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on the stability of the Pexiderized cubic functional equation in multi-normed spaces

Mahdi Nazarianpoor[1*] and Ghadir Sadeghi[2]

ABSTRACT. In this paper, we investigate the Hyers-Ulam stability of the orthogonally cubic equation and Pexiderized cubic equation

$$f(kx+y)+f(kx-y)=g(x+y)+g(x-y)+\frac{2}{k}g(kx)-2g(x),$$

in multi-normed spaces by the direct method and the fixed point method. Moreover, we prove the Hyers-Ulam stability of the 2-variables cubic equation

$$\begin{aligned}f(2x+y,2z+t)&+f(2x-y,2z-t)\\&=2f(x+y,z+t)+2f(x-y,z-t)+12f(x,z),\end{aligned}$$

and orthogonally cubic type and k-cubic equation in multi-normed spaces. A counter example for non stability of the cubic equation is also discussed.

1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [14] concerning the stability of group homomorphisms: Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a number $\delta>0$ such that if a mapping $f:G_1\to G_2$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in G_1$ then there exists a homomorphism $T:G_1\to G_2$ such that $d(f(x),T(x))<\varepsilon$ for all $x\in G_1$?
D.H. Hyers [5] gave a first affirmative answer to the question of Ulam for Banach spaces:
Let X be a normed space and Y a Banach space. Suppose that for some

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* Corresponding author.

$\varepsilon > 0$, the mapping $f : X \to Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.
Jun and Kim [6] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.1}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.1).
Every solution of the cubic functional equation is said to be a cubic mapping.

T.Z. Xu et al. [15] proved the generalized Hyers-Ulam stability of the general mixed additive-cubic functional equation

$$f(kx+y)+f(kx-y) = kf(x+y)+kf(x-y)+2f(kx)-2kf(x), \tag{1.2}$$

in quasi-Banach spaces.

In [9], S. Ostadbashi and M. Soleimaninia considered the Pexiderized cubic functional equation

$$f(kx + y) + f(kx - y) = g(x + y) + g(x - y) + \frac{2}{k}g(kx) - 2g(x), \tag{1.3}$$

and they investigated the stability of the functional equation (1.3) in topological vector spaces.

The functional equation

$$\begin{aligned} &f(2x + y, 2z + t) + f(2x - y, 2z - t) \\ &\qquad = 2f(x + y, z + t) + 2f(x - y, z - t) + 12f(x, z), \end{aligned} \tag{1.4}$$

is called the 2-variables cubic functional equation.

It is easy to see that the function $f(x, y) = x^3 + y^3$ is a solution of equation (1.4).

Every solution of the 2-variables cubic functional equation is said to be a 2-variables cubic mapping.

Chang, Jun and Jung [1] introduced a cubic type functional equation as follows:

$$\begin{aligned} &f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2) + 7[f(x_1) + f(-x_1)] \\ &\qquad = 2f(x_1 + x_2) + 4[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)], \end{aligned} \tag{1.5}$$

and they investigated the modified Hyers-Ulam-Rassias stability of this equation by using the fixed point method. Moreover I.S. Chang and Y.S. Jung [2] established the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.5).

It is easy to see that the function $f(x) = x^3 + C$, where $C \in \mathbb{R}$ is a constant is a solution of the equation (1.5).

K. Ravi, M.J. Rassias, P. Narasimman, R.K. Kumar [13] introduced the k-cubic functional equation

$$kf(x+ky)-f(kx+y)=\frac{k(k^2-1)}{2}[f(x+y)+f(x-y)] + (k^4-1)f(y)-2k(k^2-1)f(x), \tag{1.6}$$

where $k \geq 2$ and they established the generalized Hyers-Ulam stability problem for the functional equation (1.6).

Note that the function $f(x)=x^3$ is a solution of the functional equation (1.6).

Every solution of the k-cubic functional equation is said to be a k-cubic mapping.

During the last decades, several stability problems of functional equations have been investigated. The reader is referred to [3, 8, 11] and references therein for detailed information on stability of functional equations.

Pinsker [10] characterized orthogonal additive functional equation on an inner product space. The orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y), \quad x \perp y,$$

in which $\perp$ is an orthogonality relation, is first investigated by Gudder and Strawther [4]. In 1985, Ratz [12] introduced a new definition of orthogonality by using more restrictive axioms than Gudder and Strawther. Moreover, he investigated the structure of the orthogonally additive mappings.

There are several orthogonality notations on a real normed space. But here, we present the orthogonal concept introduced by Ratz. This is given in the following definition.

Definition 1.1. Suppose that X is a vector space (algebraic module) with $dimX \geq 2$, and $\perp$ is a binary relation on X with the following properties:

(i) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(ii) independence: if $x, y \in X-\{0\}$ and $x \perp y$, then x and y are linearly independent;
(iii) homogeneity: if $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(iv) Thalesian properity: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \geq 0$ then there exists $y_0 \in P$ such that $x \perp y_0$ and $x+y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonal space (resp., module). By an orthogonal normed space (normed module), we mean an orthogonal space (resp., module) having a normed (resp., normed module) structure.

Let $(X, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\mathfrak{G}_k$ the group of permutations on k symbols.

Definition 1.2. A multi-norm on $\{X^k : k \in \mathbb{N}\}$ is a sequence

$$(\|.\|_k) = (\|.\|_k : k \in \mathbb{N}),$$

such that $\|.\|_k$ is a norm on X^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in X$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(MN1)

$$\|x_{\sigma(1)}, \dots, x_{\sigma(k)}\|_k = \|x_1, \dots, x_k\|_k, \quad (\sigma \in \mathfrak{G}_k, x_1, \dots, x_k \in X);$$

(MN2)

$$\|\alpha_1 x_1, \dots, \alpha_k x_k\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|x_1, \dots, x_k\|_k,$$

$$(\alpha_1, \dots, \alpha_k \in \mathbb{C}, x_1, \dots, x_k \in X);$$

(MN3)

$$\|x_1, \dots, x_{k-1}, 0\|_k = \|x_1, \dots, x_{k-1}\|_{k-1}, \quad (x_1, \dots, x_{k-1} \in X);$$

(MN4)

$$\|x_1, \dots, x_{k-1}, x_{k-1}\|_k = \|x_1, \dots, x_{k-1}\|_{k-1}, \quad (x_1, \dots, x_{k-1} \in X).$$

In this case, we say that $((X^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Example 1.3. Let $(X, \|.\|)$ be a Banach lattice, and define

$$\|x_1, \dots, x_k\|_k := \||x_1| \vee \dots \vee |x_k|\|, \quad (x_1, \dots, x_k \in X).$$

Then $((X^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Lemma 1.4. *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping, that is,*

$$d(Jx, Jy) \leq \alpha d(x, y), \quad \forall x, y \in X,$$

for some $\alpha < 1$. Then for each element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad (n \geq 0),$$

or there exists $n_0 \geq 0$ such that:

(i) *$d(J^n x, J^{n+1} x) < \infty, (n \geq n_0)$;*
(ii) *the sequence $(J^n x)$ converges to a fixed point y^* of J;*
(iii) *y^* is the unique fixed point of J in the set*

$$U = \{y \in X : \ d(J^{n_0} x, y) < \infty\};$$

(iv) *$d(y, y^*) \leq d(y, Jy)/(1 - \alpha), (y \in U)$.*

In this paper, we investigate the Hyers-Ulam stability of the orthogonally cubic equation and Pexiderized cubic equation

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in multi-normed spaces by the direct method and the fixed point method. Moreover we prove the Hyers-Ulam stability of a 2-variables cubic equation

$$\begin{aligned}f(2x+y,2z+t)+f(2x-y,2z-t)&=2f(x+y,z+t)\\&\quad+2f(x-y,z-t)+12f(x,z),\end{aligned}$$

and orthogonally cubic type and k-cubic equation in multi-normed spaces. Throughout this paper, assume that X is an orthogonal space (except in Section 3, 4) and that Y is a Banach space. Let $((Y^n,||.||_n):n\in\mathbb{N})$ be a multi-Banach space.

2. STABILITY OF THE CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the orthogonally cubic functional equation in multi-normed spaces. For convenience, we use the following abbreviation for a given mapping $f:X\to Y$,

$$Df(x,y)=f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),$$

for all $x,y\in X$ with $x\perp y$.

Definition 2.1. A mapping $f:X\to Y$ is called an orthogonally cubic mapping if

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x),$$

for all $x,y\in X$ with $x\perp y$.

Theorem 2.2. *Let $\alpha\geq 0$ and $f:X\to Y$ be a mapping satisfying*

$$\sup_{k\in\mathbb{N}}\|Df(x_1,y_1),\ldots,Df(x_k,y_k)\|_k\leq\alpha, \tag{2.1}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k\in X$ with $x_i\perp y_i(i=1,\ldots,k)$. Then there exists a unique orthogonally cubic mapping $C:X\to Y$ such that

$$\sup_{k\in\mathbb{N}}\|f(x_1)-C(x_1),\ldots,f(x_k)-C(x_k)\|_k\leq\frac{\alpha}{14}, \tag{2.2}$$

for all $x_1,\ldots,x_k\in X$.

Proof. Let $x_1,\ldots,x_k\in X$. Letting $y_1=\cdots=y_k=0$ in (2.1), we get

$$\sup_{k\in\mathbb{N}}\|f(2x_1)-8f(x_1),\ldots,f(2x_k)-8f(x_k)\|\leq\frac{\alpha}{2}, \tag{2.3}$$

for all $x_1, \dots, x_k \in X$, since $0 \perp x_i$, $(i = 1, \dots, k)$. Replacing $x_1, \dots, x_k$ by $2^n x_1, \dots, 2^n x_k$ in relation (2.3) and dividing by 8^{n+1}, from relation (2.3) one gets

(2.4)
$$\sup_{k\in\mathbb{N}} \left\| \frac{f(2^{n+1}x_1)}{8^{n+1}} - \frac{f(2^n x_1)}{8^n}, \dots, \frac{f(2^{n+1}x_k)}{8^{n+1}} - \frac{f(2^n x_k)}{8^n} \right\|_k \le \frac{\alpha}{2}\left(\frac{1}{8^{n+1}}\right).$$

It follows from the last inequality that

$$\sup_{k\in\mathbb{N}} \left\| \frac{f(2^{n+m}x_1)}{8^{n+m}} - \frac{f(2^n x_1)}{8^n}, \dots, \frac{f(2^{n+m}x_k)}{8^{n+m}} - \frac{f(2^n x_k)}{8^n} \right\|_k \le \frac{\alpha}{2}\left(\frac{1}{8^{n+1}} + \dots + \frac{1}{8^{n+m}}\right), \tag{2.5}$$

for $n, m \in \mathbb{N}, m \ge 1$.

It follows that $\left(\frac{f(2^n x)}{8^n}\right)$ is Cauchy and so is convergent in the complete multi-normed space Y.

Set

$$C(x) := \lim_{n\to\infty} \frac{f(2^n x)}{8^n}.$$

Hence for each $\varepsilon > 0$ there exists n_0 such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{f(2^n x)}{8^n} - C(x), \dots, \frac{f(2^{n+k-1}x)}{8^{n+k-1}} - C(x) \right\|_k < \varepsilon,$$

for all $n \ge n_0$. Therefore we get

$$\lim_{n\to\infty} \left\| \frac{f(2^n x)}{8^n} - C(x) \right\| = 0, \tag{2.6}$$

for all $x \in X$. Letting $n = 0$ in (2.5), we get

$$\sup_{k\in\mathbb{N}} \left\| \frac{f(2^m x_1)}{8^m} - f(x_1), \dots, \frac{f(2^m x_k)}{8^m} - f(x_k) \right\|_k \le \frac{\alpha}{2} \sum_{i=1}^{m} \frac{1}{8^i}.$$

Letting m tend to infinity and using Lemma 1.4 and (2.6), we have

$$\sup_{k\in\mathbb{N}} \| f(x_1) - C(x_1), \dots, f(x_k) - C(x_k) \|_k \le \frac{\alpha}{14}.$$

Let $x, y \in X$ and $x \perp y$. So we have $2^n x \perp 2^n y$. Letting $x_1 = \dots = x_k = 2^n x$ and $y_1 = \dots = y_k = 2^n y$ in (2.1) and divide both sides by 8^n to obtain

$$\frac{1}{8^n} \| Df(2^n x, 2^n y) \| \le \frac{\alpha}{8^n}.$$

Taking the limit as $n \to \infty$, we get

$$DC(x, y) = 0.$$

Hence C is an orthogonally cubic mapping.
Let C' be another orthogonally cubic mapping satisfying (2.2). It is easy to see that every cubic mapping $f : X \to Y$ satisfies

$$f(kx) = k^3 f(x),$$

for all positive integer k and $x \in X$. So, we have

$$\begin{aligned}\|Cx - C'x\| &= \frac{1}{8^n}\|C(2^n x) - C'(2^n x)\| \\ &\le \frac{1}{8^n}\|C(2^n x) - f(2^n x)\| + \frac{1}{8^n}\|f(2^n x) - C'(2^n x)\| \\ &\le \frac{1}{8^n}\left(\frac{\alpha}{14} + \frac{\alpha}{14}\right).\end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. □

We prove the generalized Hyers-Ulam stability of the cubic functional equation in multi-normed spaces by using the fixed point method.

