -cubic mapping $C : X \rightarrow Y$ such that (6.2) is satisfied. If

$$\lambda = \frac{(k+1)(k^2+1)}{k^2+k+1},$$

then we have

$$\begin{aligned} & \sup_{t \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_t) - C(x_t)\|_t \\ & \leq \sup_{t \in \mathbb{N}} \|f(x_1) - C(x_1) - \lambda f(0), \dots, f(x_t) - C(x_t) - \lambda f(0)\|_t \\ & \quad + \lambda \|f(0), \dots, f(0)\|_t \\ & \leq \frac{\alpha + (k^4 - 1)\|f(0)\|}{k^3 - 1}, \end{aligned}$$

for all $x_1, \dots, x_t \in X$. \square

We prove the generalized Hyers-Ulam stability of the k -cubic functional equation in multi-normed spaces by using the fixed point method.

Theorem 6.5. *Let $k \geq 2$ and $t \in \mathbb{N}$ and $\phi : X^{2t} \rightarrow [0, \infty)$ be a function such that*

$$\phi(kx_1, \dots, kx_t, ky_1, \dots, ky_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < k^3$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f : X \rightarrow Y$ is a mapping such that $f(0) = 0$ and

$$(6.3) \quad \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$ with $x_i \perp y_i (i = 1, \dots, t)$. Then there exists a unique orthogonally k -cubic mapping $C : X \rightarrow Y$ such that

$$(6.4) \quad \|f(x_1) - C(x_1), \dots, f(x_t) - C(x_t)\|_t \leq \frac{1}{k^3 - a} \phi(x_1, \dots, x_t, 0, \dots, 0),$$

for all $x_1, \dots, x_t \in X$.

Proof. Let $E = \{f : X \rightarrow Y\}$, and consider the complete generalized metric d defined on E by

$$\begin{aligned} d(g, h) &= \inf \left\{ c \in (0, \infty) : \|g(x_1) - h(x_1), \dots, g(x_t) - h(x_t)\|_t \right. \\ & \quad \left. \leq c\psi(x_1, \dots, x_t), \forall x_1, \dots, x_t \in X \right\}, \end{aligned}$$

where $\psi : X^t \rightarrow [0, \infty)$ is a mapping defined by

$$\psi(x_1, \dots, x_t) = \phi(x_1, \dots, x_t, 0, \dots, 0).$$

We now define a mapping $J : E \rightarrow E$ by

$$Jg(x) = \frac{1}{k^3}g(kx), \quad \forall x \in X.$$

It is easy to see that J is a strictly contractive mapping with the Lipschitz constant $\frac{a}{k^3}$. Letting $y_1 = \dots = y_t = 0$ in (6.3), we get

$$(6.5) \quad \|f(kx_1) - k^3 f(x_1), \dots, f(kx_t) - k^3 f(x_t)\|_t \leq \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. It follows from (6.5) that $d(Jf, f) \leq \frac{1}{k^3}$. According to Lemma 1.4, we deduce the existence of a fixed point of J , that is, the existence of a mapping $C : X \rightarrow Y$ such that $C(kx) = k^3C(x)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \rightarrow 0$ that

$$\begin{aligned} C(x) &= \lim_{n \rightarrow \infty} (J^n f)(x) \\ &= \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}, \quad \forall x \in X. \end{aligned}$$

Moreover

$$d(f, C) \leq \frac{1}{1 - \frac{a}{k^3}} d(Jf, f),$$

implies the inequality

$$d(f, C) \leq \frac{1}{k^3 - a}.$$

So (6.4) is satisfied.

Let $x, y \in X$ and $x \perp y$. Letting $x_1 = \dots = x_t = k^n x$ and $y_1 = \dots = y_t = k^n y$ in (6.3) and dividing the both sides by k^{3n} , we get

$$\begin{aligned} \frac{1}{k^{3n}} \|Df(k^n x, k^n y)\| &\leq \frac{1}{k^{3n}} \phi(k^n x, \dots, k^n x, k^n y, \dots, k^n y) \\ &\leq \left(\frac{a}{k^3}\right)^n \phi(x, \dots, x, y, \dots, y). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we deduce that C is an orthogonally k -cubic mapping.

The uniqueness assertion is similar to the proof of Theorem 2.3. \square

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