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FIXED AND COMMON FIXED POINTS FOR (ψ, φ)-WEAKLY CONTRACTIVE MAPPINGS IN b-METRIC SPACES

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Abstract. In this paper, we give a fixed point theorem for (ψ, φ)-weakly contractive mappings in complete b-metric spaces. We also give a common fixed point theorem for such mappings in complete b-metric spaces via altering functions. The given results generalize two known results in the setting of metric spaces. Two examples are given to verify the given results.

1. Introduction

The notion of a b-metric which is, in essence, a relaxation of the triangle inequality, first introduced by Bakhtin [2] and then followed by Czerwik [7] to obtain a generalization of the Banach contraction principle. Such a relaxation for a distance is also discussed in [10] under the name nonlinear elastic matching distance. In particular, this kind of distances are used in [6, 16, 22] for trade mark shapes, to measure ice floes, and to study the optimal transport path between probability measures, respectively. Later, Khamsi and Hussain [13] reintroduced the notion of a b-metric under the name metric-type. For some recent works in b-metric spaces the reader is referred to [3, 8, 11, 18, 19, 21]. In order to present our main results, we start with the following two definitions.

Definition 1.1. Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if the following conditions hold for all $x, y, z \in X$:

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(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular inequality).

Then, the pair (X, d) is called a b-metric space with parameter s.

Definition 1.2 ([14]). (Altering Distance Function) A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and strictly increasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Alber et al., [1] introduced weakly contractive mappings and gave some fixed point results for such mappings in Hilbert spaces. Dutta and Choudhury [9] gave the following result which is a generalization of the main result given by Rhoades [20].

Theorem 1.3. *Let (X,d) be a complete metric space and $T : X \to X$ satisfies*

$$\psi\left(d(Tx, Ty)\right) \leq \psi\left(d(x, y)\right) - \varphi\left(d(x, y)\right),$$

where ψ and φ are altering distance functions and $x, y \in X$. Then T has a unique fixed point.

Chandok [4] proved the following common fixed point theorem for the generalized (ψ, φ)-weakly contractive mappings.

Theorem 1.4. *Let (X,d) be a complete metric space and $T, f : X \to X$ satisfies*

$$\psi\left(d(Tx, fy)\right) \leq \psi\left(\frac{d(x, fy) + d(y, Tx)}{2}\right) - \varphi\left(d(x, fy), d(y, Tx)\right),$$

for all $x, y \in X$, where ψ is an altering distance function and $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a lower semi-continuous mapping such that $\varphi(x, y) = 0$ if and only if $x = y = 0$. Then T and f have a unique common fixed point.

In this paper, we restate Theorems 1.3 and 1.4 in the complete b-metric spaces and obtain a generalization of them.

2. Main Results

Throughout this section, we assume that (X, d) is a complete b-metric space. We first use two notations

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) |\, \psi \text{ is an altering distance function}\},$$

and

$$\Phi_1 = \left\{\varphi : [0,\infty) \to [0,\infty) |\, \varphi \text{ is continuous, } \varphi(t) = 0 \Leftrightarrow t = 0, \text{ and } \varphi(\liminf_{n\to\infty} a_n) \le \liminf_{n\to\infty} \varphi(a_n)\right\}.$$

(see e.g., [17]).

Theorem 2.1. *Let (X,d) be a complete b-metric space with parameter $s \ge 1, T : X \to X$ be a self-mapping satisfying the (ψ,φ)-weakly contractive condition*

$$\psi\left(sd(Tx,Ty)\right) \le \psi\left(\frac{d(x,y)}{s^2}\right) - \varphi\left(d(x,y)\right), \tag{2.1}$$

for all $x, y \in X$, where $\psi \in \Psi, \varphi \in \Phi_1$. Then T has a fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Consider the iterated sequence $\{x_n\}$, where $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ We will prove that $d(x_n, x_{n+1}) \to 0$. Using (2.1), we have

$$\psi\left(sd(x_n,x_{n+1})\right) \le \psi\left(\frac{d(x_{n-1},x_n)}{s^2}\right) - \varphi\left(d(x_{n-1},x_n)\right), \quad n = 1,2,3,\dots. \tag{2.2}$$