Theorem 2.3. *Let $k \in \mathbb{N}$ and $\phi : X^{2k} \to [0, \infty)$ be a function such that*

$$\phi(2x_1, \ldots, 2x_k, 2y_1, \ldots, 2y_k) \le a\phi(x_1, \ldots, x_k, y_1, \ldots, y_k),$$

for some $a > 0$ with $a < 8$ and all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x_1, y_1), \ldots, Df(x_k, y_k)\|_k \le \phi(x_1, \ldots, x_k, y_1, \ldots, y_k), \tag{2.7}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ with $x_i \perp y_i$ $(i = 1, \ldots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x_1) - C(x_1), \ldots, f(x_k) - C(x_k)\|_k \le \frac{1}{16 - 2a}\phi(x_1, \ldots, x_k, 0, \ldots, 0), \tag{2.8}$$

for all $x_1, \ldots, x_k \in X$.

Proof. Let $E = \{f : X \to Y\}$, and introduce the generalized metric d defined on E by

$$\begin{aligned}d(g, h) = \inf\Big\{c \in (0, \infty) : \|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \\ \le c\psi(x_1, \ldots, x_k), \forall x_1, \ldots, x_k \in X\Big\},\end{aligned}$$

where $\psi : X^k \to [0, \infty)$ is a mapping defined by

$$\psi(x_1, \ldots, x_k) = \phi(x_1, \ldots, x_k, 0, \ldots, 0).$$

Then, it is easy to show that d is a complete generalized metric on E. We now define a mapping $J : E \to E$ by

$$Jg(x) = \frac{1}{8}g(2x), \quad \forall x \in X.$$

We claim that J is a strictly contractive mapping. Given $g, h \in E$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) < c$. It is easy to see that

$$\|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \leq c\psi(x_1, \ldots, x_k),$$

for all $x_1, \ldots, x_k \in X$. Therefore,

$$\begin{aligned}\|Jg(x_1) - Jh(x_1), \ldots, Jg(x_k) - Jh(x_k)\|_k \\ &= \left\| \frac{1}{8}g(2x_1) - \frac{1}{8}h(2x_1), \ldots, \frac{1}{8}g(2x_k) - \frac{1}{8}h(2x_k) \right\|_k \\ &\leq \frac{1}{8}ca\psi(x_1, \ldots, x_k),\end{aligned}$$

for all $x_1, \ldots, x_k \in X$. So we have $d(Jg, Jh) \leq \frac{1}{8}ad(g, h)$ for all $g, h \in E$. Letting $y_1 = \cdots = y_k = 0$ in (2.7), we get

$$\|f(2x_1) - 8f(x_1), \ldots, f(2x_k) - 8f(x_k)\|_k \leq \frac{1}{2}\psi(x_1, \ldots, x_k), \tag{2.9}$$

for all $x_1, ..., x_k \in X$. It follows from (2.9) that $d(Jf, f) \leq \frac{1}{16}$. According to Lemma 1.4, we deduce the existence of a fixed point of J, that is, the existence of a mapping $C : X \to Y$ such that $C(2x) = 8C(x)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \to 0$ that

$$\begin{aligned}C(x) &= \lim_{n\to\infty} (J^n f)(x) \\ &= \lim_{n\to\infty} \frac{f(2^n x)}{8^n}, \quad \forall x \in X.\end{aligned}$$

Moreover $d(f, C) \leq \frac{1}{1-\frac{a}{8}}d(Jf, f)$, implies the inequality

$$d(f, C) \leq \frac{1}{16 - 2a}.$$

So (2.8) is satisfied.

Let $x, y \in X$ and $x \perp y$. So $2^n x \perp 2^n y$. Letting $x_1 = \cdots = x_k = 2^n x$ and $y_1 = \cdots = y_k = 2^n y$ in (2.7) and dividing both sides by 8^n, we to get

$$\begin{aligned}\frac{1}{8^n}\|Df(2^n x, 2^n y)\| &\leq \frac{1}{8^n}\phi(2^n x, \ldots, 2^n x, 2^n y, \ldots, 2^n y) \\ &\leq \left(\frac{a}{8}\right)^n \phi(x, \ldots, x, y, \ldots, y).\end{aligned}$$

Taking the limit as $n \to \infty$, we deduce that C is an orthogonally cubic mapping.

The uniqueness of C follows from the fact that C is the unique fixed point of J with the property that there exists $\lambda \in (0, \infty)$ such that

$$\|f(x_1) - C(x_1), \ldots, f(x_k) - C(x_k)\|_k \leq \lambda\psi(x_1, \ldots, x_k),$$

for all $x_1, \ldots, x_k \in X$. $\square$

Corollary 2.4. *Let $k \in \mathbb{N}$ and p, α be real numbers with $\alpha > 0$ and $p < 3$, and let $f : X \to Y$ be a mapping satisfying*

$$\|Df(x_1, y_1), \ldots, Df(x_k, y_k)\|_k \leq \alpha(\|x_1\|^p + \cdots + \|x_k\|^p + \|y_1\|^p + \cdots + \|y_k\|^p),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ with $x_i \perp y_i (i = 1, \ldots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x_1) - C(x_1), \ldots, f(x_k) - C(x_k)\|_k \leq \frac{\alpha}{16 - 2^{p+1}}(\|x_1\|^p + \cdots + \|x_k\|^p),$$

where $x_1, \ldots, x_k \in X$.

Proof. Let

$$\phi(x_1, \ldots, x_k, y_1, \ldots, y_k) = \alpha(\|x_1\|^p + \cdots + \|x_k\|^p + \|y_1\|^p + \cdots + \|y_k\|^p),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ and $a = 2^p$ in Theorem 2.3, we obtain the desired result. $\square$

The following example shows that the assumption $p < 3$ cannot be omitted in Corollary 2.4. We know from Example 1.3 that if

$$\|x_1, \ldots, x_k\|_k = \sup\{|x_1|, \ldots, |x_k|\},$$

then $((\mathbb{R}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Example 2.5. Fix $k \in \mathbb{N}$. For $\epsilon > 0$ put $\lambda := \frac{7}{9216}\varepsilon$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) := \begin{cases} \lambda & x \in [1, \infty); \\ \lambda x^3 & x \in (-1, 1); \\ -\lambda & x \in (-\infty, -1]. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ to be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{8^n}, \quad x \in \mathbb{R}.$$

Then f satisfies the following functional inequality

(2.10)
$$\|Df(x_1, y_1), \ldots, Df(x_k, y_k)\|_k \leq \varepsilon(|x_1|^3 + \cdots + |x_k|^3 + |y_1|^3 + \cdots + |y_k|^3),$$

where $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$.

Proof. We claim that

$$|Df(x, y)| \leq \varepsilon(|x|^3 + |y^3|), \quad x, y \in \mathbb{R}.$$

We have

$$|f(x)| \leq \frac{8}{7}\lambda,$$

for all $x \in \mathbb{R}$. Therefore we see that f is bounded.

If $|x|^3+|y|^3=0$ or $|x|^3+|y^3|\geq \frac{1}{8}$, then

$$\begin{aligned}|Df(x,y)| &\leq \frac{144}{7}\lambda\\ &\leq \frac{144}{7}\lambda(8(|x|^3+|y^3|))\\ &\leq \varepsilon(|x|^3+|y^3|),\end{aligned}$$

for all $x,y\in\mathbb{R}$. Now, suppose that $0<|x|^3+|y^3|<\frac{1}{8}$. Then there exists a nonnegative integer k such that

$$\frac{1}{8^{k+2}}\leq |x|^3+|y^3|<\frac{1}{8^{k+1}}.$$

Hence,

$$2^kx<\frac{2}{3}\quad\text{and}\quad 2^ky<\frac{2}{3},$$

and

$$2^n(2x+y),2^n(2x-y),2^n(x+y),2^n(x-y),2^nx\in(-1,1),$$

for all $n=0,1,\ldots,k-1$. As a result, we get

$$\begin{aligned}\frac{|Df(x,y)|}{|x|^3+|y|^3} &\leq \sum_{n=k}^{\infty}\frac{18\lambda}{8^n(|x|^3+|y^3|)}\\ &\leq \sum_{n=0}^{\infty}\frac{18\lambda}{8^{n+k}(|x|^3+|y^3|)}\\ &\leq \sum_{n=0}^{\infty}\frac{18\lambda}{8^n8^{k+2}(|x|^3+|y^3|)}64\\ &\leq \sum_{n=0}^{\infty}\frac{1152\lambda}{8^n}\\ &= \frac{9216}{7}\lambda\\ &= \varepsilon.\end{aligned}$$

So

$$|Df(x,y)|\leq\varepsilon(|x|^3+|y^3|),\quad x,y\in\mathbb{R}.$$

It is easy to see that f satisfies (2.10) for all $x_1,\ldots,x_k,y_1,\ldots,y_k\in\mathbb{R}$. Now, we claim that the cubic functional equation in Corollary 2.4 is not stable for $p=3$. Suppose on the contrary that there exists a cubic mapping $C:\mathbb{R}\to\mathbb{R}$ and $\beta>0$ such that

$$\|f(x_1)-C(x_1),\ldots,f(x_k)-C(x_k)\|_k\leq\beta(|x_1|^3+\cdots+|x_k|^3),$$

where $x_1, \dots, x_k \in \mathbb{R}$. So $|f(x) - C(x)| \le \delta |x|^3$ for some constants $\delta > 0$ and all $x \in \mathbb{R}$. Then there exists a constant $\gamma \in \mathbb{R}$ for which $C(x) = \gamma x^3$ for all $x \in \mathbb{Q}$. So we get

$$\frac{|f(x)|}{x^3} \le \delta + |\gamma|, \quad x \in \mathbb{Q}.$$

Let $M \in \mathbb{N}$ with $M\lambda > \delta + |\gamma|$. If x is a rational number in $(0, \frac{1}{2^{M-1}})$, then we have $2^n x \in (0,1)$ for each $n = 0, 1, 2, \dots, M-1$. Consequently, for such an x we have

$$\begin{aligned} \frac{f(x)}{x^3} &= \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{8^n x^3} \\ &\ge \sum_{n=0}^{M-1} \frac{\lambda 8^n x^3}{8^n x^3} \\ &= M\lambda \\ &> \delta + |\gamma|, \end{aligned}$$

which yields a contradiction. □

Next, we prove the generalized Hyers-Ulam stability of the orthogonally cubic functional equation in the closed balls of X^k. Assume that $N_r(X^k)$ is the closed ball in X^k of radius r around the origin and $((X^n, ||.||_n) : n \in \mathbb{N})$ is a multi-normed space.

Theorem 2.6. *Let* $\phi_k : X^{2k} \to [0, \infty)$ $(k \in \mathbb{N})$ *be a family of functions such that* $\phi_k\left(\frac{\mathbf{x}}{2}, \frac{\mathbf{y}}{2}\right) \le a\phi_k(\mathbf{x}, \mathbf{y})$ *for some* $a > 0$ *with* $a < \frac{1}{8}$ *and all* $\mathbf{x}, \mathbf{y} \in N_r(X^k)$ *and* $k \in \mathbb{N}$. *Suppose that* $f : X \to Y$ *is a mapping satisfying*

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \le \phi_k(\mathbf{x}, \mathbf{y}), \tag{2.11}$$

for all $k \in \mathbb{N}$ *and* $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k) \in N_r(X^k)$ *with* $x_i \perp y_i (i = 1, \dots, k)$. *Then there exists a unique orthogonally cubic mapping* $C : X \to Y$ *such that*

$$\sup_{k \in \mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \le \frac{a \sup_{k \in \mathbb{N}} \phi_k(\mathbf{x}, 0)}{2(1 - 8a)},$$

where $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$.

