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Extensions of Saeidi's Propositions for Finding a Unique Solution of a Variational Inequality for (u, v)-cocoercive Mappings in Banach Spaces

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Abstract. Let C be a nonempty closed convex subset of a real Banach space E, let $B : C \to E$ be a nonlinear map, and let u, v be positive numbers. In this paper, we show that the generalized variational inequality $VI(C, B)$ is singleton for (u, v)-cocoercive mappings under appropriate assumptions on Banach spaces. The main results are extensions of the Saeidi's Propositions for finding a unique solution of the variational inequality for (u, v)-cocoercive mappings in Banach spaces.

1. Introduction

Let C be a nonempty closed convex subset of a real normed linear space E and E^* be the dual space of E. Suppose that $\langle ., . \rangle$ denotes the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}$$

for each $x \in E$. Suppose that $U = \{x \in E : \|x\| = 1\}$. A Banach space E is called smooth if for all $x \in U$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x \rangle = \|x\|$ and $\|j_x\| = 1$ (see [1]).

Recall the following definitions and exampels:

(i) Let C be a nonempty closed convex subset of a real normed linear space E. A mapping T of C into itself is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and a mapping f is an α-contraction on E if

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$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad x, y \in E \text{ and } 0 \leq \alpha < 1.$$

(ii) Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $B : C \to H$ is a nonlinear map and P_C is the projection of H onto C. The classical variational inequality problem $VI(C, B)$ is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$ (see [6]). For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad (v \in C),$$

if and only if $u = P_C z$. Therefore

$$u \in VI(C, B) \quad \Leftrightarrow \quad u = P_C(u - \lambda Bu),$$

where $\lambda > 0$ is a constant (see [6]). It is known that the projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,$$

for $x, y \in H$.

(iii) Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $B : C \to H$ is a nonlinear map. B is called v-strongly monotone if

$$\langle Bx - By, x - y\rangle \geq v\|x - y\|^2 \quad \text{for all } x, y \in C,$$

for a constant $v > 0$.

(iv) Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $B : C \to H$ is a nonlinear map. B is said to be relaxed (u, v)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$

for all $x, y \in C$. For $u = 0$, B is v-strongly monotone. Clearly, every v-strongly monotone map is a relaxed (u, v)-cocoercive map.

(v) Let E be a real Banach space with the dual space E^*. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \qquad x \neq y \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| < 1.$$

(vi) Suppose that C is a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be a best approximation to x if $\|x - y_0\| = d(x, C)$, where

$$d(x, C) = \inf_{y \in C} \|x - y\|.$$

The number $d(x, C)$ is called the distance from x to C.

The set of all best approximations from x to C is denoted by $P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$. This defines a mapping P_C from E into 2^C and is called the metric projection onto C. C is Chebyshev if $P_C(x)$ is singleton for each $x \in E$ and C is proximinal if $P_C(x) \neq \emptyset$, for all $x \in E$. Every closed convex subset C of a reflexive Banach space is proximinal and every closed convex subset C of a reflexive strictly convex Banach space is a Chebyshev set. Let C be a proximinal subset of a Banach space E, by [1, Proposition 2.10.1], C is closed, hence Chebyshev subsets of a Banach space E are closed too (for more details see [1, page 115]).

(vii) Let C be a nonempty closed subset of a Banach space E. Then a mapping $Q : E \to C$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \quad \forall x \in E, \quad \forall t \geq 0.$$

A mapping $Q : E \to C$ is said to be a retraction or a projection if $Qx = x$, $\forall x \in C$. If E is smooth then the sunny nonexpansive retraction of E onto C is uniquely decided (see [7]). Then, if E is a smooth Banach space, the sunny nonexpansive retraction of E onto C is denoted by Q_C. Let C be a nonempty closed subset of a Banach space E. Then the subset C is said to be a nonexpansive retract (resp. sunny nonexpansive retract) if there exists a nonexpansive retraction (resp. sunny nonexpansive retraction) of E onto C (see [3, 4]). Let C be a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E. Let Q_C be the sunny nonexpansive retraction of E onto C. Then we have

$$x_0 = Q_C x \quad \Leftrightarrow \quad \langle x - x_0 \,,\, J(x_0 - y)\rangle \geq 0, \tag{1.2}$$

for each $y \in C$. We have $P_C = Q_C$ in a Hilbert space (see [5]).

