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# COUPLED FIXED POINT RESULTS FOR $\alpha$-ADMISSIBLE MIZOGUCHI-TAKAHASHI CONTRACTIONS IN $b$-METRIC SPACES WITH APPLICATIONS

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Abstract. The aim of this paper is to establish some fixed point theorems for $\alpha$-admissible Mizoguchi-Takahashi contractive mappings defined on a $b$-metric space which generalize the results of Gordji and Ramezani [23]. As a result, we obtain some coupled fixed point theorems which generalize the results of Ćirić et al. [6]. We also present an application in order to illustrate the effectiveness of our results.

## 1. Introduction

Ran and Reurings initiated the studying of fixed point results on partially ordered sets in [28], where they gave many useful results in matrix equations. Existence of a fixed point for contraction type mappings in partially ordered metric spaces and its applications have been considered recently by many authors.

In [4], Bhaskar and Lakshmikantham have introduced the notions of mixed monotone mapping and coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings and discussed the existence and uniqueness of solution for periodic boundary value problem.

**Definition 1.1** ([4])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping $F$ is said to be has the mixed monotone

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property if $F(x, y)$ is monotone nondecreasing in $x$ and is monotone nonincreasing in $y$, that is, for every $x, y \in X$,

(i) for each $x_1, x_2 \in X$, if $x_1 \preceq x_2$, then $F(x_1, y) \preceq F(x_2, y)$;
(ii) for each $y_1, y_2 \in X$, if $y_1 \preceq y_2$, then $F(x, y_1) \succeq F(x, y_2)$.

**Definition 1.2** ([4]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F$, if $F(x, y) = x$ and $F(y, x) = y$.

Gnana Bhaskar and Lakshmikantham in [4] proved the following important theorem:

**Theorem 1.3** (Theorem 2.1[4]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on $X$. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u), d(y, v)], \tag{1.1}$$

*for all $x \preceq u$ and $y \succeq v$. If there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(x_0, y_0), \tag{1.2}$$

*then there exist $x, y \in X$ such that*

$$x = F(x, y) \quad \text{and} \quad y = F(y, x). \tag{1.3}$$

Consistent with [24], let $CB(X)$ be the class of all nonempty closed bounded subsets of $X$ and $K(X)$ be the class of all nonempty compact subsets of $X$. Let $H$ be the Hausdorff metric on $CB(X)$ induced by the metric $d$ of $X$ which is given by

$$H(A, B) = \left\{ \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \right\}, \tag{1.4}$$

for every $A, B \in CB(X)$.

A point $x \in X$ is called a fixed point of a multi-valued mapping $T : X \to CB(X)$ if $x \in Tx$.

In [29], Reich proved that if $(X, d)$ is a complete metric space and $T : X \to CB(X)$ satisfies

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \tag{1.5}$$

for each $x, y \in X$, where $\alpha : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{r \to t^+} \alpha(r) < 1$ for each $t \in (0, \infty)$, then $T$ has a fixed point. Reich raised the question of whether $K(X)$ can be replaced by $CB(X)$ in this result.

In [20], Mizoguchi and Takahashi gave a positive answer to the conjecture of Reich.

Let $\Theta$ be the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) $\psi(x) = 0 \quad \Leftrightarrow \quad x = 0$;
(ii) $\psi$ is lower semicontinuous and nondecreasing;
(iii) $\limsup\limits_{x\to 0} \frac{x}{\psi(x)} < \infty$ for all $x \in (0,\infty)$.

Recently, in [9], Gordji and Ramezani generalized the Mizoguchi and Takahashi's Theorem for single-valued mappings as follows:

**Theorem 1.4** ([9])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $f : X \times X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\varphi \in \Phi$ such that*

$$\varphi(d(fx, fy)) \le \alpha(\varphi(d(x,y)))\varphi(d(x,y)), \tag{1.6}$$

*for all $x, y \in X$ such that $x$ and $y$ are comparable and that $\alpha : [0,\infty) \to [0,1]$ is a function satisfying $\limsup_{s\to t^+} \alpha(s) < 1$, for all $t \in (0,\infty)$. Assume that either $f$ is continuous or $X$ is such that the following holds:*
*if an increasing sequence $x_n \to x$ in $X$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.*
*Then $f$ has a fixed point.*

In [6], Ćirić et al. derived some new coupled fixed point theorems for nonlinear contractive maps that satisfied a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Presented theorems extend and generalize many well-known results in the literature. As an application, they gave an existence and uniqueness theorem for the solution of a two-point boundary value problem.

Let $\Phi$ be the class of all functions $\beta : [0,\infty) \to [0,1)$ such that $\beta$ is a function satisfying $\limsup\limits_{x\to t^+} \beta(x) < 1$, for all $t \in (0,\infty)$.

**Theorem 1.5** (Corollary 2.1 [6])**.** *Let $(X, d, \preceq)$ be a sequentially $\preceq$-complete metric space and $A : X \times X \to X$ be a map having the mixed monotone property on X. Suppose that there exist $\psi \in \Psi$ and $\bar{\varphi} : [0,\infty) \to [0,\infty)$ with $\liminf_{s\to t^+}(\frac{\bar{\varphi(s)}}{\psi(s)}) > 0$ for all $t \ge 0$ such that for any $(x,y),(u,v) \in X \times X$ with $(u,v) \preceq (x,y)$,*

$$\begin{aligned}\psi(d(A(x,y), A(u,v))) \le \psi(\max\{d(x,u), d(y,v)\}) \\ - \bar{\varphi}(\psi(\max\{d(x,u), d(y,v)\})).\end{aligned}$$