Proof. Let $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$. Replacing $\mathbf{x}, \mathbf{y}$ in (2.11) by $\frac{\mathbf{x}}{2}, 0$ respectively, we get

$$\begin{aligned} \left\| f(x_1) - 8f\left(\frac{x_1}{2}\right), \dots, f(x_k) - 8f\left(\frac{x_k}{2}\right) \right\|_k &\le \frac{1}{2}\phi_k\left(\frac{\mathbf{x}}{2}, 0\right) \\ &\le \frac{1}{2} a\phi_k(\mathbf{x}, 0), \end{aligned} \tag{2.12}$$

since $0 \perp \frac{1}{2}x_i(i = 1, \dots, k)$. Letting $\mathbf{x}$ by $\frac{\mathbf{x}}{2^n}$ and multiplying with 8^n in (2.12) we have

$$\left\|8^n f\left(\frac{x_1}{2^n}\right) - 8^{n+1} f\left(\frac{x_1}{2^{n+1}}\right), \dots, 8^n f\left(\frac{x_k}{2^n}\right) - 8^{n+1} f\left(\frac{x_k}{2^{n+1}}\right)\right\|_k \leq \frac{a}{2}(8a)^n \phi_k(\mathbf{x}, 0).$$

Therefore the sequence $(8^n f(\frac{x}{2^n}))$ is Cauchy and so is convergent in the complete multi-norm space Y. Set

$$C'(x) = \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right), \quad (x \in N_r(X)).$$

It is easy to see that

$$\sup_{k\in\mathbb{N}} \|f(x_1) - C'(x_1), \dots, f(x_k) - C'(x_k)\|_k \leq \frac{a \sup_{k\in\mathbb{N}} \phi_k(\mathbf{x}, 0)}{2(1-8a)},$$

where $\mathbf{x} = (x_1, \dots, x_k) \in N_r(X^k)$.

Let $x \in N_r(X)$. Then we have

$$8C'\left(\frac{x}{2}\right) = \lim_{n\to\infty} 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) = C'(x).$$

So for all $m \in \mathbb{N}$ we have $8^m C'\left(\frac{x}{2^m}\right) = C'(x)$. Therefore we can define the well-defined mapping $C : X \to Y$ by $C(x) := 8^n C'\left(\frac{x}{2^n}\right)$, where n is the least non-negative integer such that $\frac{x}{2^n} \in N_r(X)$, indeed $n = \log_2 \|x\|$. It is clear that

$$C(x) = \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right), \quad (x \in X).$$

Let $x, y \in X$ and $x \perp y$. So $\frac{x}{2^n} \perp \frac{y}{2^n}$. There exists a large enough n such that $\frac{x}{2^n}, \frac{y}{2^n} \in N_r(X)$. Letting $x_1 = \dots = x_k = \frac{x}{2^n}, y_1 = \dots = y_k = \frac{y}{2^n}$ in (2.11) and multiplying both sides with 8^n, we to get

$$\left\|8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq (8a)^n \phi_k(x, \dots, x, y, \dots, y).$$

Taking the limit as $n \to \infty$, we conclude that C is an orthogonally cubic mapping.

The proof of the uniqueness assertion is similar to the ones of Theorem 2.2. □

Corollary 2.7. *Let p, α be positive real numbers with $p > 3$, and let $f : X \to Y$ be a mapping satisfying*

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \alpha(\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p),$$

for all $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k) \in N_r(X^k)$ with $x_i \perp y_i$ $(i = 1, \dots, k)$. Then there exists a unique orthogonally cubic *mapping* $C : X \to Y$ *such that*

$$\sup_{k\in\mathbb{N}} \|f(x_1) - C(x_1), \ldots, f(x_k) - C(x_k)\|_k \leq \frac{\alpha}{2^{p+1} - 16} \sup_{k\in\mathbb{N}} (\|x_1\|^p + \cdots + \|x_k\|^p),$$

where $\mathbf{x} = (x_1, \ldots, x_k) \in N_r(X^k)$.

Proof. Letting $\phi_k(\mathbf{x}, \mathbf{y}) = \alpha(\|x_1\|^p + \cdots + \|x_k\|^p + \|y_1\|^p + \cdots + \|y_k\|^p)$ for all $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k) \in X^k$ and $a = \frac{1}{2^p}$ in Theorem 2.6, we obtain the desired result. □

3. STABILITY OF THE PEXIDERIZED CUBIC FUNCTIONAL EQUATION

In this section, we prove the Hyers-Ulam stability of the Pexiderized cubic functional equation (1.3) in multi-normed spaces. For given $f, g : X \to Y$, we set

$$Df(x, y) = f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x),$$

and

$$D(f, g)(x, y) = f(kx+y)+f(kx-y)-g(x+y)-g(x-y)-\frac{2}{k}g(kx)+2g(x),$$

for all $x, y \in X$. Suppose that $k > 1$. We need the following lemma. The proof is found in [15].

Lemma 3.1. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *satisfying* (1.2). *Then the mapping* $G(x) := f(2x) - 8f(x)$ *is additive and* $H(x) := f(2x) - 2f(x)$ *is cubic.*

Theorem 3.2. *Let* $t \in \mathbb{N}$ *and* $\phi : X^{2t} \to [0, \infty)$ *be a function such that*

$$\phi(2x_1, \ldots, 2x_t, 2y_1, \ldots, 2y_t) \leq a\phi(x_1, \ldots, x_t, y_1, \ldots, y_t),$$

for some $a > 0$ *with* $a < 2$ *and all* $x_1, \ldots, x_t, y_1, \ldots, y_t \in X$. *Suppose that* $f : X \to Y$ *is a mapping such that* $f(0) = 0$ *and*

$$\|Df(x_1, y_1), \ldots, Df(x_t, y_t)\|_t \leq \phi(x_1, \ldots, x_t, y_1, \ldots, y_t), \tag{3.1}$$

for all $x_1, \ldots, x_t, y_1, \ldots, y_t \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(2x_1) - 8f(x_1) - A(x_1), \ldots, f(2x_t) - 8f(x_t) - A(x_t)\|_t \leq \frac{1}{2 - a}\psi(x_1, \ldots, x_t), \tag{3.2}$$

where $\psi : X^t \to [0, \infty)$ *is a mapping defined by*

(3.3)

$$\psi(x_1, \ldots, x_t) = \frac{1}{k^3 - k}[(1 + k)[\phi(x_1, \ldots, x_t, (2k + 1)x_1, \ldots, (2k + 1)x_t)$$

$$
\begin{aligned}
&+\phi(x_1,\dots,x_t,(2k-1)x_1,\dots,(2k-1)x_t)]\\
&+\phi(3x_1,\dots,3x_t,x_1,\dots,x_t)\\
&+(1+8k^2)\phi(x_1,\dots,x_t,x_1,\dots,x_t)\\
&+\phi(x_1,\dots,x_t,3kx_1,\dots,3kx_t)\\
&+\phi(x_1,\dots,x_t,kx_1,\dots,kx_t)\\
&+k^2\phi(2x_1,\dots,2x_t,2x_1,\dots,2x_t)\\
&+\phi(2x_1,\dots,2x_t,2kx_1,\dots,2kx_t)\\
&+2\phi(x_1,\dots,x_t,(k+1)x_1,\dots,(k+1)x_t)\\
&+2\phi(x_1,\dots,x_t,(k-1)x_1,\dots,(k-1)x_t)\\
&+2\phi(2x_1,\dots,2x_t,x_1,\dots,x_t)\\
&+2\phi(2x_1,\dots,2x_t,kx_1,\dots,kx_t)\\
&+8\phi\left(\frac{x_1}{2},\dots,\frac{x_t}{2},\frac{kx_1}{2},\dots,\frac{kx_t}{2}\right)\\
&+8k\phi\left(\frac{x_1}{2},\dots,\frac{x_t}{2},\frac{(2k-1)x_1}{2},\dots,\frac{(2k-1)x_t}{2}\right)\\
&+\frac{2}{k-1}\phi(0,\dots,0,x_1,\dots,x_t)\\
&+8\phi\left(\frac{x_1}{2},\dots,\frac{x_t}{2},\frac{3kx_1}{2},\dots,\frac{3kx_t}{2}\right)\\
&+\frac{k+1}{k-1}\phi(0,\dots,0,(k+1)x_1,\dots,(k+1)x_t)\\
&+8k\phi\left(\frac{x_1}{2},\dots,\frac{x_t}{2},\frac{(2k+1)x_1}{2},\dots,\frac{(2k+1)x_t}{2}\right)\\
&+\frac{1+8k^2}{k-1}\phi(0,\dots,0,(k-1)x_1,\dots,(k-1)x_t)\\
&+\frac{k}{k-1}\phi(0,\dots,0,(3k-1)x_1,\dots,(3k-1)x_t)\\
&+\frac{k^2}{k-1}\phi(0,\dots,0,2(k-1)x_1,\dots,2(k-1)x_t)\\
&+\frac{k^2+k-1}{k-1}\phi(0,\dots,0,2kx_1,\dots,2kx_t)\\
&+\frac{8k}{k-1}\phi\left(0,\dots,0,\frac{(3k-1)x_1}{2},\dots,\frac{(3k-1)x_t}{2}\right)\\
&+\frac{8k}{k-1}\phi\left(0,\dots,0,\frac{(k+1)x_1}{2},\dots,\frac{(k+1)x_t}{2}\right)
\end{aligned}
$$

$$+\frac{8k^2+2k-8}{k-1}\phi(0,\ldots,0,kx_1,\ldots,kx_t)$$

for all $x_1,\ldots,x_t \in X$.

Proof. Let $x_1,\ldots,x_t \in X$. Letting $x_1=\cdots=x_t=0$ in (3.1), we obtain (3.4)

$$\|f(y_1)+f(-y_1),\ldots,f(y_t)+f(-y_t)\|_t \le \frac{1}{k-1}\phi(0,\ldots,0,y_1,\ldots,y_t),$$

for all $y_1,\ldots,x_t \in X$. Letting $y_1=x_1,\ldots,y_t=x_t$ in (3.1), we get

$$\begin{aligned}(3.5)\qquad &\|f((k+1)x_1)+f((k-1)x_1)-kf(2x_1)-2f(kx_1)\\ &+2kf(x_1),\ldots,f((k+1)x_t)+f((k-1)x_t)\\ &-kf(2x_t)-2f(kx_t)+2kf(x_t)\|_t\\ &\le \phi(x_1,\ldots,x_t,x_1,\ldots,x_t).\end{aligned}$$

Hence

$$\begin{aligned}(3.6)\qquad &\|f(2(k+1)x_1)+f(2(k-1)x_1)-kf(4x_1)-2f(2kx_1)\\ &+2kf(2x_1),\ldots,f(2(k+1)x_t)+f(2(k-1)x_t)\\ &-kf(4x_t)-2f(2kx_t)+2kf(2x_t)\|_t\\ &\le \phi(2x_1,\ldots,2x_t,2x_1,\ldots,2x_t).\end{aligned}$$

Letting $y_1=kx_1,\ldots,y_t=kx_t$ in (3.1), we get

$$\begin{aligned}(3.7)\qquad &\|f(2kx_1)-kf((k+1)x_1)-kf(-(k-1)x_1)-2f(kx_1)\\ &+2kf(x_1),\ldots,f(2kx_t)-kf((k+1)x_t)\\ &-kf(-(k-1)x_t)-2f(kx_t)+2kf(x_t)\|_t\\ &\le \phi(x_1,\ldots,x_t,kx_1,\ldots,kx_t).\end{aligned}$$

Letting $y_1=(k+1)x_1,\ldots,y_t=(k+1)x_t$ in (3.1), we have

$$\begin{aligned}(3.8)\qquad &\|f((2k+1)x_1)+f(-x_1)-kf((k+2)x_1)-kf(-kx_1)\\ &-2f(kx_1)+2kf(x_1),\ldots,f((2k+1)x_t)+f(-x_t)\\ &-kf((k+2)x_t)-kf(-kx_t)-2f(kx_t)+2kf(x_t)\|_t\\ &\le \phi(x_1,\ldots,x_t,(k+1)x_1,\ldots,(k+1)x_t).\end{aligned}$$

Letting $y_1=(k-1)x_1,\ldots,y_t=(k-1)x_t$ in (3.1), we have

$$\begin{aligned}(3.9)\qquad &\|f((2k-1)x_1-(k+2)f(kx_1)-kf(-(k-2)x_1)\\ &+(2k+1)f(x_1),\ldots,f((2k-1)x_t-(k+2)f(kx_t)\end{aligned}$$

$$- kf(-(k-2)x_t) + (2k+1)f(x_t)\|_t \leq \phi(x_1, \ldots, x_t, (k-1)x_1, \ldots, (k-1)x_t).$$