(viii) Let E be a real normed linear space. Let C be a nonempty closed convex subset of E and $B : C \to E$ be a nonlinear map. B is said to be relaxed (u, v)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, j(x - y)\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$

for all $x, y \in C$ and $j(x - y) \in J(x - y)$.

Example 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H. The mapping $B : C \to H$ is said to be a relaxed (u, v)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$

for all $x, y \in C$. By [1, Example 2.4.2], in a Hilbert space H, the normalized duality mapping is the identity. Then $J(x-y) = \{x-y\}$. Therefore, the above definition extends the definition of the relaxed (u, v)-cocoercive mappings, from the real Hilbert spaces to the real normed linear spaces.

(ix) Let C be a nonempty closed convex subset of a real normed linear space E and $B : C \to E$ be a nonlinear map. B is called v-strongly monotone if there exists a constant $v > 0$ such that
$$\langle Bx - By, j(x-y)\rangle \geq v\|x-y\|^2,$$
for all $x, y \in C$ and $j(x-y) \in J(x-y)$.

Example 1.2. Let C be a nonempty closed convex subset of a real Hilbert space H.The mapping $B : C \to H$ is said to be v-strongly monotone, if there exists a constant $v > 0$ such that
$$\langle Bx - By, x-y\rangle \geq v\|x-y\|^2,$$
for all $x, y \in C$. Since H is a Hilbert space, $J(x-y) = \{x-y\}$. Therefore, the above definition extends the definition of v-strongly monotone mappings, from the real Hilbert spaces to the real normed linear spaces.

Example 1.3. Let C be a nonempty closed convex subset of a real Banach space E. Let T be an α-contraction of C into itself. Putting $B = I - T$, we have
$$\begin{aligned}\langle Bx - By, j(x-y)\rangle &= \langle (I-T)x - (I-T)y, j(x-y)\rangle \\ &= \langle (x-y) - (Tx - Ty), j(x-y)\rangle \\ &= \langle x-y, j(x-y)\rangle - \langle Tx - Ty, j(x-y)\rangle \\ &\geq \langle x-y, j(x-y)\rangle - \|Tx-Ty\|\|j(x-y)\| \\ &\geq \|x-y\|^2 - \|Tx-Ty\|\|x-y\| \\ &\geq \|x-y\|^2 - \alpha\|x-y\|^2 = (1-\alpha)\|x-y\|^2,\end{aligned}$$
for all $x, y \in C$ and $j(x-y) \in J(x-y)$. Hence $B : C \to E$ is a $(1-\alpha)$-strongly monotone mapping. Therefore B is a relaxed $(u,(1-\alpha))$-cocoercive mapping on E for each $u > 0$;

(x) The following definitions generalize the classical variational inequality problem 1.1,
   (a) Let E be a real normed linear space and C be a nonempty closed convex subset of E. Let $B : C \to E$ be a nonlinear map. The classical variational inequality problem $VI(C, B)$ is to find $u \in C$ such that
$$\langle j(Bu), v-u\rangle \geq 0, \tag{1.3}$$
   for all $v \in C$ and $j(Bu) \in J(Bu)$.

(b) Let E be a real normed linear space. Let C be a nonempty closed convex subset of E. Let $B : C \to E$ be a nonlinear map. The classical variational inequality problem $VI(C, B)$ is to find $u \in C$ such that

$$\langle Bu, j(v-u)\rangle \geq 0, \tag{1.4}$$

for all $v \in C$ and $j(v-u) \in J(v-u)$.

(xi) Let C be a nonempty Chebyshev subset of a normed linear space E such that P_C be a metric projection from E into C. Let B be a mapping from C into E. B is said to be a P_C-nonexpansive, if

$$\|P_C Bx - P_C By\| \leq \|Bx - By\|.$$

Example 1.4. Let C be a nonempty closed convex subset of a Hilbert space H and P_C, the metric projection from H onto C and B a mapping from C into H. By [1, Proposition 2.10.15], P_C is a nonexpansive projection. Thus we have

$$\|P_C Bx - P_C By\| \leq \|Bx - By\|,$$

therefore, B is P_C-nonexpansive.