*Suppose also that either $A$ is continuous or $(X, d, \preceq)$ has the following properties:*

(i) *any $\preceq$-nondecreasing sequence $(x_n)$ with $x_n \to x$ implies $x_n \preceq x$ for each $n$,*

(ii) *any $\preceq$-nonincreasing sequence $(y_n)$ with $y_n \to y$ implies $y_n \succeq y$ for each $n$.*

*If there exist $x_0, y_0 \in X$ such that $x_0 \preceq A(x_0, y_0)$ and $y_0 \preceq A(y_0, x_0)$, then there exist $a, b \in X$ such that $a = A(a, b)$ and $b = A(b, a)$.*

The concept of a $b$-metric space was introduced by Czerwik in [7]. After that, several interesting results about the existence of fixed points for single-valued and multi-valued operators in $b$-metric spaces have been obtained (see, e.g., [1]-[30]).

Consistent with [7], the following definitions and results will be needed in the sequel.

**Definition 1.6** ([7])**.** Let $X$ be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a $b$-metric if for all $x, y, z \in X$, the following conditions hold:

($b_1$) $d(x, y) = 0$ if and only if $x = y$,

($b_2$) $d(x, y) = d(y, x)$,

($b_3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair $(X, d)$ is called a $b$-metric space.

**Example 1.7.** [1] Let $(X, \rho)$ be a metric space and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then $d$ is a $b$-metric with $s = 2^{p-1}$. Condition ($b_3$) follows easily from the convexity of the function $f(x) = x^p$ $(x > 0)$.

The notions of $b$-convergent and $b$-Cauchy sequences, as well as of $b$-completeness in $b$-metric spaces are introduced in an obvious way (see, e.g., [5]).

*Remark* 1.8. In general a $b$-metric function $d(x, y)$ for $s > 1$ need not to be jointly continuous in both variables.

The following example (corrected from [11]) illustrates this fact.

**Example 1.9.** Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd} \\ & \ (m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then, considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \leq \frac{5}{2}(d(m,n) + d(n,p)).$$

Thus, $(X, d)$ is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

The following lemma plays an important role in proving the main results.

**Lemma 1.10** ([1])**.** *Let $(X, d)$ be a b-metric space with coefficient $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ b-converge to $x, y$, respectively. Then, we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

*In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have*

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

**Definition 1.11** ([16])**.** Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that $f$ is a triangular $\alpha$-admissible mapping if

(T1) $\alpha(x,y) \geq 1 \quad \text{implies} \quad \alpha(fx, fy) \geq 1, \quad x, y \in X,$

(T2) $\begin{cases} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1 \end{cases} \text{implies} \quad \alpha(x,y) \geq 1, \quad x, y, z \in X.$

**Example 1.12** ([16])**.** Let $X = \mathbb{R}$, $fx = \sqrt[3]{x}$ and $\alpha(x,y) = e^{x-y}$, then $f$ is a triangular $\alpha$-admissible mapping. Indeed, if $\alpha(x,y) = e^{x-y} \geq 1$, then $x \geq y$ which implies that $fx \geq fy$, that is, $\alpha(fx, fy) = e^{fx - fy} \geq 1$. Also, if

$$\begin{cases} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1, \end{cases}$$

then

$$\begin{cases} x - z \geq 0, \\ z - y \geq 0, \end{cases}$$

that is, $x - y \geq 0$ and so, $\alpha(x,y) = e^{x-y} \geq 1$.

**Lemma 1.13** ([16])**.** *Let $f$ be a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Samet et al. [32] defined the notion of $\alpha$-admissible mapping as follows.

**Definition 1.14.** Let $T$ be a self-mapping on $X$ and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that $T$ is an $\alpha$-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

In this paper, we obtain some fixed point theorems for the $\alpha$-admissible Mizoguchi-Takahashi contractions in $b$-metric spaces. This results generalize several comparable results in the literature.

## 2. Main results

From now on, let $\Psi$ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) $\psi(x) = 0 \quad \Leftrightarrow \quad x = 0$;
(ii) $\psi$ is lower semicontinuous and nondecreasing;
(iii) $\limsup_{x \to 0} \frac{x}{\psi(x)} < \infty$ for all $x \in (0, \infty)$,

and let $\Phi_s$ be the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ such that $\beta$ is a function satisfying for all $t \in (0, \infty)$,

$$\limsup_{x \to t^+} \beta(x) < \frac{1}{s}.$$

Let

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fx)\}.$$

First, we prove the following fixed point result.

**Theorem 2.1.** *Let $(X, d)$ be a $b$-complete $b$-metric space (with the parameter $s > 1$) and let $f : X \to X$ satisfies the following condition:*

$$\alpha(x, y)\psi(sd(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{2.1}$$

*for all $x, y \in \mathcal{X}$ where $\beta \in \Phi_s$, $\psi \in \Psi$, $\alpha : X^2 \to [0, +\infty)$ and $f$ is a triangular $\alpha$-admissible mapping.*

*Let there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

*Then, $f$ has a fixed point if,*

(I) *$f$ is continuous, or*
(II) *whenever $\{x_n\}$ in $X$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Proof.* Let $x_0 \in X$ be such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\}$ as $x_{n+1} = fx_n$ for all $n = 0, 1, 2, \ldots$.

As $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$ and since $f$ is an $\alpha$-admissible mapping, then $\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_n = x_{n+1}$, then $x_n$ is a fixed point of $f$.