Therefore,

$$\psi\left(sd(x_n,x_{n+1})\right) \le \psi\left(\frac{d(x_{n-1},x_n)}{s^2}\right), \quad n = 1,2,3,\dots.$$

Since ψ is strictly increasing, we have

$$sd(x_n,x_{n+1}) \le \frac{d(x_{n-1},x_n)}{s^2}, \quad n = 1,2,3,\dots.$$

Therefore, we get

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n), \quad n = 1,2,3,\dots.$$

Thus $\{d(x_n,x_{n+1})\}$ is a nonincreasing sequence and hence it is convergent. Let $d(x_n,x_{n+1}) \to r$, where $r \ge 0$. Letting $n \to \infty$ in (2.2) and using the continuity of ψ and φ, we obtain

$$\psi(sr) \le \psi\left(\frac{r}{s^2}\right) - \varphi(r).$$

Therefore

$$\psi\left(\frac{r}{s^2}\right) \le \psi\left(\frac{r}{s^2}\right) - \varphi(r).$$

This implies $r = 0$, that is,

$$d(x_n,x_{n+1}) \to 0. \tag{2.3}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose opposite, i.e., $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \tag{2.4}$$

and

$$d(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon. \tag{2.5}$$

Using (2.4) and (2.5), we obtain

$$\begin{aligned} \varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq s\left(d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})\right) \\ &\leq s\left(\varepsilon + d(x_{n(k)-1}, x_{n(k)})\right), \end{aligned}$$

for all $k \geq 1$. Therefore

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq s\varepsilon, \tag{2.6}$$

Moreover, for all $k \geq 1$, we have

$$\begin{aligned} \varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq s\left(d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)})\right) \\ &\leq sd(x_{m(k)}, x_{m(k)+1}) + s^2\left(d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)}, x_{n(k)+1})\right). \end{aligned}$$

Using (2.3), we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} sd(x_{m(k)+1}, x_{n(k)+1}). \tag{2.7}$$

Also letting $k \to \infty$ and using (2.4) for all $k \geq 1$, we get

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}). \tag{2.8}$$

Using (2.1) and (2.7), we have

(2.9)

$$\begin{aligned} \psi\left(\frac{\varepsilon}{s}\right) &\leq \psi\left(\limsup_{k\to\infty} sd\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \\ &= \psi\left(\limsup_{k\to\infty} sd(Tx_{m(k)}, Tx_{n(k)})\right) \\ &\leq \limsup_{k\to\infty}\left(\psi\left(\frac{d(x_{m(k)}, x_{n(k)})}{s^2}\right) - \varphi\left(d(x_{m(k)}, x_{n(k)})\right)\right) \\ &= \limsup_{k\to\infty} \psi\left(\frac{d(x_{m(k)}, x_{n(k)})}{s^2}\right) - \liminf_{k\to\infty} \varphi\left(d(x_{m(k)}, x_{n(k)})\right). \end{aligned}$$

Using (2.6) and that $\varphi \in \Phi_1$, we obtain

$$\psi\left(\frac{\varepsilon}{s}\right) \le \psi\left(\frac{\varepsilon}{s}\right) - \varphi\left(\liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)})\right).$$

Hence, we have $\varphi\left(\liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)})\right) = 0$. Since $\varphi \in \Phi_1$, we get $\liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = 0$, which contradicts (2.8). Hence $\{x_n\}$ is a Cauchy sequence . The completeness of X implies that there exists $x^* \in X$ such that $x_n \to x^*$. Using (2.1) we have

$$\begin{aligned}\psi\left(sd(Tx_n, Tx^*)\right) &\le \psi\left(\frac{d(x_n, x^*)}{s^2}\right) - \varphi\left(d(x_n, x^*)\right)\\ &\le \psi\left(\frac{d(x_n, x^*)}{s^2}\right), \quad n = 0, 1, 2, 3, \ldots.\end{aligned}$$