Example 1.5. Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. By [1, Corollary 2.10.11], there exists a metric projection mapping $P_C : X \to C$ such that $P_C(x) = x$ for all $x \in C$. Let B is a mapping from C into C, therefore, we have

$$\|P_C Bx - P_C By\| = \|Bx - By\|,$$

hence B is P_C-nonexpansive.

In this paper, we prove that $VI(C, B)$ is singleton where C is a nonempty closed convex subset of a Banach spaces E and B is a (u, v)-cocoercive mappings from C into E, under appropriate assumptions on E.

2. PRELIMINARIES

A continuous strictly increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be gauge function if $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$.

Let E be a normed space and E^* be its dual space. Let μ be a gauge function. Then the mapping $J_\mu : E \to E^*$ defined by

$$J_\mu(x) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*,\ \|j\|_* = \mu(\|x\|)\},$$

for all $x \in E$, J_μ is called the duality mapping with gauge function μ. In the particular case $\mu(t) = t$, the duality mapping $J_\mu = J$ is the normalized duality mapping [1].

Theorem 2.1 ([1]). *Let C be a nonempty convex subset of a smooth Banach space E and let $x \in E$ and $y \in C$. Then the following statements are equivalent:*

(a) *y is a best approximation to x: $\|x - y\| = d(x, C)$.*
(b) *y is a solution of the variational inequality:*
$\langle y - z, J_\mu(x - y)\rangle \geq 0$, for all $z \in C$, where J_μ is a duality mapping with gauge function μ.

Remark 2.2. Let C be a nonempty convex Chebyshev subset of a real smooth Banach space E. Putting $\mu(t) = t$, from Theorem 2.1, we have

$$u \in VI(C, B) \quad \Leftrightarrow \quad u = P_C(u - \lambda Bu). \tag{2.1}$$

Remark 2.3. In a Banach space, a metric projection mapping is not nonexpansive, in general. However, the existence of nonexpansive projections from a Banach space even into a nonconvex subset Ω, is discussed in [2]. Let C is a Chebyshev subset of a Banach space E and B a mapping from C into E. If a metric projection P_C from a Banach space into C is nonexpansive, then we have $\|P_C Bx - P_C By\| \leq \|Bx - By\|$, therefore, B is P_C-nonexpansive.

Remark 2.4. Let C be a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E. Let Q_C be a sunny nonexpansive retraction. By (1.2), we have

$$u \in VI(C, B) \quad \Leftrightarrow \quad u = Q_C(u - \lambda Bu). \tag{2.2}$$

3. MAIN RESULTS

In this section, we deal with some results to prove that $VI(C, B)$ is singleton when, $B : C \to E$ is a relaxed (u, v)-cocoercive and $0 <$ μ-Lipschitzian mapping and C is a nonempty convex subset of a real smooth Banach space E.

We will make use of the following Theorem.

Theorem 3.1. *Let E be a Banach space. Then for all $x, y \in E$, we have*

$$\langle x - y, j(x - y)\rangle \leq \langle x - y, x^* - y^*\rangle + 4\|x\|\|y\|,$$

for all $x^* \in J(x), y^* \in J(y), j(x - y) \in J(x - y)$.

Proof. For $x = y$, obviously the inequality holds. Let $x^* \in J(x), y^* \in J(y)$ and $x \neq y$. As in the proof of [8, Theorem 4.2.4], we have

$$\langle x - y, x^* - y^*\rangle \geq (\|x\| - \|y\|)^2 + (\|x\| + \|y\|)(\|x\| + \|y\| - \|x + y\|).$$