Now, assume that $x_n \neq x_{n+1}$ for all $n$, that is,

$$d(x_n, x_{n+1}) > 0, \tag{2.2}$$

for all $n$. Let $d_n = d(x_n, x_{n+1})$. Then, from (2.1) we obtain that

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(sd(x_{n+1}, x_{n+2}))\\ &\leq \alpha(x_n, x_{n+1})\psi(sd(x_{n+1}, x_{n+2}))\\ &= \alpha(x_n, x_{n+1})\psi(sd(fx_n, fx_{n+1}))\\ &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1}))\\ &\leq \psi(M(x_n, x_{n+1})).\end{aligned}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_n)\}\\ &= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1})\}\\ &= d(x_n, x_{n+1}),\end{aligned}$$

that is, the sequence $\{\psi(d(x_n, x_{n+1}))\}$ is non-increasing and so there exists $r \geq 0$ such that $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = r \geq 0$.

Suppose that $r > 0$. By the definition of $\beta$, we have $\limsup_{x\to r^+} \beta(x) < \frac{1}{s}$ and $\beta(r) < \frac{1}{s}$. Therefore, there exist $r_1 \in [0, \frac{1}{s})$ and $\varepsilon_1 > 0$ such that $\beta(x) \leq r_1$ for all $x \in [r, r + \varepsilon_1)$. We can take $n_0 \in \mathbb{N}$ such that $r \leq \psi(d(x_{n+1}, x_{n+2})) \leq r + \varepsilon_1$, for all $n \geq n_0$. Since,

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(sd(x_{n+1}, x_{n+2}))\\ &\leq \alpha(x_n, x_{n+1})\psi(sd(x_{n+1}, x_{n+2}))\\ &= \alpha(x_n, x_{n+1})\psi(sd(fx_n, fx_{n+1}))\\ &\leq \beta(\psi(d(x_n, x_{n+1})))\psi(d(x_n, x_{n+1}))\\ &\leq r_1\psi(d(x_n, x_{n+1})),\end{aligned} \tag{2.3}$$

for every $n \geq n_0$, thus, we have $r \leq r_1 r$. This implies that $r = 0$.

Since $\psi(d(x_{n+1}, x_{n+2}))$ is decreasing and $\psi$ is increasing, then there exists nonnegative number $u$ such that $\{d(x_{n+1}, x_{n+2})\}$ converges to it. According to conditions on $\psi$, we can write $\psi(u) \leq \psi(d(x_{n+1}, x_{n+2}))$, for every $n \in \mathbb{N}$. This implies

$$\psi(u) \leq \lim_{n\to\infty} \psi(d(x_{n+1}, x_{n+2})) = r = 0. \tag{2.4}$$

Therefore, $u = 0$.

By 2.3, we have

$$\sum_{n=1}^{\infty} s^n\psi(d(x_{n+1}, x_{n+2})) \leq \sum_{n=1}^{n_0} s^n\psi(d(x_{n+1}, x_{n+2}))$$

$$+\sum_{n=1}^{\infty}(sr_1)^n\psi(d(x_{n_0+1},x_{n_0+2}))<\infty.$$

Since

$$\limsup_{n\to\infty}\frac{s^n d(x_{n+1},x_{n+2})}{s^n\psi(d(x_{n+1},x_{n+2}))}\leq\limsup_{x\to 0^+}\frac{x}{\psi(x)}<\infty,$$

then by Limit Comparison Test Theorem,

$$\sum_{n=1}^{\infty}s^n d(x_{n+1},x_{n+2})<\infty.$$

This means that the sequence $\{x_n\}$ is a $b$-Cauchy sequence. Since $(X,d)$ is a $b$-complete $b$-metric space, then there exists $x\in X$ such that

$$\lim_{n\to\infty}x_n=x. \tag{2.5}$$

Finally, we claim that $f(x)=x$.

If $f$ be a continuous function, then obviously $f(x)=x$.

Now, let (II) holds. Assume that $fx\neq x$. Then from Lemma 1.10, as $\psi$ is increasing,

$$\begin{aligned}\psi(d(x,fx))&=\psi(s\cdot\frac{1}{s}d(x,fx))\\&\leq\psi(s\liminf_{n\to\infty}d(x_{n+1},fx))\\&\leq\liminf_{n\to\infty}\psi(sd(x_{n+1},fx))\\&\leq\liminf_{n\to\infty}\alpha(x_n,x)\psi(sd(fx_n,fx))\\&\leq\liminf_{n\to\infty}\beta(\psi(M(x_n,x)))\psi(M(x_n,x))\\&=0,\end{aligned}$$

as

$$\lim_{n\to\infty}M(x_n,x)=\lim_{n\to\infty}\max\{d(x_n,x),d(x_n,x_{n+1}),d(x,x_{n+1})\}=0,$$

which is a contradiction. This implies that $fx=x$. □

As a result of the above theorem, we have the following consequence in ordered $b$-metric spaces.

**Theorem 2.2.** *Let $(X,d,\preceq)$ be an ordered $b$-complete $b$-metric space and let $f:X\to X$ satisfies the following condition:*

$$\psi(sd(fx,fy))\leq\beta(\psi(M(x,y)))\psi(M(x,y)), \tag{2.6}$$

*for all $x,y\in X$, with $x\preceq y$ where $\beta\in\Phi_s$ and $\psi\in\Psi$.*
*Then, $f$ has a fixed point if,*

(i) *$f$ is nondecreasing with respect to $\preceq$;*
(ii) *there exists $x_0\in X$ such that $x_0\preceq fx_0$;*

(iii) *$f$ is continuous, or*
(iii′) *$(X, d, \preceq)$ is regular.*

*Proof.* Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, \text{ if } x \preceq y, \\ \\ 0, \text{ otherwise.} \end{cases}$$