Replacing $x_1 = 2x_1, \ldots, x_t = 2x_t, y_1 = x_1, \ldots, y_t = x_t$ in (3.1), we have

$$\begin{aligned}(3.10)\quad &\|f((2k+1)x_1) + f((2k-1)x_1) - 2f(2kx_1) - kf(3x_1)\\ &+ 2kf(2x_1) - kf(x_1), \ldots, f((2k+1)x_t) + f((2k-1)x_t)\\ &- 2f(2kx_t) - kf(3x_t) + 2kf(2x_t) - kf(x_t)\|_t\\ &\leq \phi(2x_1, \ldots, 2x_t, x_1, \ldots, x_t).\end{aligned}$$

Letting $x_1 = 3x_1, \ldots, x_t = 3x_t, y_1 = x_1, \ldots, y_t = x_t$ in (3.1), we obtain

$$\begin{aligned}(3.11)\quad &\|f((3k+1)x_1 + f((3k-1)x_1) - 2f(3kx_1) - kf(4x_1)\\ &- kf(2x_1) + 2kf(3x_1), \ldots, f((3k+1)x_t + f((3k-1)x_t)\\ &- 2f(3kx_t) - kf(4x_t) - kf(2x_t) + 2kf(3x_t)\|_t\\ &\leq \phi(3x_1, \ldots, 3x_t, x_1, \ldots, x_t).\end{aligned}$$

Setting $x_1 = 2x_1, \ldots, x_t = 2x_t, y_1 = kx_1, \ldots, y_t = kx_t$ in (3.1), we obtain

$$\begin{aligned}(3.12)\quad &\|f(3kx_1) + f(kx_1) - kf((k+2)x_1) - kf(-(k-2)x_1)\\ &- 2f(2kx_1) + 2kf(2x_1), \ldots, f(3kx_t) + f(kx_t) - kf((k+2)x_t)\\ &- kf(-(k-2)x_t) - 2f(2kx_t) + 2kf(2x_t)\|_t\\ &\leq \phi(2x_1, \ldots, 2x_t, kx_1, \ldots, kx_t).\end{aligned}$$

Letting $y_1 = (2k+1)x_1, \ldots, y_t = (2k+1)x_t$ in (3.1), we have

$$\begin{aligned}(3.13)\quad &\|f((3k+1)x_1) + f(-(k+1)x_1) - kf(2(k+1)x_1)\\ &- kf(-2kx_1) - 2f(kx_1) + 2kf(x_1), \ldots, f((3k+1)x_t)\\ &+ f(-(k+1)x_t) - kf(2(k+1)x_t)\\ &- kf(-2kx_t) - 2f(kx_t) + 2kf(x_t)\|_t\\ &\leq \phi(x_1, \ldots, x_t, (2k+1)x_1, \ldots, (2k+1)x_t).\end{aligned}$$

Letting $y_1 = (2k-1)x_1, \ldots, y_t = (2k-1)x_t$ in (3.1), we have

$$\begin{aligned}(3.14)\quad &\|f((3k-1)x_1) + f(-(k-1)x_1) - kf(-2(k-1)x_1)\\ &- kf(2kx_1) - 2f(kx_1) + 2kf(x_1), \ldots, f((3k-1)x_t)\end{aligned}$$

$$\begin{aligned}&+ f(-(k-1)x_t) - kf(-2(k-1)x_t)\\&- kf(2kx_t) - 2f(kx_t) + 2kf(x_t)\|_t\\&\le \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t).\end{aligned}$$

Putting $y_1 = 3kx_1, \dots, y_t = 3kx_t$ in (3.1), we have

$$\begin{aligned}\|&f(4kx_1) + f(-2kx_1) - kf((3k+1)x_1) - kf(-(3k-1)x_1)\\&- 2f(kx_1) + 2kf(x_1), \dots, f(4kx_t) + f(-2kx_t)\\&- kf((3k+1)x_t) - kf(-(3k-1)x_t)\\&- 2f(kx_t) + 2kf(x_t)\|_t\\&\le \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t).\end{aligned} \tag{3.15}$$

It follows from (3.4), (3.5), (3.11), (3.13) and (3.14) that

$$\begin{aligned}\|&kf(2(k+1)x_1) + kf(-2(k-1)x_1) + 6f(kx_1) - 2f(3kx_1)\\&- kf(4x_1) + 2kf(3x_1) - 6kf(x_1), \dots, kf(2(k+1)x_t)\\&+ kf(-2(k-1)x_t) + 6f(kx_t) - 2f(3kx_t)\\&- kf(4x_t) + 2kf(3x_t) - 6kf(x_t)\|_t\\&\le \phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t)\\&+ \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t)\\&+ \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t)\phi(x_1, \dots, x_t, x_1, \dots, x_t)\\&+ \frac{1}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t)\\&\times \frac{1}{k-1}\phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t)\\&+ \frac{k}{k-1}\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t).\end{aligned} \tag{3.16}$$

By using (3.4), (3.8) and (3.9) we deduce that

$$\begin{aligned}\|&f((2k+1)x_1) + f((2k-1)x_1) - kf((k+2)x_1) - kf(-(k-2)x_1)\\&- 4f(kx_1) + 4kf(x_1), \dots, f((2k+1)x_t) + f((2k-1)x_t)\\&- kf((k+2)x_t) - kf(-(k-2)x_t) - 4f(kx_t) + 4kf(x_t)\|_t\\&\le \phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t)\\&+ \phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t)\\&+ \frac{1}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) + \frac{k}{k-1}\phi(0, \dots, 0, kx_1, \dots, kx_t).\end{aligned} \tag{3.17}$$

It follows from (3.10) and (3.17) that

$$\begin{aligned}(3.18)\quad &\|kf((k+2)x_1)+kf(-(k-2)x_1)-2f(2kx_1)+4f(kx_1)\\ &-kf(3x_1)+2kf(2x_1)-5kf(x_1),\dots,kf((k+2)x_t)\\ &+kf(-(k-2)x_t)-2f(2kx_t)+4f(kx_t)\\ &-kf(3x_t)+2kf(2x_t)-5kf(x_t)\|_t\\ &\le \phi(x_1,\dots,x_t,(k+1)x_1,\dots,(k+1)x_t)\\ &+\phi(x_1,\dots,x_t,(k-1)x_1,\dots,(k-1)x_t)\\ &+\phi(2x_1,\dots,2x_t,x_1,\dots,x_t)+\frac{1}{k-1}\phi(0,\dots,0,x_1,\dots,x_t)\\ &+\frac{k}{k-1}\phi(0,\dots,0,kx_1,\dots,kx_t).\end{aligned}$$

From (3.12) and (3.18) we have

$$\begin{aligned}(3.19)\quad &\|f(3kx_1)-4f(2kx_1)+5f(kx_1)-kf(3x_1)+4kf(2x_1)\\ &-5kf(x_1),\dots,f(3kx_t)-4f(2kx_t)+5f(kx_t)\\ &-kf(3x_t)+4kf(2x_t)-5kf(x_t)\|_t\\ &\le \phi(x_1,\dots,x_t,(k+1)x_1,\dots,(k+1)x_t)\\ &+\phi(x_1,\dots,x_t,(k-1)x_1,\dots,(k-1)x_t)\\ &+\phi(2x_1,\dots,2x_t,x_1,\dots,x_t)+\phi(2x_1,\dots,2x_t,kx_1,\dots,kx_t)\\ &+\frac{1}{k-1}\phi(0,\dots,0,x_1,\dots,x_t)+\frac{k}{k-1}\phi(0,\dots,0,kx_1,\dots,kx_t).\end{aligned}$$

It follows from (3.4), (3.13), (3.14) and (3.15) that

$$\begin{aligned}(3.20)\quad &\|kf(-(k+1)x_1)-kf(-(k-1)x_1)-k^2f(2(k+1)x_1)\\ &+k^2f(-2(k-1)x_1)+k^2f(2kx_1)-(k^2-1)f(-2kx_1)\\ &+f(4kx_1)-2f(kx_1)+2kf(x_1),\dots,kf(-(k+1)x_t)\\ &-kf(-(k-1)x_t)-k^2f(2(k+1)x_t)+k^2f(-2(k-1)x_t)\\ &+k^2f(2kx_t)-(k^2-1)f(-2kx_t)+f(4kx_t)\\ &-2f(kx_t)+2kf(x_t)\|_t\\ &\le k\phi(x_1,\dots,x_t,(2k+1)x_1,\dots,(2k+1)x_t)\\ &+k\phi(x_1,\dots,x_t,(2k-1)x_1,\dots,(2k-1)x_t)\\ &+\phi(x_1,\dots,x_t,3kx_1,\dots,3kx_t)\\ &+\frac{k}{k-1}\phi(0,\dots,0,(3k-1)x_1,\dots,(3k-1)x_t).\end{aligned}$$

By (3.4), (3.6), (3.7) and (3.20), we get

$$\begin{aligned}(3.21)\qquad &\|f(4kx_1) - 2f(2kx_1) - k^3 f(4x_1) + 2k^3 f(2x_1), \ldots, f(4kx_t)\\ &\quad - 2f(2kx_t) - k^3 f(4x_t) + 2k^3 f(2x_t)\|_t\\ &\le k\phi(x_1, \ldots, x_t, (2k+1)x_1, \ldots, (2k+1)x_t)\\ &\quad + k\phi(x_1, \ldots, x_t, (2k-1)x_1, \ldots, (2k-1)x_t)\\ &\quad + \phi(x_1, \ldots, x_t, 3kx_1, \ldots, 3kx_t)\\ &\quad + \phi(x_1, \ldots, x_t, kx_1, \ldots, kx_t)\\ &\quad + k^2\phi(2x_1, \ldots, 2x_t, 2x_1, \ldots, 2x_t)\\ &\quad + \frac{k}{k-1}\phi(0, \ldots, 0, (3k-1)x_1, \ldots, (3k-1)x_t)\\ &\quad + \frac{k}{k-1}\phi(0, \ldots, 0, (k+1)x_1, \ldots, (k+1)x_t)\\ &\quad + \frac{k^2}{k-1}\phi(0, \ldots, 0, 2(k-1)x_1, \ldots, 2(k-1)x_t)\\ &\quad + (k+1)\phi(0, \ldots, 0, 2kx_1, \ldots, 2kx_t).\end{aligned}$$

Therefore we have

$$\begin{aligned}(3.22)\qquad &\|f(2kx_1) - 2f(kx_1) - k^3 f(2x_1) + 2k^3 f(x_1), \ldots, f(2kx_t)\\ &\quad - 2f(kx_t) - k^3 f(2x_t) + 2k^3 f(x_t)\|_t\\ &\le k\phi\left(\frac{x_1}{2}, \ldots, \frac{x_t}{2}, \frac{(2k+1)x_1}{2}, \ldots, \frac{(2k+1)x_t}{2}\right)\\ &\quad + k\phi\left(\frac{x_1}{2}, \ldots, \frac{x_t}{2}, \frac{(2k-1)x_1}{2}, \ldots, \frac{(2k-1)x_t}{2}\right)\\ &\quad + \phi\left(\frac{x_1}{2}, \ldots, \frac{x_t}{2}, \frac{3kx_1}{2}, \ldots, \frac{3kx_t}{2}\right)\\ &\quad + \phi\left(\frac{x_1}{2}, \ldots, \frac{x_t}{2}, \frac{kx_1}{2}, \ldots, \frac{kx_t}{2}\right)\\ &\quad + k^2\phi(x_1, \ldots, x_t, x_1, \ldots, x_t)\\ &\quad + \frac{k}{k-1}\phi\left(0, \ldots, 0, \frac{(3k-1)x_1}{2}, \ldots, \frac{(3k-1)x_t}{2}\right)\\ &\quad + \frac{k}{k-1}\phi\left(0, \ldots, 0, \frac{(k+1)x_1}{2}, \ldots, \frac{(k+1)x_t}{2}\right)\\ &\quad + \frac{k^2}{k-1}\phi(0, \ldots, 0, (k-1)x_1, \ldots, (k-1)x_t)\end{aligned}$$

$$+ (k+1)\phi(0,\dots,0,kx_1,\dots,kx_t).$$

By using (3.7), we obtain

$$\begin{aligned}
&\|f(4kx_1) - kf(2(k+1)x_1) - kf(-2(k-1)x_1) \\
&\quad - 2f(2kx_1) + 2kf(2x_1),\dots,f(4kx_t) - kf(2(k+1)x_t) \\
&\quad - kf(-2(k-1)x_t) - 2f(2kx_t) + 2kf(2x_t)\|_t \\
&\le \phi(2x_1,\dots,2x_t,2kx_1,\dots,2kx_t).
\end{aligned} \tag{3.23}$$