Since ψ is strictly increasing, we have

$$sd(Tx_n, Tx^*) \le \frac{d(x_n, x^*)}{s^2}, \quad n = 0, 1, 2, \ldots.$$

Passing to limit when $n \to \infty$, we obtain $Tx_n \to Tx^*$. We have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*, \tag{2.10}$$

i.e., x^* is a fixed point of T. To see the uniqueness of the fixed point x^*, assume to the contrary that $Ty^* = y^*$ and $x^* \ne y^*$. From (2.1),

$$\psi\left(sd(Tx^*, Ty^*)\right) \le \psi\left(\frac{d(x^*, y^*)}{s^2}\right) - \varphi\left(d(x^*, y^*)\right).$$

Then

$$\psi\left(\frac{d(x^*, y^*)}{s^2}\right) \le \psi\left(\frac{d(x^*, y^*)}{s^2}\right) - \varphi\left(d(x^*, y^*)\right). \tag{2.11}$$

Hence $\varphi\left(d(x^*, y^*)\right) = 0$, which implies that $x^* = y^*$. □

In Theorem 2.1, if $\psi(t) = t$ and $\varphi(t) = \left(\frac{1}{s^2} - \alpha\right)t$, where $\alpha \in [0, \frac{1}{s^2})$, we get the following result which is also a generalization of the Banach contraction principle.

Corollary 2.2. *Let (X, d) be a complete b-metric space with the parameter $s \ge 1$, $\alpha \in [0, \frac{1}{s^2})$ and T be a self-mapping on X satisfying $d(Tx, Ty) \le \frac{\alpha}{s} d(x, y)$, for all $x, y \in X$. Then T has a unique fixed point.*

Example 2.3. Let $X = [0, 1]$ and d be defined by $d(x, y) = (x - y)^2$, for all $x, y \in [0, 1]$. It is easy to check that (X, d) is a b-metric space with parameter $s = 2$. We set $Tx = \frac{x}{8}$ for all $x \in X$. Define $\psi : [0, \infty) \to$

$[0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = 2t$ and $\varphi(t) = \frac{t}{4}$. Then for $x, y \in X$, we have

$$\psi\left(2d(Tx,Ty)\right) = \psi\left(2\left(\frac{x}{8} - \frac{y}{8}\right)^2\right) = \frac{4}{64}(x-y)^2,$$

and

$$\psi\left(\frac{d(x,y)}{s^2}\right) - \varphi\left(d(x,y)\right) = \frac{(x-y)^2}{4} > \frac{4(x-y)^2}{64}.$$

Hence

$$\psi\left(2d(Tx,Ty)\right) \leq \psi\left(\frac{d(x,y)}{s^2}\right) - \varphi\left(d(x,y)\right),$$

for all $x, y \in [0,1]$.

3. A common fixed point theorem

In the section, we give a common fixed point theorem in the b-metric spaces. In fact, motivated by the results given in [4], we give a common fixed point theorem for self-mappings satisfying a (ψ,φ)-generalized Chatterjea-type contractive condition in b-metric spaces. The following notation will be needed, (see e.g., [17]):

$$\Phi_2 = \Big\{\varphi : [0,\infty) \times [0,\infty) \to [0,\infty) | \varphi(x,y) = 0 \Leftrightarrow x = y = 0,$$
$$\varphi\left(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right) \leq \liminf_{n\to\infty} \varphi(a_n, b_n)\Big\}.$$

Theorem 3.1. *Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and $T, f : X \to X$ satisfy the (ψ,φ)-generalized Chatterjea-type contractive condition*

$$\psi\left(sd(Tx,fy)\right) \leq \psi\left(\frac{d(x,fy) + \frac{d(y,Tx)}{s^3}}{s+1}\right) - \varphi\left(d(x,fy), d(y,Tx)\right), \tag{3.1}$$

for all $x, y \in X$ and for some $\psi \in \Psi, \varphi \in \Phi_2$. If T or f are continuous, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X, x_1 = Tx_0$ and $x_2 = fx_1$. Consider the sequence $\{x_n\}$ in which $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$ for every $n \geq 0$. We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Using Condition (3.1), for $n \geq 0$ we obtain