First, we prove that $f$ is a triangular $\alpha$-admissible mapping. Assume that $\alpha(x, y) \geq 1$. Therefore, we have $x \preceq y$. Since $f$ is nondecreasing with respect to $\preceq$, we get $fx \preceq fy$, that is, $\alpha(fx, fy) \geq 1$. Also, let $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x \preceq z$ and $z \preceq y$. Consequently, we deduce that $x \preceq y$, that is, $\alpha(x, y) \geq 1$. Thus, $f$ is a triangular $\alpha$-admissible mapping. Since

$$\psi(sd(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{2.7}$$

for all $x, y \in X$, with $x \preceq y$, then

$$\alpha(x, y)\psi(sd(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)). \tag{2.8}$$

Moreover, from (ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, that is, $\alpha(x_0, fx_0) \geq 1$. Hence, all the conditions of Theorem 2.1 are satisfied and therefore $f$ has a fixed point. □

## 3. Coupled fixed point results

We will use the following simple lemma in proving our next results.

**Lemma 3.1** ([27]). *Let $(\mathcal{X}, d)$ be a b-metric space (with the parameter $s$) and let $f : \mathcal{X}^2 \to \mathcal{X}$. Suppose that $F : \mathcal{X}^2 \to \mathcal{X}^2$ is given by*

$$FX = (f(x, y), f(y, x)), \quad X = (x, y) \in \mathcal{X}^2.$$

(a) *If a mapping $\Omega_2^m : \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ is given by*

$$\Omega_2^m(X, U) = \max\{d(x, u), d(y, v)\},$$
$$X = (x, y) \text{ and } U = (u, v) \in \mathcal{X}^2,$$

*then $(\mathcal{X}^2, \Omega_2^m)$ is a b-metric space (with the same parameter $s$). The space $(\mathcal{X}^2, \Omega_2^m)$ is b-complete iff $(\mathcal{X}, d)$ is b-complete.*

(b) *If $f$ is continuous from $(\mathcal{X}^2, \Omega_2^m)$ to $(\mathcal{X}, d)$ then $F$ is continuous in $(\mathcal{X}^2, \Omega_2^m)$.*

(c) *The mapping $f : X^2 \to X$ is $\alpha$-admissible, that is, for every $x, y, u, v \in X$,*

$$\alpha((x, y), (u, v)) \geq 1 \quad \Rightarrow \quad \alpha((f(x, y), f(y, x)), (f(u, v), f(v, u))) \geq 1,$$

*if and only if the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is $\alpha$-admissible, i.e.,*

$$X, U \in X^2, \quad \alpha(X, U) \geq 1 \quad \Rightarrow \quad \alpha(FX, FU) \geq 1,$$

*where* $\alpha : X^2 \times X^2 \to [0,\infty)$ *is a function.*

(d) *The statement (c) holds if we replace the* $\alpha$*-admissibility by triangular* $\alpha$*-admissibility.*

Let $(\mathcal{X}, d)$ be a $b$-metric space and $f : \mathcal{X}^2 \to \mathcal{X}$. In the rest of this paper, unless otherwise stated, for all $x, y, u, v \in \mathcal{X}$, let

$$N_f^m(x,y,u,v) = \max\{d(f(x,y), f(u,v)), d(f(y,x), f(v,u))\},$$

and

$$N^m(x,y,u,v) = \max\big\{d(x,u), d(y,v), d(x,f(x,y)), d(y,f(y,x)), d(u,f(x,y)), d(v,f(y,x))\big\}.$$

Obviously,

$$N^m(x,y,u,v) = \max\{\Omega_2^m(X,U), \Omega_2^m(X,FX), \Omega_2^m(U,FX)\} = M_{\Omega_2^m}(X,U).$$

Now, we have the following coupled fixed point result.

**Theorem 3.2.** *Let* $(\mathcal{X}, d)$ *be a b-complete b-metric space with the parameter s and let* $f : \mathcal{X}^2 \to \mathcal{X}$*. Assume that*

$$\alpha((x,y),(u,v))\psi(sN_f^m(x,y,u,v)) \le \beta(\psi(N^m(x,y,u,v)))\psi(N^m(x,y,u,v)), \tag{3.1}$$

*for all* $x, y, u, v \in \mathcal{X}$*, where* $\beta \in \Phi_s$*,* $\psi \in \Psi$ *and* $\alpha : (X^2)^2 \to [0,\infty)$ *and* $f$ *is a triangular* $\alpha$*-admissible mapping.*
*Assume also that there exist* $x_0, y_0 \in \mathcal{X}$ *such that*

$$\alpha((x_0,y_0),(f(x_0,y_0),f(y_0,x_0))) \ge 1 \text{ and } \alpha((y_0,x_0),(f(y_0,x_0),f(x_0,y_0))) \ge 1.$$

*Also, suppose that either*

(a) $f$ *is continuous, or,*

(b) *whenever* $\{x_n\}$ *and* $\{y_n\}$ *in* $X$ *be sequences such that*

$$\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \ge 1 \text{ and } \alpha((y_n,x_n),(y_{n+1},x_{n+1})) \ge 1,$$

*for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *and* $y_n \to y$ *as* $n \to +\infty$*, we have*

$$\alpha((x_n,y_n),(x,y)) \ge 1 \text{ and } \alpha((y_n,x_n),(y,x)) \ge 1,$$

*for all* $n \in \mathbb{N} \cup \{0\}$.