It follows from (3.21) and (3.23) that

$$\begin{aligned}
&\|kf(2(k+1)x_1) + kf(-2(k-1)x_1) - k^3 f(4x_1) \\
&\quad + (2k^3-2k)f(2x_1),\dots,kf(2(k+1)x_t) \\
&\quad + kf(-2(k-1)x_t) - k^3 f(4x_t) + (2k^3-2k)f(2x_t)\|_t \\
&\le k\phi(x_1,\dots,x_t,(2k+1)x_1,\dots,(2k+1)x_t) \\
&\quad + k\phi(x_1,\dots,x_t,(2k-1)x_1,\dots,(2k-1)x_t) \\
&\quad + \phi(x_1,\dots,x_t,3kx_1,\dots,3kx_t) \\
&\quad + \phi(x_1,\dots,x_t,kx_1,\dots,kx_t) \\
&\quad + k^2\phi(2x_1,\dots,2x_t,2x_1,\dots,2x_t) \\
&\quad + \phi(2x_1,\dots,2x_t,2kx_1,\dots,2kx_t) \\
&\quad + \frac{k}{k-1}\phi(0,\dots,0,(3k-1)x_1,\dots,(3k-1)x_t) \\
&\quad + \frac{k}{k-1}\phi(0,\dots,0,(k+1)x_1,\dots,(k+1)x_t) \\
&\quad + \frac{k^2}{k-1}\phi(0,\dots,0,2(k-1)x_1,\dots,2(k-1)x_t) \\
&\quad + (k+1)\phi(0,\dots,0,2kx_1,\dots,2kx_t).
\end{aligned} \tag{3.24}$$

Using (3.16) and (3.24), we deduce that

$$\begin{aligned}
&\|2f(3kx_1) - 6f(kx_1) + (k-k^3)f(4x_1) - 2kf(3x_1) \\
&\quad + (2k^3-2k)f(2x_1) + 6kf(x_1),\dots,2f(3kx_t) \\
&\quad - 6f(kx_t) + (k-k^3)f(4x_t) - 2kf(3x_t) \\
&\quad + (2k^3-2k)f(2x_t) + 6kf(x_t)\|_t \\
&\le (1+k)[\phi(x_1,\dots,x_t,(2k+1)x_1,\dots,(2k+1)x_t) \\
&\quad + \phi(x_1,\dots,x_t,(2k-1)x_1,\dots,(2k-1)x_t)] \\
&\quad + \phi(3x_1,\dots,3x_t,x_1,\dots,x_t) + \phi(x_1,\dots,x_t,x_1,\dots,x_t)
\end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&+ \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
&+ \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
&+ k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
&+ \frac{1}{k-1}\phi(0, \dots, 0, (k-1)x_1, \dots, (k-1)x_t) \\
&+ \frac{k+1}{k-1}\phi(0, \dots, 0, (k+1)x_1, \dots, (k+1)x_t) \\
&+ \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\
&+ \frac{(k^2+k-1)}{k-1}\phi(0, \dots, 0, 2kx_1, \dots, 2kx_t) \\
&+ \frac{k}{k-1}\phi(0, \dots, 0, (3k-1)x_1, \dots, (3k-1)x_t) \\
&+ \frac{k^2}{k-1}\phi(0, \dots, 0, 2(k-1)x_1, \dots, 2(k-1)x_t).
\end{aligned}
$$

Moreover from (3.19) and (3.25), we have

$$
\begin{aligned}
(3.26)\quad &\|8f(2kx_1) - 16f(kx_1) + (k-k^3)f(4x_1) + (2k^3-10k)f(2x_1) \\
&+ 16kf(x_1), \dots, f(2kx_t) - 16f(kx_t) + (k-k^3)f(4x_t) \\
&+ (2k^3-10k)f(2x_t) + 16kf(x_t)\|_t \\
&\le (1+k)[\phi(x_1, \dots, x_t, (2k+1)x_1, \dots, (2k+1)x_t) \\
&+ \phi(x_1, \dots, x_t, (2k-1)x_1, \dots, (2k-1)x_t)] \\
&+ \phi(3x_1, \dots, 3x_t, x_1, \dots, x_t) \\
&+ \phi(x_1, \dots, x_t, x_1, \dots, x_t) \\
&+ \phi(x_1, \dots, x_t, 3kx_1, \dots, 3kx_t) \\
&+ \phi(x_1, \dots, x_t, kx_1, \dots, kx_t) \\
&+ k^2\phi(2x_1, \dots, 2x_t, 2x_1, \dots, 2x_t) \\
&+ \phi(2x_1, \dots, 2x_t, 2kx_1, \dots, 2kx_t) \\
&+ 2\phi(x_1, \dots, x_t, (k+1)x_1, \dots, (k+1)x_t) \\
&+ 2\phi(x_1, \dots, x_t, (k-1)x_1, \dots, (k-1)x_t) \\
&+ 2\phi(2x_1, \dots, 2x_t, x_1, \dots, x_t) \\
&+ 2\phi(2x_1, \dots, 2x_t, kx_1, \dots, kx_t) \\
&+ \frac{2}{k-1}\phi(0, \dots, 0, x_1, \dots, x_t) \\
&+ \frac{2k}{k-1}\phi(0, \dots, 0, kx_1, \dots, kx_t)
\end{aligned}
$$

$$+\frac{k+1}{k-1}\phi(0,\dots,0,(k+1)x_1,\dots,(k+1)x_t)$$
$$+\frac{1}{k-1}\phi(0,\dots,0,(k-1)x_1,\dots,(k-1)x_t)$$
$$+\frac{k^2+k-1}{k-1}\phi(0,\dots,0,2kx_1,\dots,2kx_t)$$
$$+\frac{k}{k-1}\phi(0,\dots,0,(3k-1)x_1,\dots,(3k-1)x_t)$$
$$+\frac{k^2}{k-1}\phi(0,\dots,0,2(k-1)x_1,\dots,2(k-1)x_t).$$

So by (3.22) and (3.26), we get

$$\|f(4x_1)-10f(2x_1)+16f(x_1),\dots,f(4x_t)-10f(2x_t)+16f(x_t)\|_t \leq \psi(x_1,\dots,x_t),$$

for all $x_1,\dots,x_t \in X$. Let $g : X \to Y$ be the mapping defined by $g(x) := f(2x) - 8f(x)$. Therefore we have

$$\|g(2x_1)-2g(x_1),\dots,g(2x_t)-2g(x_t)\|_t \leq \psi(x_1,\dots,x_t),$$

for all $x_1,\dots,x_t \in X$. It is easy to see that the mapping $A : X \to Y$ given by

$$A(x) := \lim_{n\to\infty}\frac{g(2^n x)}{2^n},$$

satisfies

$$\|g(x_1)-A(x_1),\dots,g(x_t)-A(x_t)\|_t \leq \frac{1}{2-a}\psi(x_1,\dots,x_t),$$

for all $x_1,\dots,x_t \in X$. So (3.2), is satisfied. Moreover the mapping A satisfies (1.2) and $A(2x) = 2A(x)$ for all $x \in X$. By Lemma 3.1 the mapping A is additive. The uniqueness assertion is easy and we omit its proof. □

Theorem 3.3. *Let $t \in \mathbb{N}$ and $\phi : X^{2t} \to [0,\infty)$ be a function such that*

$$\phi(2x_1,\dots,2x_t,2y_1,\dots,2y_t) \leq a\phi(x_1,\dots,x_t,y_1,\dots,y_t),$$

for some $a > 0$ with $a < 8$ and all $x_1,\dots,x_t,y_1,\dots,y_t \in X$. Suppose that $f : X \to Y$ is a mapping such that $f(0) = 0$ and

$$\|Df(x_1,y_1),\dots,Df(x_t,y_t)\|_t \leq \phi(x_1,\dots,x_t,y_1,\dots,y_t),$$

for all $x_1,\dots,x_t,y_1,\dots,y_t \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x_1)-2f(x_1)-C(x_1),\dots,f(2x_t)-2f(x_t)-C(x_t)\|_t \leq \frac{1}{8-a}\psi(x_1,\dots,x_t), \tag{3.27}$$

where $\psi : X^t \to [0, \infty)$ *is a mapping defined by* (3.3) *for all* $x_1, \dots, x_t \in X$.

Proof. We know from Theorem 3.2 that

$$\|f(4x_1) - 10f(2x_1) + 16f(x_1), \dots, f(4x_t) - 10f(2x_t) + 16f(x_t)\|_t \leq \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. Let $h : X \to Y$ be the mapping defined by $h(x) := f(2x) - 2f(x)$. Therefore, we have

$$\|h(2x_1) - 8h(x_1), \dots, h(2x_t) - 8h(x_t)\|_t \leq \psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. It is easy to see that the mapping $C : X \to Y$ given by

$$C(x) := \lim_{n\to\infty} \frac{h(2^n x)}{8^n},$$

satisfies

$$\|h(x_1) - C(x_1), \dots, h(x_t) - C(x_t)\|_t \leq \frac{1}{8-a}\psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. So (3.27), is satisfied. Moreover the mapping C satisfies (1.2) and $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 3.1 the mapping C is cubic. The uniqueness assertion is easy and we omit its proof. □

Theorem 3.4. *Let* $t \in \mathbb{N}$ *and* $\phi : X^{2t} \to [0, \infty)$ *be a function such that*

$$\phi(2x_1, \dots, 2x_t, 2y_1, \dots, 2y_t) \leq a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ *with* $a < 2$ *and all* $x_1, \dots, x_t, y_1, \dots, y_t \in X$. *Suppose that* $f : X \to Y$ *is a mapping such that* $f(0) = 0$ *and*

$$\|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *and a unique cubic mapping* $C : X \to Y$ *such that*

$$\|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \leq \frac{1}{6}\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi(x_1, \dots, x_t), \tag{3.28}$$

where $\psi : X^t \to [0, \infty)$ *is a mapping defined by* (3.3) *for all* $x_1, \dots, x_t \in X$.

Proof. By Theorems 3.2 and 3.3, there exist a unique additive mapping $A_0 : X \to Y$ and a cubic mapping $C_0 : X \to Y$ such that

$$\|f(2x_1) - 8f(x_1) - A_0(x_1), \dots, f(2x_t) - 8f(x_t) - A_0(x_t)\|_t \leq \frac{1}{2-a}\psi(x_1, \dots, x_t),$$

and

$$\|f(2x_1) - 2f(x_1) - C_0(x_1), \dots, f(2x_t) - 2f(x_t) - C_0(x_t)\|_t \\ \leq \frac{1}{8-a}\psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. Hence

$$\|f(x_1) + \frac{1}{6}A_0(x_1) - \frac{1}{6}C_0(x_1), \dots, f(x_t) + \frac{1}{6}A_0(x_t) - \frac{1}{6}C_0(x_t)\|_t \\ \leq \frac{1}{6}\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi(x_1, \dots, x_t),$$

for all $x_1, \dots, x_t \in X$. Letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$, we obtain (3.28). For the uniqueness assertion, let A_1, C_1 be another additive and cubic mapping satisfying (3.28). Let $A' = A - A_1$ and $C' = C - C_1$. So

(3.29)

$$\|A'(x) + C'(x)\| \leq \|f(x) - A(x) - C(x)\| + \|f(x) - A_1(x) - C_1(x)\| \\ \leq \frac{1}{3}\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi(x, \dots, x),$$

for all $x \in X$. So

$$\frac{1}{8^n}\|A'(2^n x) + C'(2^n x)\| \leq \left(\frac{a}{8}\right)^n \left(\frac{1}{3}\right)\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi(x, \dots, x),$$

for all $x \in X$. Hence $C' = 0$. It follows from (3.29) that

$$\|A'(x)\| \leq \frac{1}{3}\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi(x, \dots, x),$$

for all $x \in X$. Hence $A' = 0$. □

Following the same idea as in the proof of Theorems 3.2, 3.3 and 3.4, we get the following corollary.

Corollary 3.5. *Let $\alpha \geq 0$ and $f : X \to Y$ be a mapping such that $f(0) = 0$ and*

$$\sup_{t \in \mathbb{N}} \|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \leq \alpha,$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\sup_{t \in \mathbb{N}} \|f(x_1) - A(x_1) - C(x_1), \dots, f(x_t) - A(x_t) - C(x_t)\|_t \\ \leq \frac{4}{21}\left(\frac{\alpha}{k^3 - k}\right)\left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1}\right],$$

for all $x_1, \ldots, x_t \in X$.

