Sahand Communications in Mathematical Analysis (SCMA) Vol. 9 No. 1 (2018), 143-150
http://scma.maragheh.ac.ir

A cone theoretic Krein-Milman theorem in semi topological cones

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Abstract. In this paper, a Krein-Milman type theorem in T_0 semi topological cone is proved, in general. In fact, it is shown that in any locally convex T_0 semi topological cone, every convex compact saturated subset is the compact saturated convex hull of its extreme points, which improves the results of Larrecq.

1. Introduction

A branch of order theory called domain theory was initiated in the early 1970s with the pioneering work of Dana S. Scott on a model of untyped lambda-calculus [14]. Progress in this domain rapidly required a lot of materials on (non-Hausdorff) topologies. After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other [3, 5].

One of the Klaus Keimel's mathematical interests is the interaction between order theory and functional analysis. In recent years this has led to the beginnings of the domain-theoretic functional analysis, which may be considered to be a topic within positive analysis in the sense of Jimmie Lawson [9]. In the latter, notions of positivity and order play a key role, as do lower semicontinuity and so do T_0 spaces. Some basic functional analytic tools were developed by Roth and Tix and later on Plotkin and Keimel for these structures. Roth has written several papers in this area including his papers [11, 12] on Hahn-Banach type theorems for locally convex cones. Tix in her Ph.D. thesis gave a domain-theoretic version of these theorems in the framework of d-cones (see [15, 16]). Plotkin subsequently gave another separation theorem, which was incorporated,

2010 *Mathematics Subject Classification.* 54D10, 46B22, 47L07.
Key words and phrases. T_0 topology, Extreme Point, Krein-Milman type theorem
Received: 26 November 2016, Accepted: 12 February 2017.
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together with other improvements, into a revised version of Tix's thesis [10, 17]. Finally, Keimel [7] improved the Hahn-Banach theorems to semi topological cones.

The theory of locally convex cones, with applications to Korovkin type approximation theory for positive operators and to vector-measure theory, developed in the books by Keimel and Roth [8] and Roth [13], respectively.

The extreme points of a convex set are of interest primarily because of the Krein-Milman theorem and its generalizations. The Krein-Milman theorem asserts that a compact convex subset K of a locally convex Hausdorff space is the closed convex hull of its extreme points [2].

In 2008, Goubault-Larrecq [4], proved a Krein-Milman type theorem for non-Hausdorff cones (in the sense of Keimel [7]). In fact, he proved the following analogue of the Krein-Milman theorem: in any locally convex T_0 topological cone C, every convex compact saturated subset is the compact saturated convex hull of its extreme points. In this paper, our aim is to prove the same theorem in semi topological cones, in general.

2. Preliminaries

For convenience of the reader, we give a survey of the relevant materials from [1, 2, 5, 7], without proofs, thus making our exposition self-contained.

For a subset A of a partially ordered set P, we use the following notations:

$$\downarrow A =: \{x \in P | x \leq a \text{ for some } a \in A\},$$
$$\uparrow A =: \{x \in P | x \geq a \text{ for some } a \in A\}.$$

We call A, a lower or upper set, if $\downarrow A = A$ or $\uparrow A = A$, respectively. A subset of P is saturated iff it is an upper set.

We denote by $\mathbb{R}_+$ the subset of all nonnegative reals. Further, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. Addition, multiplication and the order are extended to $\overline{\mathbb{R}}$ in the usual way. In particular, $+\infty$ becomes the greatest element and we put $0 \cdot (+\infty) = 0$.

According to [7], a *cone* is a set C, together with two operations $+ : C \times C \to$ C and $\cdot : \mathbb{R}_+ \times C \to C$ and a neutral element $0 \in C$, satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}_+$:

$$\begin{aligned} &0 + v = v, && 1v = v, \\ &v + (w + u) = (v + w) + u, && (\lambda\mu)v = \lambda(\mu v), \\ &v + w = w + v, && (\lambda + \mu)v = \lambda v + \mu v, \\ & && \lambda(v + w) = \lambda v + \lambda w. \end{aligned}$$

An *ordered cone* C is a cone endowed with a partial order $\leq$ such that the addition and multiplication by fixed scalars $r \in \mathbb{R}_+$ are order preserving, that is, for all $x, y, z \in C$ and all $r \in \mathbb{R}_+$:

$$x \leq y \quad \Rightarrow \quad x + z \leq y + z \;\text{ and }\; rx \leq ry.$$

Let us recall that a *linear function* from a cone $(C, +, \cdot)$ to a cone $(C', +, \cdot)$ is a function $f : C \to C'$ such that $f(v + w) = f(v) + f(w)$ and $f(\lambda v) = \lambda f(v)$, for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$.