*Then,* $f$ *has a coupled fixed point in* $\mathcal{X}$*.*

*Proof.* Let $\Omega_2^m$ be the $b$-metric on $\mathcal{X}^2$ defined in Lemma 3.1. Also, define the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ by $FX = (f(x,y), f(y,x))$, $X = (x,y)$ as in Lemma 3.1. Then, $(\mathcal{X}^2, \Omega_2^m)$ is a $b$-metric space (with the same parameter $s$ as $\mathcal{X}$). Moreover, the contractive condition (3.1) implies that

$$\alpha(X,U)\psi(s\Omega_2^m(FX,FU)) \leq \beta(\psi(M_{\Omega_2^m}(X,U)))\psi(M_{\Omega_2^m}(X,U)), \tag{3.2}$$

holds for all $X, U \in \mathcal{X}^2$. Also, one can shows that all conditions of Theorem 2.1 satisfied for $F$ and we have proved in Theorem 2.1 that under these conditions, it follows that $F$ has a fixed point $\overline{X} = (\bar{x}, \bar{y}) \in \mathcal{X}^2$ which is obviously a coupled fixed point of $f$. □

In the following theorem, we give a sufficient condition for the uniqueness of the coupled fixed point (see also [21]).

**Theorem 3.3.** *In addition to the hypotheses of Theorem 3.2, suppose that for all $(x,y)$ and $(x^*,y^*) \in \mathcal{X}^2$, there exists $(u,v) \in \mathcal{X}^2$, such that $\alpha((x,y),(u,v)) \geq 1$ and $\alpha((x^*,y^*),(u,v)) \geq 1$. Then, $f$ has a unique coupled fixed point of the form $(a,a)$.*

*Proof.* It was proved in Theorem 3.2 that the set of coupled fixed points of $f$, i.e., the set of fixed points of $F$ in $\mathcal{X}^2$ is nonempty. We shall show that if $X$ and $X^*$ are fixed points of $F$, that is,

$$X = FX \text{ and } X^* = FX^*,$$

then $X = X^*$.

Choose an element $U = (u,v) \in \mathcal{X}^2$ such that $\alpha((x,y),(u,v)) \geq 1$ and $\alpha((x^*,y^*),(u,v)) \geq 1$. Let $U_0 = U$ and choose $U_1 \in \mathcal{X}^2$ so that $U_1 = FU_0$. Then, we can inductively define a sequence $\{U_n\}$ such that $U_{n+1} = FU_n$.

As $\alpha((x,y),(u,v)) \geq 1$ and $f$ is triangular $\alpha$-admissible, then

$$\alpha((f(x,y), f(y,x)), (f(u,v), f(v,u))) \geq 1,$$

i.e., $\alpha(X, U_0) \geq 1$ which yields that

$$\alpha(FX, FU) = \alpha(FX, FU_0) = \alpha(X, U_1) \geq 1.$$

Therefore, by the mathematical induction, we obtain that $\alpha(X, U_n) \geq 1$, for all $n \geq 0$.

Applying (3.1), one obtains that

$$\begin{aligned} \psi(s\Omega_2^m(FX, U_{n+1}) &\leq \alpha(X,U_n)\psi(s\Omega_2^m(FX, FU_n) \\ &\leq \beta(\psi(M_{\Omega_2^m}(X,U_n)))\psi(M_{\Omega_2^m}(X,U_n)) \\ &\leq \psi(M_{\Omega_2^m}(X,U_n)). \end{aligned} \tag{3.3}$$

From the properties of $\psi$, if we proceed as in Theorem 2.1 we deduce that the sequence $\{\Omega_2^m(X,U_n)\}$ is nonincreasing. Hence, as in Theorem 2.1, we can show that

$$\lim_{n\to\infty} \Omega_2^m(X,U_n) = 0,$$

that is, $\{U_n\}$ is $b$-convergent to $X$.

Similarly, we can show that $\{U_n\}$ is $b$-convergent to $X^*$. Since the limit is unique, it follows that $X = X^*$. Hence, the coupled fixed point is unique. Also, if $(a, b)$ is a coupled fixed point of $f$, then $(b, a)$ is also a coupled fixed point of $f$. Uniqueness of the coupled fixed point yields that $a = b$. □

Let $\Omega_2^a : \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ is given by

$$\Omega_2^a(X, U) = \frac{d(x, u) + d(y, v)}{2}, \qquad X = (x, y), \qquad U = (u, v) \in \mathcal{X}^2,$$

then $(\mathcal{X}^2, \Omega_2^a)$ is a $b$-metric space (with the same parameter $s$).

Let $(\mathcal{X}, d)$ be a $b$-metric space, $f : \mathcal{X}^2 \to \mathcal{X}$. For all $x, y, u, v \in \mathcal{X}$, let

$$N_f^a(x, y, u, v) = \frac{d(f(x, y), f(u, v)) + d(f(y, x), f(v, u))}{2}$$

and

$$N^a(x, y, u, v) = \max\left\{\frac{d(x, u) + d(y, v)}{2}, \frac{d(x, f(x, y)) + d(y, f(y, x))}{2}, \frac{d(u, f(x, y)) + d(v, f(y, x))}{2}\right\}.$$

Obviously,

$$N^m(x, y, u, v) = \max\{\Omega_2^a(X, U), \Omega_2^a(X, FX), \Omega_2^a(U, FX)\} = M_{\Omega_2^a}(X, U).$$

*Remark* 3.4. The results of Theorems 3.2 and 3.3 hold, if we replace $\Omega_2^m$, $N_f^m$ and $N^m$ by $\Omega_2^a$, $N_f^a$ and $N^a$, respectively.

## 4. Coupled fixed point results in partially ordered $b$-metric spaces

We will use the following simple lemma in proving our results.