Theorem 3.6. *Let* $\alpha \geq 0$ *and* $f, g : X \to Y$ *be two mappings such that* $f(0) = g(0) = 0$ *and*

$$\sup_{t\in\mathbb{N}} \|D(f,g)(x_1,y_1), \ldots, D(f,g)(x_t,y_t)\|_t \leq \alpha,$$

for all $x_1, \ldots, x_t, y_1, \ldots, y_t \in X$. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique cubic mapping* $C : X \to Y$ *such that*

$$\sup_{t\in\mathbb{N}} \|f(x_1) - A(x_1) - C(x_1), \ldots, f(x_t) - A(x_t) - C(x_t)\|_t$$
$$\leq \frac{4}{21}\left(\frac{2\alpha}{k(k-1)}\right)\left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1}\right],$$

and

$$\sup_{t\in\mathbb{N}} \|g(x_1) - kA(x_1) - kC(x_1), \ldots, g(x_t) - kA(x_t) - kC(x_t)\|_t$$
$$\leq \frac{4}{21}\left(\frac{2\alpha}{k^2-1}\right)\left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1}\right],$$

where $x_1, \ldots, x_t \in X$.

Proof. It is easy to see that

$$(3.30)\quad Df(x,y) = D(f,g)(x,y) - D(f,g)(x,0) - \frac{k}{2}D(f,g)\left(\frac{x+y}{k}, 0\right) - \frac{k}{2}D(f,g)\left(\frac{x-y}{k}, 0\right) + kD(f,g)\left(\frac{x}{k}, 0\right),$$

for all $x, y \in X$. Using (3.30), we get

$$\sup_{t\in\mathbb{N}} \|Df(x_1,y_1), \ldots, Df(x_t,y_t)\|_t \leq 2(k+1)\alpha,$$

for all $x_1, \ldots, x_t, y_1, \ldots, y_t \in X$. It follows from Theorem 3.5 that there exist a unique additive mapping $A_1 : X \to Y$ and a unique cubic mapping $C_1 : X \to Y$ such that

$$\sup_{t\in\mathbb{N}} \|f(x_1) - A_1(x_1) - C_1(x_1), \ldots, f(x_t) - A_1(x_t) - C_1(x_t)\|_t$$
$$\leq \frac{4}{21}\left(\frac{2\alpha}{k(k-1)}\right)\left[9k^2 + 18k + 31 + \frac{18k^2 + 21k - 5}{k-1}\right],$$

for all $x_1, \ldots, x_t \in X$. Moreover we see that

$$(3.31)\qquad Dg(x,y) = kD(f,g)(x,y) - \frac{k}{2}D(f,g)\left(\frac{kx+y}{k}, 0\right)$$

$$-\frac{k}{2}D(f,g)\left(\frac{kx-y}{k},0\right),$$

for all $x, y \in X$. It follows from (3.31) that

$$\sup_{t\in\mathbb{N}}\|Dg(x_1,y_1),\dots,Dg(x_t,y_t)\|_t \le 2k\alpha,$$

for all $x_1,\dots,x_t,y_1,\dots,y_t \in X$. Again using Theorem 3.5, we conclude that there exist a unique additive mapping $A_2 : X \to Y$ and a unique cubic mapping $C_2 : X \to Y$ such that

$$\sup_{t\in\mathbb{N}}\|g(x_1)-A_2(x_1)-C_2(x_1),\dots,g(x_t)-A_2(x_t)-C_2(x_t)\|_t$$
$$\le \frac{4}{21}\left(\frac{2\alpha}{k^2-1}\right)\left[9k^2+18k+31+\frac{18k^2+21k-5}{k-1}\right],$$

for all $x_1,\dots,x_t \in X$.
We show that $A_2 = kA_1$. To this end we have

$$\frac{k}{2}D(f,g)\left(\frac{x}{k},0\right) = kf(x)-g(x),$$

for all $x \in X$. Therefore, we get

$$\|kA_1(x)-A_2(x)\| \le \left(\frac{1}{6}\right)\left(\frac{1}{2^n}\right)\lim_{n\to\infty}\frac{k}{2}\left\|D(f,g)\left(\frac{2^{n+1}x}{k},0\right)\right\|$$
$$+8\frac{k}{2}\left\|D(f,g)\left(\frac{2^n x}{k},0\right)\right\|.$$

So $A_2 = kA_1$. Similarly $C_2 = kC_1$. The proof of the uniqueness assertion is similar to the ones of Theorem 3.4. □

We prove the generalized Hyers-Ulam stability of the Pexiderized cubic functional equation in multi-normed spaces

Theorem 3.7. *Let $t \in \mathbb{N}$ and $\phi : X^{2t} \to [0,\infty)$ be a function such that*

$$\phi(2x_1,\dots,2x_t,2y_1,\dots,2y_t) \le a\phi(x_1,\dots,x_t,y_1,\dots,y_t),$$

for some $a > 0$ with $a < 2$ and all $x_1,\dots,x_t,y_1,\dots,y_t \in X$. Suppose that $f, g : X \to Y$ are two mappings such that $f(0) = g(0) = 0$ and

$$\|D(f,g)(x_1,y_1),\dots,D(f,g)(x_t,y_t)\|_t \le \phi(x_1,\dots,x_t,y_1,\dots,y_t),$$

and $\phi_1, \phi_2 : X^{2t} \to [0,\infty)$ are two mappings defined by

$$\phi_1(x_1,\dots,x_t,y_1,\dots,y_t) := \phi(x_1,\dots,x_t,y_1,\dots,y_t)$$
$$+\phi(x_1,\dots,x_t,0,\dots,0)$$
$$+\frac{k}{2}\phi\left(\frac{x_1+y_1}{k},\dots,\frac{x_t+y_t}{k},0,\dots,0\right)$$

$$+\frac{k}{2}\phi\left(\frac{x_1-y_1}{k},\dots,\frac{x_t-y_t}{k},0,\dots,0\right)$$
$$+k\phi\left(\frac{x_1}{k},\dots,\frac{x_t}{k},0,\dots,0\right),$$

and

$$\phi_2(x_1,\dots,x_t,y_1,\dots,y_t):=k\phi(x_1,\dots,x_t,y_1,\dots,y_t)$$
$$+\frac{k}{2}\phi\left(\frac{kx_1+y_1}{k},\dots,\frac{kx_t+y_t}{k},0,\dots,0\right)$$
$$+\frac{k}{2}\phi\left(\frac{kx_1-y_1}{k},\dots,\frac{kx_t-y_t}{k},0,\dots,0\right),$$

for all $x_1,\dots,x_t,y_1,\dots,y_t\in X$. Then there exist a unique additive mapping $A:X\to Y$ and a unique cubic mapping $C:X\to Y$ such that

$$\|f(x_1)-A(x_1)-C(x_1),\dots,f(x_t)-A(x_t)-C(x_t)\|_t$$
$$\le\frac{1}{6}\left(\frac{1}{2-a}+\frac{1}{8-a}\right)\psi_1(x_1,\dots,x_t),$$

and

$$\|g(x_1)-kA(x_1)-kC(x_1),\dots,g(x_t)-kA(x_t)-kC(x_t)\|_t$$
$$\le\frac{1}{6}\left(\frac{1}{2-a}+\frac{1}{8-a}\right)\psi_2(x_1,\dots,x_t),$$

for all $x_1,\dots,x_t\in X$ where $\psi_1,\psi_2:X^t\to[0,\infty)$ are two mappings defined similar to the definition of ψ in Theorem 3.2 with this difference that ϕ (in the definition of ψ) is replaced by ϕ_1 and ϕ_2 anywhere respectively.

Proof. It follows from (3.30) that

$$\|Df(x_1,y_1),\dots,Df(x_t,y_t)\|_t\le\phi_1(x_1,\dots,x_t,y_1,\dots,y_t),$$

for all $x_1,\dots,x_t,y_1,\dots,y_t\in X$. Using Theorem 3.4, we conclude that there exist a unique additive mapping $A_1:X\to Y$ and a unique cubic mapping $C_1:X\to Y$ such that

$$\|f(x_1)-A_1(x_1)-C_1(x_1),\dots,f(x_t)-A_1(x_t)-C_1(x_t)\|_t$$
$$\le\frac{1}{6}\left(\frac{1}{2-a}+\frac{1}{8-a}\right)\psi_1(x_1,\dots,x_t),$$

for all $x_1,\dots,x_t\in X$. Moreover, using (3.31), we get

$$\|Dg(x_1,y_1),\dots,Dg(x_t,y_t)\|_t\le\phi_2(x_1,\dots,x_t,y_1,\dots,y_t),$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Using Theorem 3.4 again, we deduce that there exist a unique additive mapping $A_2 : X \to Y$ and a unique cubic mapping $C_2 : X \to Y$ such that

$$\begin{aligned}\|f(x_1) - A_2(x_1) - C_2(x_1), \dots, f(x_t) - A_2(x_t) - C_2(x_t)\|_t \\ \le \frac{1}{6}\left(\frac{1}{2-a} + \frac{1}{8-a}\right)\psi_2(x_1, \dots, x_t),\end{aligned}$$

for all $x_1, \dots, x_t \in X$.

Similar to the proof of Theorem 3.6, we have $A_2 = kA_1$ and $C_2 = kC_1$. The proof of the uniqueness assertion is similar to ones proof of Theorem 3.4. □

4. Stability of the 2-variables cubic functional equation

In this section, we prove the Hyers-Ulam stability of the 2-variables, cubic functional equation

$$\begin{aligned}f(2x+y, 2z+t) + f(2x-y, 2z-t) = 2f(x+y, z+t) \\ + 2f(x-y, z-t) + 12f(x, z),\end{aligned} \tag{4.1}$$

in multi-normed spaces. Throughout this section, assume that X is a vector space.

Lemma 4.1. *If $f : X^2 \to Y$ is a 2-variables cubic mapping, then $f(2^n x, 2^n y) = 8^n f(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$.*

Proof. Letting $y = t = 0$ in (4.1), we get $f(2x, 2z) = 8f(x, z)$ for all $x, z \in X$. So by induction we have $f(2^n x, 2^n y) = 8^n f(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$. □

Given a mapping $f : X^2 \to Y$, we set

$$\begin{aligned}Df(x, y, z, t) = f(2x+y, 2z+t) + f(2x-y, 2z-t) - 2f(x+y, z+t) \\ - 2f(x-y, z-t) - 12f(x, z),\end{aligned}$$

for all $x, y, z, t \in X$.

Theorem 4.2. *Let $\alpha \ge 0$ and $f : X \to Y$ be a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1, z_1, t_1), \dots, Df(x_k, y_k, z_k, t_k)\|_k \le \alpha, \tag{4.2}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, t_1, \dots, t_k \in X$. Then there exists a unique 2-variable cubic mapping $C : X \to Y$ such that

$$\sup_{k\in\mathbb{N}} \|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k \le \frac{\alpha}{14}, \tag{4.3}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$.

Proof. Let $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Letting $z_1 = y_1, z_2 = y_2, \dots, z_k = y_k$ and $y_1 = \cdots = y_k = t_1 = \cdots = t_k = 0$ in (4.2), we get

$$\sup_{k \in \mathbb{N}} \|f(2x_1, 2y_1) - 8f(x_1, y_1), \dots, f(2x_k, 2y_k) - 8f(x_k, y_k)\|_k \leq \frac{\alpha}{2},$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$.

Replacing $x_1, \dots, x_k, y_1, \dots, y_k$ by $2^n x_1, \dots, 2^n x_k, 2^n y_1, \dots, 2^n y_k$ and dividing the last enequality by 8^{n+1}, one gets

$$\sup_{k \in \mathbb{N}} \|\frac{f(2^{n+1}x_1, 2^{n+1}y_1)}{8^{n+1}} - \frac{f(2^n x_1, 2^n y_1)}{8^n}, \dots, \frac{f(2^{n+1}x_k, 2^{n+1}y_k)}{8^{n+1}} - \frac{f(2^n x_k, 2^n y_k)}{8^n}\|_k \leq \frac{\alpha}{2}\left(\frac{1}{8^{n+1}}\right).$$

It follows that $\left(\frac{f(2^n x, 2^n y)}{8^n}\right)$ is Cauchy and so is convergent in the complete multi-normed space Y. Set

$$C(x, y) := \lim_{n \to \infty} \frac{f(2^n x, 2^n y)}{8^n}.$$

It is easy to see that

$$\sup_{k \in \mathbb{N}} \|f(x_1, y_1) - C(x_1, y_1), \dots, f(x_k, y_k) - C(x_k, y_k)\|_k \leq \frac{\alpha}{14}.$$

Let $x, y, z, t \in X$. Replacing $x_1 = \cdots = x_k = 2^n x, y_1 = \cdots = y_k = 2^n y, z_1 = \cdots = z_k = 2^n z, t_1 = \cdots = t_k = 2^n t$ in (4.2) and dividing the both sides by 8^n, we obtain

$$\frac{1}{8^n}\|Df(2^n x, 2^n y, 2^n z, 2^n t)\| \leq \frac{\alpha}{8^n}.$$

Taking the limit as $n \to \infty$, we get

$$DC(x, y, z, t) = 0.$$

Hence C is a 2-variables cubic mapping. Let C' be another 2-variables cubic mapping satisfying (4.3). So by Lemma 4.1 we have

$$\begin{aligned}
\|C(x, y) - C'(x, y)\| &= \frac{1}{8^n}\|C(2^n x, 2^n y) - C'(2^n x, 2^n y)\| \\
&\leq \frac{1}{8^n}\|C(2^n x, 2^n y) - f(2^n x, 2^n y)\| \\
&\quad + \frac{1}{8^n}\|f(2^n x, 2^n y) - C'(2^n x, 2^n y)\| \\
&\leq \frac{1}{8^n}\left(\frac{\alpha}{14} + \frac{\alpha}{14}\right).
\end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. $\square$

We prove the generalized Hyers-Ulam stability of the 2-variables cubic functinoal equation in multi-normed spaces by using the fixed point method.