$$\begin{aligned}
(3.2)\quad \psi\left(sd(x_{2n+1}, x_{2n+2})\right) &= \psi\left(sd(Tx_{2n}, fx_{2n+1})\right) \\
&\le \psi\left(\frac{d(x_{2n}, fx_{2n+1}) + \frac{d(x_{2n+1}, Tx_{2n})}{s^3}}{s+1}\right) \\
&\quad - \varphi\left(d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, Tx_{2n})\right) \\
&= \psi\left(\frac{d(x_{2n}, x_{2n+2})}{s+1}\right) \\
&\quad - \varphi\left(d(x_{2n}, x_{2n+2}), 0\right).
\end{aligned}$$

Since φ is nonnegative, we have

$$\psi\left(sd(x_{2n+1}, x_{2n+2})\right) \le \psi\left(\frac{d(x_{2n}, x_{2n+2})}{s+1}\right).$$

This implies that

$$\begin{aligned}
(3.3)\qquad sd(x_{2n+1}, x_{2n+2}) &\le \frac{d(x_{2n}, x_{2n+2})}{s+1} \\
&\le \frac{s}{s+1}\left(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right),
\end{aligned}$$

for $n \ge 0$. So we obtain

$$(3.4)\qquad d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1}), \quad n = 0, 1, 2, \dots.$$

Similarly, we have

$$(3.5)\qquad d(x_{2n+2}, x_{2n+3}) \le d(x_{2n+1}, x_{2n+2}), \quad n = 0, 1, 2, \dots.$$

Using (3.4) and (3.5), by induction we get

$$(3.6)\qquad d(x_n, x_{n+1}) \le d(x_{n-1}, x_n), \quad n = 1, 2, 3, \dots.$$

Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Hence there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (3.3), we have

(3.7)

$$\begin{aligned}
sd(x_{2n+1}, x_{2n+2}) &\le \frac{d(x_{2n}, x_{2n+2})}{s+1} \\
&\le \frac{s}{2}\left(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right),
\end{aligned}$$

for $n = 0, 1, 2, \dots$. Passing to the limit as $n \to \infty$ we have

$$sr \le \frac{1}{s+1} \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) \le \frac{s}{2}(r + r) = sr.$$

Therefore

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = (s+1)sr. \tag{3.8}$$

From (3.2), we get

$$\begin{aligned}\psi\left(\limsup_{n\to\infty} sd(x_{2n+1}, x_{2n+2})\right) &\le \limsup_{n\to\infty} \psi\left(\frac{d(x_{2n}, x_{2n+2})}{s+1}\right)\\ &\quad - \liminf_{n\to\infty} \varphi\left(d(x_{2n}, x_{2n+2}), 0\right)\\ &\le \psi\left(\frac{\limsup_{n\to\infty} d(x_{2n}, x_{2n+2})}{s+1}\right)\\ &\quad - \varphi\left(\liminf_{n\to\infty} d(x_{2n}, x_{2n+2}), 0\right).\end{aligned}$$

Then

$$\psi(sr) \le \psi\left(\frac{(s+1)sr}{s+1}\right) - \varphi\left((s+1)sr, 0\right),$$

and so $\varphi\left((s+1)sr, 0\right) = 0$. Since $\varphi \in \Phi_2$ we get $r = 0$. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.9}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. It suffices to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, and

$$d(x_{2m(k)}, x_{2n(k)}) \ge \varepsilon, \tag{3.10}$$

and

$$d(x_{2m(k)}, x_{2n(k)-2}) \le \varepsilon. \tag{3.11}$$

From (3.10) and the b-triangular inequality, we have

$$\begin{aligned}\varepsilon &\le d(x_{2m(k)}, x_{2n(k)})\\ &\le s\left(d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)})\right)\\ &\le s\varepsilon + s^2\left(d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)})\right),\end{aligned}$$

for all $k \ge 1$. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, passing to the limit as $k \to \infty$ we obtain

$$\varepsilon \le \limsup_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) \le s\varepsilon. \tag{3.12}$$