**Lemma 4.1.** *Let $(\mathcal{X}, d, \preceq)$ be an ordered $b$-metric space (with the parameter $s$) and let $f : \mathcal{X}^2 \to \mathcal{X}$. Let $F : \mathcal{X}^2 \to \mathcal{X}^2$ is given by*

$$FX = (f(x, y), f(y, x)), \qquad X = (x, y) \in \mathcal{X}^2.$$

(a) *If a relation $\sqsubseteq_2$ is defined on $\mathcal{X}^2$ by*

$$X \sqsubseteq_2 U \quad \Leftrightarrow \quad x \preceq u \wedge y \succeq v, \qquad X = (x, y), \qquad U = (u, v) \in \mathcal{X}^2,$$

*then $(\mathcal{X}^2, \Omega_2^m, \sqsubseteq_2)$ and $(\mathcal{X}^2, \Omega_2^a, \sqsubseteq_2)$ are ordered $b$-metric spaces (with the same parameter $s$).*

(b) *If the mapping $f$ has the mixed monotone property then the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is nondecreasing w.r.t. $\sqsubseteq_2$, i.e.*

$$X \sqsubseteq_2 U \quad \Rightarrow \quad FX \sqsubseteq_2 FU.$$

**Theorem 4.2.** *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered b-complete b-metric space with the parameter $s$ and let $f : \mathcal{X}^2 \to \mathcal{X}$. Assume that*

$$\psi(sN_f^m(x, y, u, v)) \leq \beta(\psi(N^m(x, y, u, v)))\psi(N^m(x, y, u, v)), \tag{4.1}$$

*for all $x, y, u, v \in \mathcal{X}$ with $x \preceq u$ and $y \succeq v$, where $\beta \in \Phi_s$ and $\psi \in \Psi$. Assume also that*

1. *$f$ has the mixed monotone property;*
2. *there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$.*

*Also, suppose that either*

(a) *$f$ is continuous, or,*
(b) *$(\mathcal{X}, d)$ is regular.*
*Then, $f$ has a coupled fixed point in $\mathcal{X}$.*

*Proof.* By Lemma 4.1, $(\mathcal{X}^2, \Omega_2^m, \sqsubseteq_2)$ is an ordered $b$-metric space (with the same parameter $s$).

Define $\alpha : X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v)) = \begin{cases} 1, \text{ if } (x, y) \sqsubseteq_2 (u, v), \\ \\ 0, \text{ otherwise.} \end{cases}$$

First, we prove that $F$ is a triangular $\alpha$-admissible mapping. Hence, we assume that $\alpha(X, U) \geq 1$, where $X = (x, y)$ and $U = (u, v)$. Therefore, we have $X \sqsubseteq_2 U$. Since $f$ has the mixed monotone property, then from Lemma 4.1, the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ is nondecreasing w.r.t. $\sqsubseteq_2$, i.e.,

$$FX \sqsubseteq_2 FU,$$

that is, $\alpha(FX, FU) \geq 1$. Also, let $\alpha(X, A) \geq 1$ and $\alpha(A, U) \geq 1$, then $X \sqsubseteq_2 A$ and $A \sqsubseteq_2 U$. Consequently, we deduce that $X \sqsubseteq_2 U$, that is, $\alpha(X, U) \geq 1$. Thus, $F$ is a triangular $\alpha$-admissible mapping.

From (4.1) and the definition of $\alpha$ and $\sqsubseteq_2$, we have

$$\psi(s\Omega_2^m(FX, FU)) \leq \beta(\psi(M_{\Omega_2^m}(X, U)))\psi(M_{\Omega_2^m}(X, U)), \tag{4.2}$$

for all $X, U \in X^2$ with $X \sqsubseteq_2 U$. Moreover, from (2) there exists $(x_0, y_0) \in X^2$ such that

$$(x_0, y_0) \sqsubseteq_2 (f(x_0, y_0), f(y_0, x_0)) = F(x_0, y_0).$$

Hence, all the conditions of Theorem 2.2 are satisfied and so $F$ has a fixed point $\overline{X} = (\bar{x}, \bar{y}) \in \mathcal{X}^2$ which is a coupled fixed point of $f$. $\square$

In the following theorem, we give a sufficient condition for the uniqueness of the coupled fixed point.

**Theorem 4.3.** *In addition to the hypotheses of Theorem 4.2, suppose that for all* $(x, y)$ *and* $(x^*, y^*) \in \mathcal{X}^2$*, there exists* $(u, v) \in \mathcal{X}^2$*, such that* $(u, v)$ *is comparable with* $(x, y)$ *and* $(x^*, y^*)$*. Then,* $f$ *has a unique coupled fixed point of the form* $(a, a)$*.*

*Proof.* It was proved in Theorem 4.2 that the set of coupled fixed points of $f$, i.e., the set of fixed points of $F$ in $\mathcal{X}^2$ is nonempty. We shall show that if $X$ and $X^*$ are fixed points of $F$, that is,

$$X = FX, \qquad X^* = FX^*,$$

then $X = X^*$. Let there exists $(u, v) \in \mathcal{X}^2$, which is comparable with $(x, y)$ and $(x^*, y^*)$. Without any loss of generality, we may assume that $(x, y) \sqsubseteq_2 (u, v)$ and $(x^*, y^*) \sqsubseteq_2 (u, v)$. According to the definition of $\alpha$ in the above theorem, $\alpha((x, y), (u, v)) \geq 1$ and $\alpha((x^*, y^*), (u, v)) \geq 1$.