Theorem 4.3. *Let $k \in \mathbb{N}$ and $\phi : X^{4k} \to [0, \infty)$ be a function such that*

$$\phi(2x_1, \ldots, 2x_k, 2y_1, \ldots, 2y_k, 2z_1, \ldots, 2z_k, 2t_1, \ldots, 2t_k) \\ \leq a\phi(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_k),$$

for some $a > 0$ with $a < 8$ and all

$$x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_k \in X.$$

Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x_1, y_1, z_1, t_1), \ldots, Df(x_k, y_k, z_k, t_k)\|_k \\ \leq \phi(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, t_1, \ldots, t_k), \tag{4.4}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Then there exists a unique 2-variable cubic mapping $C : X \to Y$ such that

$$\|f(x_1, y_1) - C(x_1, y_1), \ldots, f(x_k, y_k) - C(x_k, y_k)\|_k \\ \leq \frac{1}{16 - 2a}\phi(x_1, \ldots, x_k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0), \tag{4.5}$$

for all $x_1, \ldots, x_k \in X$.

Proof. Let $E = \{f : X^2 \to Y\}$, and consider the generalized metric d defined on E by

$$d(g, h) = \inf \Big\{c > 0, \|g(x_1, y_1) - h(x_1, y_1), \ldots g(x_k, y_k) - h(x_k, y_k)\|_k \\ \leq c\psi(x_1, \ldots, x_k, y_1, \ldots, y_k), \forall x_1, \ldots, x_k, y_1, \ldots, y_k \in X\Big\},$$

where $\psi : X^{2k} \to [0, \infty)$ is a mapping defined by

$$\psi(x_1, \ldots, x_k, y_1, \ldots, y_k) = \phi(x_1, \ldots, x_k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Then, it is easy to show that d is a complete generalized metric on E. We now define a mapping $J : E \to E$ by

$$Jg(x, y) = \frac{1}{8}g(2x, 2y), \quad \forall x, y \in X.$$

We claim that J is a strictly contractive mapping. Given $g, h \in E$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) < c$. It is easy to see that

$$\|g(x_1, y_1) - h(x_1, y_1), \ldots, g(x_k, y_k) - h(x_k, y_k)\|_k \\ \leq c\psi(x_1, \ldots, x_k, y_1, \ldots, y_k),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Therefore,

$$\begin{aligned}
&\|Jg(x_1, y_1) - Jh(x_1, y_1), \ldots, Jg(x_k, y_k) - Jh(x_k, y_k)\|_k \\
&\quad = \left\| \frac{1}{8} g(2x_1, 2y_1) - \frac{1}{8} h(2x_1, 2y_1), \ldots, \frac{1}{8} g(2x_k, 2y_k) - \frac{1}{8} h(2x_k, 2y_k) \right\|_k \\
&\quad \le \frac{1}{8} c\psi(2x_1, \ldots, 2x_k, 2y_1, \ldots, 2y_k) \le \frac{1}{8} ac\psi(x_1, \ldots, x_k, y_1, \ldots, y_k),
\end{aligned}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. So $d(Jg, Jh) \le \frac{a}{8} d(g, h)$. Letting $z_1 = y_1, z_2 = y_2, \ldots, z_k = y_k$ and $y_1 = \cdots = y_k = t_1 = \cdots = t_k = 0$ in (4.4), we get

$$\begin{aligned}
(4.6) \qquad &\|f(2x_1, 2y_1) - 8f(x_1, y_1), \ldots, f(2x_k, 2y_k) - 8f(x_k, y_k)\|_k \\
&\qquad \le \frac{1}{2} \psi(x_1, \ldots, x_k, y_1, \ldots, y_k),
\end{aligned}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. It follows from (4.6) that $d(Jf, f) \le \frac{1}{16}$. According to Lemma 1.4, we deduce the existence of a fixed point of J, that is, the existence of a mapping $C : X \to Y$ such that $C(2x, 2y) = 8C(x, y)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \to 0$ that

$$\begin{aligned}
C(x, y) &= \lim_{n\to\infty} (J^n f)(x, y) \\
&= \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{8^n}, \quad \forall x \in X.
\end{aligned}$$

Moreover $d(f, C) \le \frac{1}{1-\frac{a}{8}} d(Jf, f)$, implies the inequality

$$d(f, C) \le \frac{1}{16 - 2a}.$$

So (4.5) is satisfied.

Let $x, y, z, t \in X$. Letting $x_1 = \cdots = x_k = 2^n x, y_1 = \cdots = y_k = 2^n y, z_1 = \cdots = z_k = 2^n z, t_1 = \cdots = t_k = 2^n t$ in (4.4) and dividing the both sides by 8^n, we obtain

$$\begin{aligned}
\frac{1}{8^n} \|Df(2^n x, 2^n y, 2^n z, 2^n t)\| &\le \frac{1}{8^n} \phi(2^n x, \ldots, 2^n x, 2^n y, \ldots, 2^n y, \\
&\qquad 2^n z, \ldots, 2^n z, 2^n t, \ldots, 2^n t) \\
&\le \left(\frac{a}{8}\right)^n \phi(x, \ldots, x, y, \ldots, y, z, \ldots, z, t, \ldots, t).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$DC(x, y, z, t) = 0.$$

Hence C is a 2-variables cubic mapping.

The uniqueness of C follows from the fact that C is the unique fixed point of J with the property that there exists $\lambda \in (0, \infty)$ such that

$$\|f(x_1, y_1) - C(x_1, y_1), \ldots, f(x_k, y_k) - C(x_k, y_k)\|_k$$

$$\leq \lambda\psi(x_1,\dots,x_k,y_1,\dots,y_k),$$

for all $x_1,\dots,x_k,y_1,\dots,y_k \in X$. □

5. STABILITY OF THE CUBIC TYPE FUNCTIOAL EQUATION

In this section, we prove the Hyers-Ulam stability of the orthogonally cubic type functional equation in multi-normed spaces. Given a mapping $f : X \to Y$, we set

$$\begin{aligned} Df(x_1,x_2,x_3) = {} & f(x_1+x_2+2x_3) + f(x_1+x_2-2x_3) \\ & + f(2x_1) + f(2x_2) + 7[f(x_1)+f(-x_1)] - 2f(x_1+x_2) \\ & - 4[f(x_1+x_3)+f(x_1-x_3)+f(x_2+x_3)+f(x_2-x_3)], \end{aligned}$$

for all $x_1,x_2,x_3 \in X$ with $x_i \perp x_j (i,j=1,2,3)$.
We need the following lemma. The proof is found in [2].

Lemma 5.1. *Let X and Y be an orthogonal space and a real vector space, respectively. If a mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x_1,x_2,x_3 \in X$ with $x_i \perp x_j (i,j=1,2,3)$, then C is orthogonally cubic, where $C : X \to Y$ is a mapping defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Theorem 5.2. *Let $\alpha \geq 0$ and $f : X \to Y$ be a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|Df(x_{11},x_{21},x_{31}),\dots,Df(x_{1k},x_{2k},x_{3k})\|_k \leq \alpha, \tag{5.1}$$

for all $x_{11},\dots,x_{1k},x_{21},\dots,x_{2k},x_{31},\dots,x_{3k} \in X$ with $x_{in} \perp x_{jn} (i,j = 1,2,3, n=1,\dots,k)$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\sup_{k\in\mathbb{N}} \|f(x_{11}) - C(x_{11}) - f(0),\dots,f(x_{1k}) - C(x_{1k}) - f(0)\|_k \leq \frac{\alpha}{7}, \tag{5.2}$$

for all $x_{11},\dots,x_{1k} \in X$.

Proof. Let $x_{11},\dots,x_{1k} \in X$. Let F be a function on X defined by

$$F(x) = f(x) - f(0),$$

for all $x \in X$. Then we have $F(0) = 0$. Note that $x \perp 0$ for all $x \in X$. Letting $x_{11} = \cdots = x_{1k} = x_{31} = \cdots = x_{3k} = 0$ and $x_{21} = x_{11},\dots,x_{2k} = x_{1k}$ in (5.1), we get

$$\sup_{k\in\mathbb{N}} \|F(2x_{11}) - 8F(x_{11}),\dots,F(2x_{1k}) - 8F(x_{1k})\|_k \leq \alpha.$$

Following the same approach as in the proof of Theorem 2.2, we can define a mapping $C : X \to Y$ by setting

$$C(x) := \lim_{n\to\infty} \frac{F(2^n x)}{8^n}.$$

Moreover, we have

$$\sup_{k\in\mathbb{N}} \|F(x_{11}) - C(x_{11}), \ldots, F(x_{1k}) - C(x_{1k})\|_k \leq \frac{\alpha}{7},$$

which implies the inequality (5.2).
Let $x_1, x_2, x_3 \in X$ and $x_i \perp x_j$ $(i, j = 1, 2, 3)$. So we have $2^n x_i \perp 2^n x_j (i, j = 1, 2, 3)$. Letting $x_{11} = \cdots = x_{1k} = 2^n x_1$, $x_{21} = \cdots = x_{2k} = 2^n x_2$ and $x_{31} = \cdots = x_{3k} = 2^n x_3$ in (5.1) and dividing by 8^n, then it follows that

$$\frac{1}{8^n}\|Df(2^n x_1, 2^n x_2, 2^n x_3)\| \leq \frac{1}{8^n}\alpha.$$

Taking the limit as $n \to \infty$ we get $DC(x_1, x_2, x_3) = 0$. Since $C(0) = 0$, by Lemma 5.1 we conclude that C is an orthogonally cubic mapping.
Let C' be another orthogonally cubic mapping satisfying (5.2). Then we have

$$\begin{aligned}\|C(x) - C'(x)\| &= \frac{1}{8^n}\|C(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n}\|C(2^n x) - f(2^n x) + f(0)\| \\ &\quad + \frac{1}{8^n}\|f(2^n x) - C'(2^n x) - f(0)\| \\ &\leq \frac{1}{8^n}\left(\frac{\alpha}{7} + \frac{\alpha}{7}\right).\end{aligned}$$

Hence $C = C'$. This proves the uniqueness assertion. □

Corollary 5.3. *Let $\alpha \geq 0$ and $f : X \to Y$ be a mapping satisfying (5.1) for all $x_{11}, \ldots, x_{1k}, x_{21}, \ldots, x_{2k}, x_{31}, \ldots, x_{3k} \in X$ with $x_{in} \perp x_{jn}$ $(i, j = 1, 2, 3, n = 1, \ldots, k)$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|f(x_{11}) - C(x_{11}), \ldots, f(x_{1k}) - C(x_{1k})\|_k \leq \frac{\alpha + 7\|f(0)\|}{7}.$$

Proof. In view of Theorem 5.2, there exists a unique orthogonally cubic mapping $C : X \to Y$ such that (5.2) is satisfied. We have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|f(x_{11}) - C(x_{11}), \ldots, f(x_{1k}) - C(x_{1k})\|_k \\ &\qquad\leq \sup_{k\in\mathbb{N}} \|f(x_{11}) - C(x_{11}) - f(0), \ldots, f(x_{1k}) \\ &\qquad\quad - C(x_{1k}) - f(0)\|_k + \|f(0), \ldots, f(0)\|_k \\ &\qquad\leq \frac{\alpha + 7\|f(0)\|}{7},\end{aligned}$$

for all $x_{11}, \ldots, x_{1k} \in X$. □

We prove the generalized Hyers-Ulam stability of the orthogonally cubic type functional equation in multi-normed spaces by using the fixed point method.