Moreover from (3.10) and the b-triangular inequality we get

$$\begin{aligned}\varepsilon &\le d(x_{2m(k)}, x_{2n(k)})\\ &\le s\left(d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)})\right),\end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$, we have

$$\varepsilon \leq \limsup_{k\to\infty} sd(x_{2m(k)+1}, x_{2n(k)}). \tag{3.13}$$

On the other hand,

$$\begin{aligned} d(x_{2n(k)-1}, x_{2m(k)+1}) &\leq s\left(d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)+1})\right) \\ &\leq sd(x_{2n(k)-1}, x_{2n(k)}) + s^2\big(d(x_{2n(k)}, x_{2m(k)}) \\ &\quad + d(x_{2m(k)}, x_{2m(k)+1})\big), \end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$, we have

$$\limsup_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) \leq s^3\varepsilon. \tag{3.14}$$

Also from (3.10) one can show that

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}). \tag{3.15}$$

Using (3.1) and (3.12)-(3.14), we have

$$\begin{aligned} \psi(\varepsilon) &\leq \psi\left(\limsup_{k\to\infty} sd\left(x_{2m(k)+1}, x_{2n(k)}\right)\right) \\ &= \psi\left(\limsup_{k\to\infty} sd\left(Tx_{2m(k)}, fx_{2n(k)-1}\right)\right) \\ &\leq \limsup_{k\to\infty} \psi\left(\frac{d(x_{2m(k)}, fx_{2n(k)-1}) + \dfrac{d\left(x_{2n(k)-1}, Tx_{2m(k)}\right)}{s^3}}{s+1}\right) \\ &\quad - \liminf_{k\to\infty} \varphi\left(d(x_{2m(k)}, fx_{2n(k)-1}), d(x_{2n(k)-1}, Tx_{2m(k)})\right) \\ &\leq \psi\left(\frac{\limsup_{k\to\infty}\left(d(x_{2m(k)}, x_{2n(k)}) + \dfrac{d(x_{2n(k)-1}, x_{2m(k)+1})}{s^3}\right)}{s+1}\right) \\ &\quad - \varphi\left(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\right) \\ &\leq \psi\left(\frac{s\varepsilon + \varepsilon}{s+1}\right) \\ &\quad - \varphi\left(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\right) \\ &= \psi(\varepsilon) - \varphi\left(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\right). \end{aligned}$$

Consequently

$$\varphi\left(\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}), \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1})\right) = 0. \tag{3.16}$$

Because $\varphi \in \Phi_2$, we have

$$\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \liminf_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) = 0, \tag{3.17}$$

which contradicts (3.15). This implies that $\{x_{2n}\}$ is a Cauchy sequence and so is $\{x_n\}$. There exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. If T is continuous, we have

$$Tx^* = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = x^*, \tag{3.18}$$

i.e., x^* is a fixed point of T. Moreover, from (3.1) we have

$$\begin{aligned}
\psi\left(sd(x^*, fx^*)\right) &= \psi\left(sd(Tx^*, fx^*)\right)\\
&\le \psi\left(\frac{d(x^*, fx^*) + \dfrac{d(x^*, Tx^*)}{s^3}}{s+1}\right)\\
&\quad - \varphi\left(d(x^*, fx^*), d(x^*, Tx^*)\right)\\
&= \psi\left(\frac{d(x^*, fx^*)}{s+1}\right) - \varphi\left(d(x^*, fx^*), 0\right)\\
&\le \psi\left(\frac{d(x^*, fx^*)}{s+1}\right).
\end{aligned}$$

Since ψ is a strictly increasing function, we have

$$sd(x^*, fx^*) \le \frac{d(x^*, fx^*)}{s+1}.$$

Therefore $fx^* = x^*$. Hence x^* is a common fixed point of T and f.