Now, following the proof of Theorem 3.3, one can obtains that $X = X^*$. The remainder part of the proof is analogous to the proof of Theorem 3.3 and so we omit it. □

*Remark* 4.4. In Theorem 4.2, we can replace the contractive condition (4.1) by the following:

$$\psi(sN^a_f(x, y, u, v)) \leq \beta(\psi(N^a(x, y, u, v)))\psi(N^a(x, y, u, v)). \tag{4.3}$$

## 5. Application To Ordinary Differential Equations

Consider the periodic boundary value problem (PBVP)

$$\begin{cases} u' = h(t, u), & t \in I = [0, T]; \\ u(0) = u(T), \end{cases} \tag{5.1}$$

where $T > 0$ and $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We assume that there exist continuous functions $f, g$ such that

$$h(t, u) = f(t, u) + g(t, u), \quad t \in [0, T].$$

Existence of a unique solution to a periodic boundary value problem for mixed monotone mapping on partially ordered metric spaces was studied in [4]. Also, recently this equation is considered for single-valued mappings in [9], for generalization of Mizoguchi and Takahashi Theorem. In this section, we study the existence of a solution to equation (5.1).

Consider the space $C(I, \mathbb{R})$ of continuous functions defined on $I = [0, T]$. We endow X with the b-metric

$$d(u, v) = \sup_{t \in [0,T]} |u(t) - v(t)|^p,$$

for all $u, v \in X$ where $p > 1$.

Clearly, $(X, d)$ is a $b$-complete b-metric space with parameter $s = 2^{p-1}$. The metric space $C(I, \mathbb{R})$ can also be equipped with a partial order given by

$$x, y \in C(I), \quad x \preceq y \quad \Leftrightarrow \quad x(t) \leq y(t), \quad (t \in I). \tag{5.2}$$

Also, $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set if we define the following order relation:

$$(x, y) \sqsubseteq_2 (u, v), \quad \Leftrightarrow \quad x(t) \leq u(t), \qquad y(t) \geq v(t), \quad (t \in I). \tag{5.3}$$

Also, $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a b-complete b-metric space by following meter:

$$D((x, y), (u, v)) = \sup_{t \in I} |x(t) - u(t)|^p + \sup_{t \in I} |y(t) - v(t)|^p \ (x, y, u, v \in C(I, \mathbb{R})). \tag{5.4}$$

**Assumption 5.1.** *Suppose that there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ such that for each $u, v \in C(I, \mathbb{R})$ with $u \preceq v$*

$$0 \leq (f(s, u) + \lambda_1 u) - (f(s, v) + \lambda_1 v) \leq \sqrt[p]{\mu_1 \ln(d(u, v) + 1)}, \tag{5.5}$$

*and*

$$-\sqrt[p]{\mu_2 \ln(d(u, v) + 1)} \leq (g(s, u) - \lambda_2 u) - (g(s, v) - \lambda_2 v) \leq 0, \tag{5.6}$$

*where*

$$2^p \frac{\max\{\mu_1, \mu_2\}}{(\lambda_1 + \lambda_2)^p} \leq 1.$$

We shall obtain the unique solution of PBVP (3) and in several steps. As a first step, we study the existence of a solution of the following periodic system:

$$\begin{cases} u'(t) + \lambda_1 u - \lambda_2 v = f(t, u(t)) + g(t, v(t)) + \lambda_1 u - \lambda_2 v, \\ v'(t) + \lambda_1 v - \lambda_2 u = f(t, v(t)) + g(t, u(t)) + \lambda_1 v - \lambda_2 u, \\ u(0) = u(T) \text{ and } v(0) = v(T), \end{cases} \tag{5.7}$$

This problem is equivalent to the integral equations:

$$\begin{cases} u(t) &= \int_0^T G_1(t, s)[f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \\ & \quad + G_2(t, s)[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds, \\ v(t) &= \int_0^T G_1(t, s)[f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \\ & \quad + G_2(t, s)[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds, \end{cases} \tag{5.8}$$

where

$$G_1(t, s) = \begin{cases} \frac{1}{2}\left[\frac{e^{-(\lambda_1 + \lambda_2)(t-s)}}{1 - e^{-(\lambda_1 + \lambda_2)T}} + \frac{e^{(\lambda_2 - \lambda_1)(t-s)}}{1 - e^{(\lambda_2 - \lambda_1)T}}\right], & 0 \leq s < t \leq T, \\ \\ \frac{1}{2}\left[\frac{e^{-(\lambda_1 + \lambda_2)(t-s+T)}}{1 - e^{-(\lambda_1 + \lambda_2)T}} + \frac{e^{(\lambda_2 - \lambda_1)(t-s+T)}}{1 - e^{(\lambda_2 - \lambda_1)T}}\right], & 0 \leq t < s \leq T, \end{cases}$$

and

$$G_2(t,s)=\begin{cases}\frac{1}{2}\left[\frac{e^{(\lambda_2-\lambda_1)(t-s)}}{1-e^{(\lambda_2-\lambda_1)T}}-\frac{e^{-(\lambda_1+\lambda_2)(t-s)}}{1-e^{-(\lambda_1+\lambda_2)T}}\right], & 0\le s<t\le T,\\ \frac{1}{2}\left[\frac{e^{(\lambda_2-\lambda_1)(t-s+T)}}{1-e^{(\lambda_2-\lambda_1)T}}-\frac{e^{-(\lambda_1+\lambda_2)(t-s+T)}}{1-e^{-(\lambda_1+\lambda_2)T}}\right], & 0\le t<s\le T.\end{cases}$$

Lemma 3.2 of [4], Guarantees that $G_1(t,s)\ge 0$ and $G_2(t,s)\le 0$ for all $0\le t,s\le T$. Also, from Remark 3.3 of [4] $C(I,\mathbb{R})$ is regular.