Hence, we have

$$\langle x - y, x^* - y^*\rangle \geq (\|x\| - \|y\|)^2 + (\|x\| + \|y\|)(\|x\| + \|y\| - \|x + y\|)$$

$$
\begin{aligned}
&= (\|x\| - \|y\|)^2 + (\|x\| + \|y\|)^2 - \|x + y\|(\|x\| + \|y\|) \\
&\geq (\|x\| - \|y\|)^2 + \|x - y\|^2 - (\|x\| + \|y\|)^2 \\
&= \|x - y\|^2 - 4\|x\|\|y\| \\
&= \langle x - y, j(x - y)\rangle - 4\|x\|\|y\|,
\end{aligned}
$$

therefore,

$$
\langle x - y, j(x - y)\rangle \leq \langle x - y, x^* - y^*\rangle + 4\|x\|\|y\|.
$$

□

We now state the following important result:

Theorem 3.2. *Let C be a nonempty convex Chebyshev subset of a real smooth Banach space E. Suppose that μ, v, u be real numbers such that $\mu > 0$, and $v > u\mu^2 + 5\mu$. Let $B : C \to E$ be a relaxed (u, v)-cocoercive and μ-Lipschitzian mapping. Let P_C be a metric projection mapping from E into C such that $I - \lambda B$ be a P_C-nonexpansive mapping, for all $\lambda > 0$. Then, in the sense of* (1.3), *$VI(C, B)$ is singleton.*

Proof. Let λ be a real number such that

$$
0 < \lambda < \frac{v - u\mu^2 - 5\mu}{\mu^2}, \qquad \lambda\mu^2 \left[\frac{v - u\mu^2 - 5\mu}{\mu^2} - \lambda\right] < 1.
$$

Then, by Theorem 3.1, for every $x, y \in C$, we have

$$
\begin{aligned}
\|P_C(I - \lambda B)x - P_C(I - \lambda B)y\|^2 &\leq \|(I - \lambda B)x - (I - \lambda B)y\|^2 \\
&= \|(x - y) - \lambda(Bx - By)\|^2 \\
&= \|j\left[(x - y) - \lambda(Bx - By)\right]\|^2 \\
&= \langle (x - y) - \lambda(Bx - By), j\big[(x - y) \\
&\quad - \lambda(Bx - By)\big]\rangle \\
&\leq \langle x - y - \lambda(Bx - By), j(x - y) \\
&\quad - \lambda j(Bx - By)\rangle \\
&\quad + 4\lambda\|x - y\|\|Bx - By\| \\
&= \langle x - y, j(x - y)\rangle \\
&\quad f - \lambda\langle Bx - By, j(x - y)\rangle \\
&\quad + \lambda\langle y - x, j(Bx - By)\rangle \\
&\quad + \lambda^2\langle (Bx - By), j(Bx - By)\rangle \\
&\quad + 4\lambda\|x - y\|\|Bx - By\| \\
&\leq \|x - y\|^2 + \lambda u\|Bx - By\|^2 - \lambda v\|x - y\|^2 \\
&\quad + \lambda^2\|Bx - By\|^2 + 5\lambda\|x - y\|\|Bx - By\| \\
&\leq \|x - y\|^2 + \lambda u\mu^2\|x - y\|^2 - \lambda v\|x - y\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda^2\mu^2\|x-y\|^2 + 5\lambda\mu\|x-y\|^2 \\
&\le \left(1+\lambda u\mu^2 - \lambda v + \lambda^2\mu^2 + 5\lambda\mu\right)\|x-y\|^2 \\
&\le \left(1 - \lambda\mu^2\left[\frac{v-u\mu^2-5\mu}{\mu^2} - \lambda\right]\right)\|x-y\|^2.
\end{aligned}
$$

Now, since

$$1 - \lambda\mu^2\left[\tfrac{v-u\mu^2-5\mu}{\mu^2} - \lambda\right] < 1,$$

the mapping $P_C(I-\lambda B) : C \to C$ is a contraction and the Banach's Contraction Mapping Principle guarantees that it has a unique fixed point u; i.e., $P_C(I-\lambda B)u = u$, which, by 2.1, is the unique solution of $VI(C,B)$. □

Proposition 2 in [6] can be concluded from Theorem 3.2 for $v > u\mu^2 + 5\mu$ as the following Corollary:

Corollary 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let $B : C \to H$ be a relaxed (u,v)-cocoercive and $0 <$ μ-Lipschitzian mapping such that $v > u\mu^2 + 5\mu$. Then $VI(C,B)$ is singleton.*

Remark 3.4. Since v-strongly monotone mappings are relaxed (u,v)-cocoercive for any positive number u, Theorem 3.2 holds for v-strongly monotone mappings as follows: Let C be a nonempty convex Chebyshev subset of a real smooth Banach space E. Suppose that μ, v be real numbers such that $\mu > 0$ and $v > 5\mu$. Let $B : C \to E$ be a v-strongly monotone, μ-Lipschitzian mapping. Let P_C be a metric projection mapping from E into C such that $I - \lambda B$ be a P_C-nonexpansive mapping, for all $\lambda > 0$. Then $VI(C,B)$ is singleton.

Proposition 3 in [6] can be concluded from Remark 3.4 for $v > 5\mu$ as the following Corollary:

Corollary 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H and let $B : C \to H$ be a v-strongly monotone and $0 <$ μ-Lipschitzian mapping such that $v > 5\mu$. Then $VI(C,B)$ is singleton.*

We now state another important result:

Theorem 3.6. *Let C be a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E. Suppose that $\mu > 0$, and $v > u\mu^2 + 5\mu$. Let $B : C \to E$ be a relaxed (u,v)-cocoercive, μ-Lipschitzian mapping. Let Q_C be a sunny nonexpansive retraction from E onto C. Then, in the sense of* (1.4)*, $VI(C,B)$ is singleton.*

Proof. Let λ be a real number such that

$$0 < \lambda < \frac{v - u\mu^2 - 5\mu}{\mu^2}, \qquad \lambda\mu^2 \left[\frac{v - u\mu^2 - 5\mu}{\mu^2} - \lambda\right] < 1.$$

Then, as in the proof of theorem 3.2, for every $x, y \in C$, we have

$$\|Q_C(I - \lambda B)x - Q_C(I - \lambda B)y\|^2 \leq \left(1 - \lambda\mu^2 \left[\frac{v - u\mu^2 - 5\mu}{\mu^2} - \lambda\right]\right) \|x - y\|^2,$$

and, since

$$1 - \lambda\mu^2 \left[\tfrac{v - u\mu^2 - 5\mu}{\mu^2} - \lambda\right] < 1,$$

the mapping $Q_C(I - \lambda B) : C \to C$ is a contraction and the Banach's Contraction Mapping Principle guarantees that it has a unique fixed point u; i.e., $Q_C(I - \lambda B)u = u$, which, by 2.2, is the unique solution of $VI(C, B)$. □

Remark 3.7. Since v-strongly monotone mappings are relaxed (u, v)-cocoercive for any positive number u, Theorem 3.6 holds for v-strongly monotone mappings as follows: Let C be a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E. Let Q_C be a sunny nonexpansive retraction. Suppose that $\mu > 0$, and $v > u\mu^2 + 5\mu$. Let $B : C \to E$ be a v-strongly monotone, μ-Lipschitzian mapping. Then $VI(C, B)$ is singleton.

Remark 3.8. Proposition 2 in [6] for $v > u\mu^2 + 5\mu$ and Proposition 3 in [6] for $v > 5\mu$, follow from Theorem 3.6 and Remark 3.7, too.

Remark 3.9. S. Saeidi, in the proof of Proposition 2 in [6], proves that

$$\|P_C(I - sA)x - P_C(I - sA)y\|^2 \leq \left(1 - s\mu^2 \left[\frac{2(r - \gamma\mu^2)}{\mu^2} - s\right]\right) \|x - y\|^2,$$

when

$$0 < s < \tfrac{2(r - \gamma\mu^2)}{\mu^2},$$

and $r > \gamma\mu^2$. Putting $r = \gamma = s = 1$ and $\mu = \frac{1}{10}$, we have

$$\left(1 - s\mu^2 \left[\tfrac{2(r - \gamma\mu^2)}{\mu^2} - s\right]\right) < 0,$$

which is a contradiction. We have modified this contradiction in the proof of the Theorems 3.2 and 3.6.

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