If f is continuous, then by a similar argument to that of above one can show that T, f have a common fixed point. To see the uniqueness of the common fixed points of T and f, assume on the contrary that $Tu = fu = u$ and $Tv = fv = v$ but $u \ne v$. Consider

$$\begin{aligned}
\psi\left(sd(u, v)\right) &= \psi\left(sd(Tu, fv)\right)\\
&\le \psi\left(\frac{d(u, fv) + \dfrac{d(v, Tu)}{s^3}}{s+1}\right) - \varphi\left(d(u, fv), d(v, Tu)\right).
\end{aligned}$$

Since $s \ge 1$, we have

$$\psi\left(sd(u, v)\right) \le \psi\left(\frac{d(u, v) + d(v, u)}{2}\right) - \varphi\left(d(u, v), d(v, u)\right).$$

Then

$$\psi\left(d(u,v)\right) \le \psi\left(d(u,v)\right) - \varphi\left(d(u,v), d(v,u)\right).$$

Therefore $\varphi\left(d(u,v), d(v,u)\right) = 0$. This implies that $u = v$. □

In Theorem 3.1, if $T = f$, we have the following corollary.

Corollary 3.2. *Let (X,d) be a complete b-metric space with the parameter $s \ge 1$ and T is a self-mapping on X. Suppose that T is continuous and satisfies*

$$\psi\left(sd(Tx,Ty)\right) \le \psi\left(\frac{d(x,Ty) + \frac{d(y,Tx)}{s^3}}{s+1}\right) - \varphi\left(d(x,Ty), d(y,Tx)\right), \tag{3.19}$$

for all $x, y \in X$ and for some $\psi \in \Psi, \varphi \in \Phi_2$. Then T has a unique fixed point.

In Theorem 3.1, if $\psi(t) = t$ and

$$\varphi(u,v) = \left(\frac{1}{s+1} - \alpha\right)\left(u + \frac{v}{s^3}\right),$$

where $\alpha \in [0, \frac{1}{s+1})$, we have the following corollary.

Corollary 3.3. *Let (X,d) be a complete b-metric space with the parameter $s \ge 1$ and T, f be self-mappings on X satisfying*

$$sd(Tx,fy) \le \alpha\left(d(x,fy) + \frac{d(y,Tx)}{s^3}\right), \tag{3.20}$$

where $\alpha \in [0, \frac{1}{s+1})$ and $x, y \in X$. If T or f is continuous, then T and f have a unique common fixed point.

For $s = 1$ and $T = f$, Corollary 3.3 is a generalization of the Chatterjea theorem [5].

Theorem 3.4. *(Chatterjea theorem) Let (X,d) be a complete metric space and $T : X \to X$ satisfies*

$$d(Tx,Ty) \le \alpha\left[d(x,Ty) + d(y,Tx)\right],$$

where $0 < \alpha < \frac{1}{2}$ and $x, y \in X$. Then T has a unique fixed point.

Example 3.5. Consider the b-metric space given in Example 2.3. Set $Tx = 0$ and $fx = \frac{x^4}{8}$ for all $x \in X$. Define $\psi : [0,\infty) \to [0,\infty)$ and

$\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{3}{2}t$ and $\varphi(u,v) = \dfrac{u + \frac{v}{8}}{64}$. Then for $x, y \in X$, we have

$$\psi\left(2d(Tx,fy)\right) = \psi\left(2d\left(0, \frac{y^4}{8}\right)\right) = \frac{3}{2}\left(\frac{2y^8}{64}\right) = \frac{3y^8}{64},$$

and

$$\begin{aligned}
&\psi\left(\frac{d(x,fy) + \dfrac{d(y,Tx)}{s^3}}{s+1}\right) - \varphi\left(d(x,fy), d(y,Tx)\right) \\
&\qquad = \psi\left(\frac{1}{3}\left(d(x, \frac{y^4}{8}) + \frac{1}{8}d(y,0)\right)\right) \\
&\qquad\quad - \varphi\left(d(x, \frac{y^4}{8}), d(y,0)\right) \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\left((x - \frac{y^4}{8})^2 + \frac{y^2}{8}\right)\right) - \frac{\left((x - \frac{y^4}{8})^2 + \frac{y^2}{8}\right)}{64} \\
&\qquad = \frac{31}{64}\left((x - \frac{y^4}{8})^2 + \frac{y^2}{8}\right) \geq \frac{3}{64}y^8 \\
&\qquad = \psi\left(sd(Tx,fy)\right).
\end{aligned}$$

Hence, the conditions of Theorem 3.1 are satisfied.

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