**Definition 5.2** ([4])**.** An element $(\alpha,\beta)\in X\times X$ is called a coupled lower and upper solution of the PBVP (3) if

$$\alpha'(t)\le f(t,\alpha(t))+g(t,\beta(t)),\qquad \beta'(t)\ge f(t,\beta(t))+g(t,\alpha(t)),$$

together with the periodicity conditions,

$$\alpha(0)\le\alpha(T),\qquad \beta(0)\ge\beta(T).$$

To prove the main result of this section, we need the following Lemma from [4]:

**Lemma 5.3.** *If*

$$\lambda_1(\alpha(T)-\alpha(0))+\lambda_2(\beta(0)-\beta(T))\le\frac{\alpha(T)-\alpha(0)}{T},\tag{5.9}$$

*and*

$$\lambda_1(\beta(0)-\beta(T))+\lambda_2(\alpha(T)-\alpha(0))\le\frac{\beta(0)-\beta(T)}{T},\tag{5.10}$$

*then $\alpha(t)\le F(\alpha(t),\beta(t))$ and $\beta(t)\ge F(\beta(t),\alpha(t))$, for $t\in(0,T)$, where*

$$\begin{aligned}F(u,v)(t)=\int_0^T & G_1(t,s)[f(s,u)+g(s,v)+\lambda_1u-\lambda_2v]\\ &+G_2(t,s)[f(s,v)+g(s,u)+\lambda_1v-\lambda_2u]ds.\end{aligned}$$

Now, we ready to prove the following Theorem:

**Theorem 5.4.** *Consider the problem 5.8 with $f,g\in C(I\times\mathbb{R},\mathbb{R})$. Then, having Assumption 5.1, the existence of a coupled lower and upper solution for 5.8 provides the existence of a unique solution of 5.8.*

*Proof.* The mapping $F$ enjoys the mixed monotone property [4].

For all $(x,y)\sqsubseteq_2(u,v)$,

$$\ln(2^{p-1}d(F(u,v),F(x,y))+1)=\ln(2^{p-1}\sup_{t\in I}|F(u,v)(t)-F(x,y)(t)|^p+1)$$

$$\begin{aligned}\le\ln\Big(2^{p-1}\sup_{t\in I}\Big|\int_0^T G_1(t,s)\Big(&[f(s,u)+g(s,v)+\lambda_1u-\lambda_2v]\\ -[f(s,x)+g(s,y)+\lambda_1x-\lambda_2y]\Big)&\end{aligned}$$

$$
\begin{aligned}
&+ G_2(t,s)\Big([f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u]\\
&- [f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x]\Big)ds\Big|^p+1\Big)\\
&= \ln\Big(2^{p-1}\sup_{t\in I}\Big|\int_0^T G_1(t,s)\Big([f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v]\\
&- [f(s,x)+g(s,y)+\lambda_1 x-\lambda_2 y]\Big)\\
&- G_2(t,s)\Big([f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x]\\
&- [f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u]\Big)ds\Big|^p+1\Big)\\
&\le \ln\Big(2^{p-1}\sup_{t\in I}\Big|\int_0^T G_1(t,s)\Big[\sqrt[p]{\mu_1\ln(d(u,x)+1)}+\sqrt[p]{\mu_2\ln(d(v,y)+1)}\Big]\\
&- G_2(t,s)\Big[\sqrt[p]{\mu_1\ln(d(v,y)+1)}+\sqrt[p]{\mu_2\ln(d(u,x)+1)}\Big]ds\Big|^p+1\Big)\\
&\le \ln\Big(2^p\max\{\mu_1,\mu_2\}\ln(N^m(x,y,u,v)+1)\\
&\sup_{t\in I}\Big|\int_0^T\Big(G_1(t,s)-G_2(t,s)\Big)ds\Big|^p+1\Big)\\
&= \ln\Big(2^p\max\{\mu_1,\mu_2\}\ln(N^m(x,y,u,v)+1)\\
&\sup_{t\in I}\Big|\int_0^t\frac{e^{-(\lambda_1+\lambda_2)(t-s)}}{1-e^{-(\lambda_1+\lambda_2)T}}ds+\int_t^T\frac{e^{-(\lambda_1+\lambda_2)(t-s+T)}}{1-e^{-(\lambda_1+\lambda_2)T}}ds\Big|^p+1\Big)\\
&= \ln\Big(2^p\max\{\mu_1,\mu_2\}\ln(N^m(x,y,u,v)+1)\frac{1}{(\lambda_1+\lambda_2)^p}+1\Big)\\
&\le \frac{\ln(\ln(N^m(x,y,u,v)+1))+1}{\ln(N^m(x,y,u,v)+1)}\ln(N^m(x,y,u,v)+1)\\
&= \beta(\ln(N^m(x,y,u,v)+1))\ln(N^m(x,y,u,v)+1).
\end{aligned}
$$

Put $\psi(x)=\ln(x+1)$ and $\beta(x)=\frac{\psi(x)}{x}$. So, $\psi$ is a continuous, increasing and positive function in $(0,\infty)$ with $\psi(0)=0$ and

$$\limsup_{x\to 0}\frac{x}{\psi(x)}<\infty.$$

Also, $\beta\in\Phi_s$.

Finally, let $(\alpha,\beta)$ be a coupled upper and lower solution of equation (3). Then by Lemma 5.3, we have

$$\alpha(t)\le F(\alpha(t),\beta(t)),\qquad \beta(t)\ge F(\beta(t),\alpha(t)). \tag{5.11}$$

Now, we see that $F$ satisfies the hypotheses of Theorems 4.2 and 4.3 and so, it follows that F has a unique fixed point. $\square$

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