Theorem 5.4. *Let $k \in \mathbb{N}$ and $\phi : X^{3k} \to [0,\infty)$ be a function such that*

$$\phi(2x_{11},\dots,2x_{1k},2x_{21},\dots,2x_{2k},2x_{31},\dots,2x_{3k}) \leq a\phi(x_{11},\dots,x_{1k},x_{21},\dots,x_{2k},x_{31},\dots,x_{3k}),$$

for some $a>0$ with $a<8$ and all $x_{11},\dots,x_{1k},x_{21},\dots,x_{2k},x_{31},\dots,x_{3k} \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x_{11},x_{21},x_{31}),\dots,Df(x_{1k},x_{2k},x_{3k})\|_k \leq \phi(x_{11},\dots,x_{1k},x_{21},\dots,x_{2k},x_{31},\dots,x_{3k}), \tag{5.3}$$

for all $x_{11},\dots,x_{1k},x_{21},\dots,x_{2k},x_{31},\dots,x_{3k} \in X$ with $x_{in} \perp x_{jn}$ $(i,j = 1,2,3, n = 1,\dots,k)$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x_{11}) - C(x_{11}) - f(0),\dots,f(x_{1k}) - C(x_{1k}) - f(0)\|_k \leq \frac{1}{8-a}\phi(0,\dots,0,x_{11},\dots,x_{1k},0,\dots,0),$$

for all $x_{11},\dots,x_{1k} \in X$.

Proof. Let $E = \{f : X \to Y\}$, and consider the generalized metric d defined on E by

$$d(g,h) = \inf\Big\{c>0, \|g(x_{11}) - h(x_{11}),\dots g(x_{1k}) - h(x_{1k})\|_k \leq c\psi(x_{11},\dots,x_{1k}), \forall x_{11},\dots,x_{1k} \in X\Big\},$$

where $\psi : X^k \to [0,\infty)$ is a mapping defined by

$$\psi(x_{11},\dots,x_{1k}) = \phi(0,\dots,0,x_{11},\dots,x_{1k},0,\dots,0),$$

for all $x_{11},\dots,x_{1k} \in X$. Then, it is easy to show that d is a complete generalized metric on E. We now define a mapping $J : E \to E$ by

$$Jg(x) = \frac{1}{8}g(2x), \quad \forall x \in X.$$

Let F be a function on X defined by

$$F(x) = f(x) - f(0),$$

for all $x \in X$. Letting $x_{11} = \cdots = x_{1k} = x_{31} = \cdots = x_{3k} = 0$ and $x_{21} = x_{11},\dots,x_{2k} = x_{1k}$ in (5.3), we get

$$\|F(2x_{11}) - 8F(x_{11}),\dots,F(2x_{1k}) - 8F(x_{1k})\|_k \leq \psi(x_{11},\dots,x_{1k}),$$

for all $x_{11},\dots,x_{1k} \in X$. The rest of the proof is similar to the ones of Theorem 2.3. □

6. STABILITY OF THE k-CUBIC FUNCTIOAL EQUATION

In this section, we prove the Hyers-Ulam stability of the k-cubic functional equation in multi-normed spaces. For a given mapping $f : X \to Y$, we set

$$\begin{aligned} Df(x,y) = kf(x+ky) - f(kx+y) - \frac{k(k^2-1)}{2}[f(x+y)+f(x-y)] \\ - (k^4-1)f(y) + 2k(k^2-1)f(x), \end{aligned}$$

for all $x, y \in X$ with $x \perp y$.

Definition 6.1. A mapping $f : X \to Y$ is called an orthogonally k-cubic mapping if

$$\begin{aligned} kf(x+ky) - f(kx+y) = \frac{k(k^2-1)}{2}[f(x+y)+f(x-y)] + (k^4-1)f(y) \\ - 2k(k^2-1)f(x), \end{aligned}$$

for all $x, y \in X$ with $x \perp y$.

We state the following lemma. The proof is obvious.

Lemma 6.2. *If $k \geq 2$ and $f : X \to Y$ is a k-cubic mapping then we have $f(kx) = k^3 f(x)$ for all $x \in X$.*

Theorem 6.3. *Let $\alpha \geq 0$ and $k \geq 2$ and $f : X \to Y$ be a mapping satisfying*

$$\sup_{t\in\mathbb{N}} \|Df(x_1,y_1),\dots,Df(x_t,y_t)\|_t \leq \alpha, \tag{6.1}$$

for all $x_1,\dots,x_t,y_1,\dots,y_t \in X$ with $x_i \perp y_i (i = 1,\dots,t)$. If

$$\lambda = \frac{(k+1)(k^2+1)}{k^2+k+1},$$

then there exists a unique orthogonally k-cubic mapping $C : X \to Y$ such that

$$\sup_{t\in\mathbb{N}} \|f(x_1) - C(x_1) - \lambda f(0),\dots,f(x_t) - C(x_t) - \lambda f(0)\|_t \leq \frac{\alpha}{k^3-1}, \tag{6.2}$$

for all $x_1,\dots,x_t \in X$.

Proof. Let $x_1,\dots,x_t \in X$ and $\beta = k^4 - 1$. Letting $y_1 = \cdots = y_t = 0$ in (6.1), we obtain

$$\sup_{t\in\mathbb{N}} \|f(kx_1) - k^3 f(x_1) + \beta f(0),\dots,f(kx_t) - k^3 f(x_t) + \beta f(0)\|_t \leq \alpha,$$

since $0 \perp x_i (i = 1, \ldots, t)$. So the sequence $\left(\frac{f(k^n x)}{k^{3n}}\right)$ is Cauchy and so is convergent in the complete multi-normed space Y. Set

$$C(x) := \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}.$$

Similar to the proof of Theorem 2.2, we can conclude the inequality (6.2).

Let $x, y \in X$ and $x \perp y$. Letting $x_1 = \cdots = x_t = k^n x$, $y_1 = \cdots = y_t = k^n y$ in (6.1) and dividing the both sides by k^{3n}, we obtain

$$\frac{1}{k^{3n}} \|Df(k^n x, k^n y)\| \leq \frac{\alpha}{k^{3n}}.$$

Taking the limit as $n \to \infty$, we get

$$DC(x, y) = 0.$$

So C is an orthogonally k-cubic mapping.

Let C' be another orthogonally k-cubic mapping satisfying (6.2). So by Lemma 6.2 and induction, we have

$$C(k^n x) = k^{3n} C(x), \qquad C'(k^n x) = k^{3n} C'(x),$$

for all $x \in X$ and $n \in \mathbb{N}$. So we have

$$\begin{aligned} \|C(x) - C'(x)\| &= \frac{1}{k^{3n}} \|C(k^n x) - C'(k^n x)\| \\ &\leq \frac{1}{k^{3n}} \|C(k^n x) - f(k^n x) + \lambda f(0)\| \\ &\quad + \frac{1}{k^{3n}} \left\|C'(k^n x) - f(k^n x) + \lambda f(0)\right\| \\ &\leq \frac{1}{k^{3n}} \left(\frac{\alpha}{k^3 - 1} + \frac{\alpha}{k^3 - 1}\right). \end{aligned}$$

Hence $C = C'$. This completes the proof. □

Corollary 6.4. *Let $\alpha \geq 0$ and $k \geq 2$ and $f : X \to Y$ be a mapping satisfying (6.1) for all $x_1, \ldots, x_t, y_1, \ldots, y_t \in X$ with $x_i \perp y_i$ $(i = 1, \ldots, t)$. Then there exists a unique orthogonally k-cubic mapping $C : X \to Y$ such that*

$$\sup_{t\in\mathbb{N}} \|f(x_1) - C(x_1), \ldots, f(x_t) - C(x_t)\|_t \leq \frac{\alpha + (k^4 - 1)\|f(0)\|}{k^3 - 1}.$$

Proof. By Theorem 6.3, we conclude the existence of an orthogoally k-cubic mapping $C : X \to Y$ such that (6.2) is satisfied. If

$$\lambda = \frac{(k+1)(k^2+1)}{k^2+k+1},$$

then we have

$$\begin{aligned}
&\sup_{t\in\mathbb{N}} \|f(x_1) - C(x_1), \dots, f(x_t) - C(x_t)\|_t \\
&\quad \le \sup_{t\in\mathbb{N}} \|f(x_1) - C(x_1) - \lambda f(0), \dots, f(x_t) - C(x_t) - \lambda f(0)\|_t \\
&\qquad + \lambda \|f(0), \dots, f(0)\|_t \\
&\quad \le \frac{\alpha + (k^4 - 1)\|f(0)\|}{k^3 - 1},
\end{aligned}$$

for all $x_1, \dots, x_t \in X$. □

We prove the generalized Hyers-Ulam stability of the k-cubic functional equation in multi-normed spaces by using the fixed point method.

Theorem 6.5. *Let $k \ge 2$ and $t \in \mathbb{N}$ and $\phi : X^{2t} \to [0, \infty)$ be a function such that*

$$\phi(kx_1, \dots, kx_t, ky_1, \dots, ky_t) \le a\phi(x_1, \dots, x_t, y_1, \dots, y_t),$$

for some $a > 0$ with $a < k^3$ and all $x_1, \dots, x_t, y_1, \dots, y_t \in X$. Suppose that $f : X \to Y$ is a mapping such that $f(0) = 0$ and

$$\|Df(x_1, y_1), \dots, Df(x_t, y_t)\|_t \le \phi(x_1, \dots, x_t, y_1, \dots, y_t), \tag{6.3}$$

for all $x_1, \dots, x_t, y_1, \dots, y_t \in X$ with $x_i \perp y_i (i = 1, \dots, t)$. Then there exists a unique orthogonally k-cubic mapping $C : X \to Y$ such that

$$\|f(x_1) - C(x_1), \dots, f(x_t) - C(x_t)\|_t \le \frac{1}{k^3 - a}\phi(x_1, \dots, x_t, 0, \dots, 0), \tag{6.4}$$

for all $x_1, \dots, x_t \in X$.

Proof. Let $E = \{f : X \to Y\}$, and consider the complete generalized metric d defined on E by

$$\begin{aligned}
d(g, h) = \inf \Big\{ c \in (0, \infty) : &\|g(x_1) - h(x_1), \dots, g(x_t) - h(x_t)\|_t \\
&\le c\psi(x_1, \dots, x_t), \forall x_1, \dots, x_t \in X \Big\},
\end{aligned}$$

where $\psi : X^t \to [0, \infty)$ is a mapping defined by

$$\psi(x_1, \dots, x_t) = \phi(x_1, \dots, x_t, 0, \dots, 0).$$

We now define a mapping $J : E \to E$ by

$$Jg(x) = \frac{1}{k^3} g(kx), \quad \forall x \in X.$$

It is easy to see that J is a strictly contractive mapping with the Lipschitz constant $\frac{a}{k^3}$. Letting $y_1 = \cdots = y_t = 0$ in (6.3), we get

$$\|f(kx_1) - k^3 f(x_1), \dots, f(kx_t) - k^3 f(x_t)\|_t \le \psi(x_1, \dots, x_t), \tag{6.5}$$

for all $x_1, \dots, x_t \in X$. It follows from (6.5) that $d(Jf, f) \leq \frac{1}{k^3}$. According to Lemma 1.4, we deduce the existence of a fixed point of J, that is, the existence of a mapping $C : X \to Y$ such that $C(kx) = k^3 C(x)$ for all $x \in X$. It follows from the condition $d(J^n f, C) \to 0$ that

$$\begin{aligned} C(x) &= \lim_{n\to\infty} (J^n f)(x) \\ &= \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}, \quad \forall x \in X. \end{aligned}$$

Moreover

$$d(f, C) \leq \frac{1}{1 - \frac{a}{k^3}} d(Jf, f),$$

implies the inequality

$$d(f, C) \leq \frac{1}{k^3 - a}.$$

So (6.4) is satisfied.

Let $x, y \in X$ and $x \perp y$. Letting $x_1 = \dots = x_t = k^n x$ and $y_1 = \dots = y_t = k^n y$ in (6.3) and dividing the both sides by k^{3n}, we get

$$\begin{aligned} \frac{1}{k^{3n}} \|Df(k^n x, k^n y)\| &\leq \frac{1}{k^{3n}} \phi(k^n x, \dots, k^n x, k^n y, \dots, k^n y) \\ &\leq \left(\frac{a}{k^3}\right)^n \phi(x, \dots, x, y, \dots, y). \end{aligned}$$

Taking the limit as $n \to \infty$, we deduce that C is an orthogonally k-cubic mapping.

The uniqueness assertion is similar to the proof of Theorem 2.3. □

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[1]Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

E-mail address: `mehdi.nazarianpoor@yahoo.com`

[2] Department of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran.

E-mail address: `ghadir54@gmail.com`