A subset D of a cone C is said to be *convex*, if for all $u, v \in D$ and $\lambda \in [0, 1], \lambda u + (1 - \lambda)v \in D$. The convex closure of a set D is defined to be the smallest convex set containing D.

For example, $\mathbb{R}^n_+$ is a cone, with the coordinate-wise operations. On $\mathbb{R}_+$, the order is just the usual order $\leq$ of the reals. On $\mathbb{R}^n_+$, it is the coordinate-wise order.

Recall that a partially ordered set $(A, \leq)$ is called directed if for every $a, b \in A$ there exits $c \in A$ with $a, b \leq c$. A partially ordered set $(D, \leq)$ is called a *cpo* if every directed subset A of D has a least upper bound in D. The least upper bound of a directed subset A is denoted by $\sqcup^{\uparrow} A$, and it is also called the directed supremum, or sometimes the limit of A.

Any T_0 space X comes with an intrinsic order, the *specialization order* which is defined by $x \leq y$ if the element x is contained in the closure of the singleton $\{y\}$ or, equivalently, if every open set containing x also contains y.

In any T_0 space X, the downward closure $\downarrow E$ is closed whenever E is finite [4, P. 2].

Given any ordering $\leq$ on a set X, there are at least two topologies with $\leq$ as specialization ordering, the coarsest possible one (upper topology: a base is given by the complements of sets of the form $\downarrow E$, for E a finite subset of X) and the finest possible one (Alexandroff topology) (see [5, Sec. 4.2.2] for more details). Additionally, there are some other interesting topologies in between. An important example of a topology that sits in between is the Scott topology.

Let D be a partially ordered set. A subset A is called Scott closed if it is a lower set and is closed under supremum of directed subsets, as far as these suprema exist. Complements of Scott closed sets are called Scott open. The collection of all Scott opens is a topology, called the *Scott topology* on D [5, Prop. 4.2.18]. We write D_σ for the set D with the Scott topology.

The basic notion is that of a *Scott continuous* function: A function f from a partially ordered set P to a partially ordered set Q is called Scott continuous if it is order preserving and if, for every directed subset

D of P which has a least upper bound in P, the image $f(D)$ has a least upper bound in Q and $f(\sqcup^{\uparrow} D) = \sqcup^{\uparrow} f(D)$.

Let P and Q be two partially ordered set. A map $f : P_\sigma \to Q_\sigma$ is continuous iff $f : P \to Q$ is Scott continuous [5, Prop. 4.3.5].

On the extended reals $\overline{\mathbb{R}}$ and on its subsets $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ we use the *upper topology*, for which the only open sets are the open intervals $\{s : s > r\}$. This upper topology is T_0, but far from being Hausdorff.

According to [6], a *topological cone* is a cone C with a T_0 topology such that the addition and scalar multiplication are separately continuous, that is:

$$\text{Scalar multiplication } (r, a) \mapsto ra : \mathbb{R}_+ \times C \to C \text{ is jointly continuous,}$$
$$\text{Adition } (a, b) \mapsto a + b : C \times C \to C \text{ is jointly continuous.}$$

According to [7], a *semi topological cone* is a cone with a T_0 topology such that the addition and scalar multiplication are separately continuous, that is:

$$\begin{aligned} &a \mapsto ra : C \to C, && \text{is continuous for every fixed } r > 0,\\ &r \mapsto ra : \mathbb{R}_+ \to C, && \text{is continuous for every fixed } a \in C,\\ &b \mapsto a + b : C \to C, && \text{is continuous for every fixed } a \in C. \end{aligned}$$

An *s-cone* is a cone with a partial order such that addition and scalar multiplication:

$$(a, b) \mapsto a + b : C \times C \to C, \qquad (r, a) \mapsto ra : \mathbb{R}_+ \times C \to C,$$

are Scott continuous. An s-cone is called a [b]*d-cone* if its order is [bounded] directed complete, i.e., if each [upper bounded] directed subset has a least upper bound.

Note that every s-cone is a semi topological cone with respect to its Scott topology [7, Prop. 6.3].

For example, $\mathbb{R}_+^n$ is a topological cone, $\overline{\mathbb{R}}_+$ is also a topological cone, and $\overline{\mathbb{R}}_+^n$ as well. Again, $\overline{\mathbb{R}}_+$ is equipped with its Scott topology [4].

A cone C with a topology is called *locally convex*, if each point has a neighborhood basis of open convex neighborhoods.

We shall use the following *separation theorem* [7, Theorem 9.1]: in a semi topological cone C consider a nonempty convex subset A and an open convex subset U. If A and U are disjoint, then there exists a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(a) \leq 1 < f(u)$ for all $a \in A$ and all $u \in U$. We shall especially use the following Separation Corollary [7, Corollary 9.3]: For elements a, b in a convex T_0 semi topological cone C with $a \not\geq b$, there is a linear continuous functional $f : C \to \mathbb{R}_+$ such that $f(a) < f(b)$.

Finally, we shall use the following *geometric separation theorem* [7, Theorem 10.2]: Let C be a locally convex semi topological cone. Suppose that K is a compact convex set and that F is a nonempty closed convex set disjoint from K. Then there is a convex open set U including K and disjoint from F.

Let C be a cone. For any two points x, y of C, let $]x, y[$ be the set of points of the form $r \cdot x + (1-r) \cdot y$, with $0 < r < 1$. It is tempting to call this, the open line segment between x and y, however be aware that it is generally not open.

Let Q be a subset of a T_0 topological cone C, with specialization ordering.

An extreme point of Q is any element $x \in Q$ that is minimal in Q, and such that there are no two distinct points x_1 and x_2 of Q such that $x \in]x_1, x_2[$.

In the sequel, we shall need the notion of a closed subset of Q. This is by definition the intersection of a closed subset of C with Q.

Call a face A of Q any non-empty closed subset of Q such that, for any $x_1, x_2 \in Q$, if $]x_1, x_2[$ intersects A, then $]x_1, x_2[$ is entirely contained in A.

A cone-theoretic version of the Krein-Milman theorem [4]: Let C be a locally convex T_0 topological cone, Q a convex compact saturated subset of C. Then Q is the smallest convex compact saturated subset of C containing the extreme points of Q.

3. Main results

The purpose of this section is to study the concept of extreme points and related topics in a T_0 semi topological cone. In particular, we prove the Krein-Milman theorem in a T_0 semi topological cone, and so we improve the results of Larrecq [4].

It is clear that any topological cone is a semi topological cone, but the converse is not true, in general. For example, an s-cone need not be a topological cone with respect to the Scott topology (see [7, P. 124]), but every s-cone is a semi topological cone with respect to its Scott topology.

We know that, if Q is a subset of a T_0 topological cone C and A is a closed subset of Q, and $]x_1, x_2[$ is contained in A, then x_1 and x_2 are also in A [4, Lemma 3.2]. In the following lemma, we show that the similar result is true in T_0 semi topological cone.

Lemma 3.1. *Let Q be a subset of a T_0 semi topological cone C. If A is a closed subset of Q, and $]x_1, x_2[$ is contained in A, then x_1 and x_2 are also in A.*

Proof. We show that for given $x_1, x_2 \in C$, the map $f_{x_1,x_2} : r \in [0,1]^m \mapsto r \cdot x_1 + (1-r) \cdot x_2$ is continuous from $[0,1]^m$ to C, where $[0,1]^m$ denotes $[0,1]$ together with its usual, metric topology.

By the definition of a semi topological cone, the map $s \in [0,1] \mapsto s \cdot x_2$ is continuous and the map $r \in [0,1] \mapsto 1-r$ is clearly continuous too, so the composition of them (i.e., $r \in [0,1] \mapsto (1-r) \cdot x_2$) is continuous. It follows that, the map $r \in [0,1] \mapsto r \cdot x_1 + (1-r) \cdot x_2$ is continuous. Since the map f_{x_1,x_2} is composition of the continuous map $r \in [0,1] \mapsto r \cdot x_1 + (1-r) \cdot x_2$, and the continuous map $r \in [0,1]^m \mapsto r \in [0,1]$, so it is continuous too. Assume $]x_1, x_2[\subset A$, that is, $r \cdot x_1 + (1-r) \cdot x_2 \in A$ for all $r, 0 < r < 1$. So $f^{-1}_{x_1,x_2}(A)$ contains the open interval $(0,1)$. Remember that A is closed in Q, so we may write A as $F \cap Q$, where F is closed in C. Then $f^{-1}_{x_1,x_2}(F)$ contains $(0,1)$. However f_{x_1,x_2} is continuous. Since F is closed, so $f^{-1}_{x_1,x_2}(F)$ is closed in $[0,1]^m$. But the only closed set in $[0,1]^m$ containing $(0,1)$ is $[0,1]$ itself. So both 0 and 1 are in $f^{-1}_{x_1,x_2}(F)$, i.e., x_1 and x_2 are in F. Since $x_1, x_2 \in Q$, hence they are in A, too. □

By applying Lemma 3.1 and considering [4, P. 5,6]), the following statements are true in semi topological cones.

Lemma 3.2.
i. *Any point $x \in C$ is an extreme point of Q if and only if $\{x\}$ is a face of Q.*
ii. *If Q is convex and non-empty, then Q itself is a face of Q.*
iii. *If Q is compact, then $Face(Q)$ is a cpo.*
iv. *If Q is compact, then every face of Q contains a minimal face (for set inclusion).*
v. *Let K be a non-empty compact subset of C, and f any continuous map from C to $\mathbb{R}_+$. Then the greatest lower bound $a = \inf_{z \in K} f(z)$ is attained, i.e., there is a element $z \in K$ such that $f(z) = a$.*
vi. *If Q is compact, then for any face A of Q, and any linear continuous map $f : C \to \mathbb{R}_+$, the greatest lower bound $inf_{z \in K} f(z)$ is attained, write it $min_A f$. Then the set $argmin_A f = \{x \in A | f(x) = min_A f\}$ is a face of Q.*
vii. *Let C be a locally convex T_0 topological cone and Q a compact subset of C. Any minimal face of Q (for set inclusion) is of the form $\{x\}$, with x extreme in Q.*

Now, we can prove the following cone-theoretic version of the Krein-Milman theorem for semi topological cones.

Theorem 3.3. *Let C be a locally convex T_0 semi topological cone and Q be a convex compact saturated subset of C. Then Q is the smallest convex compact saturated subset of C containing the extreme points of Q.*

Proof. If Q is empty, the result is clear, so assume $Q \neq \varnothing$. Clearly, Q is convex, compact, saturated, and contains all the extreme points of Q. For the converse, let Q' be any convex compact saturated subset of C containing the extreme points of Q, and assume, by contradiction, that there is a point x in Q that is not in Q'. Note that the set $\downarrow x$ is a closed and convex subset and since Q' is upper, so $\downarrow x \cap Q' = \varnothing$. Hence, we can apply the Geometric Separation theorem, stated in Preliminary section, with $K = Q'$, $F = \downarrow x$. So there is a convex open set U containing Q' and disjoint from $\downarrow x$, i.e., not containing x.

We now use Keimels Separation theorem, as stated in Preliminary, with $E = \{x\}$ and U just given: there is a continuous linear map $f : C \to \bar{R}_+$ such that $f(x) \leq 1$, and $f(y) > 1$ for every $y \in U$. Let $a = inf_{z \in Q'} f(z)$. By Lemma 3.2 (v), a is attained, say $a = f(z_0)$; since $Q' \subset U$, $f(z_0) > 1$, so $a > 1$. By Lemma 3.2 (ii), $A = Q$ is a face of Q. By Lemma 3.2 (vi), $argmin_A f$ is then a smaller face of Q. Any element z of $argmin_A f$ is such that $f(z) = min_A f = min_{y \in Q} f(y) \leq f(x) \leq 1$. Since $f(z) \geq a > 1$ for every $z \in Q'$, we conclude that $argmin_A f$ does not intersect Q. Let A be a minimal face of Q contained in $argmin_A f$. The latter exists by Lemma 3.2 (iv). By Lemma 3.2 (vii), A is of the form $\{y\}$, with y extreme in Q. Since $y \in A \subset argmin_A f$ and $argmin_A f$ does not intersect Q', y is an extreme point not contained in Q' and this leads to a contradiction. □

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