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Existence of three solutions for a class of quasilinear elliptic systems involving the $p(x)$-Laplace operator

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Abstract. The aim of this paper is to obtain three weak solutions for the Dirichlet quasilinear elliptic systems on a bonded domain. Our technical approach is based on the general three critical points theorem obtained by Ricceri.

1. Introduction

In this paper, we consider the quasilinear elliptic systems

(1.1)
$$\begin{cases} -\Delta_{p_1(x)}u_1 + |u_1|^{p_1(x)-2}u_1 \\ \quad = \lambda F_{u_1}(x,u_1,\dots,u_n) \\ \qquad +\mu\left(|u_1|^{\alpha_1(x)-2}u_1 + |u_1|^{\beta_1(x)-2}u_1\right), & \text{in } \Omega, \\ \qquad -\Delta_{p_2(x)}u_2 + |u_1|^{p_1(x)-2}u_1 \\ \quad = \lambda F_{u_2}(x,u_1,\dots,u_n) \\ \qquad +\mu\left(|u_2|^{\alpha_2(x)-2}u_2 + |u_2|^{\beta_2(x)-2}u_2\right), & \text{in } \Omega, \\ \qquad \vdots \\ \qquad -\Delta_{p_n(x)}u_n + |u_n|^{p_n(x)-2}u_n \\ \quad = \lambda F_{u_n}(x,u_1,\dots,u_n) \\ \qquad +\mu\left(|u_n|^{\alpha_n(x)-2}u_n + |u_n|^{\beta_n(x)-2}u_n\right), & \text{in } \Omega, \\ u_i = 0 & \text{for } 1 \le i \le n, \text{ on } \partial\Omega, \end{cases}$$

where $-\Delta_{p_i(x)}u_i = -\text{div}\left(|\nabla u_i|^{p_i(x)-2}\nabla u_i\right)$ is the $p_i(x)$-Laplacian operator, $\lambda,\mu \in [0,\infty), \Omega \subset \mathbb{R}^N (N \ge 1)$ is a nonempty bounded open set with smooth boundary $\partial\Omega, F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

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$F(\cdot, t_1, \ldots, t_N)$ is measurable in Ω for all $(t_1, \ldots, t_N) \in \mathbb{R}^N$ and the mapping $(t_1, \ldots, t_N) \to F(x, t_1, \ldots, t_N)$ is in C^1 in $\mathbb{R}^n$ for a.e. $x \in \Omega, F_{u_i}$ denotes the partial derivative of F with respect to $u_i, p_i \in C(\overline{\Omega})$,

$$1 < p_i^- = \inf_{x\in\overline{\Omega}} p_i(x) \leq p_i^+ = \sup_{x\in\overline{\Omega}} p_i(x) < +\infty \quad \text{for} \quad 1 \leq i \leq n.$$

Moreover,

$$p_i^*(x) = \begin{cases} \frac{Np_i(x)}{N-p_i(x)} & p_i(x) < N, \\ \infty & p_i(x) \geq N, \end{cases}$$

for $1 \leq i \leq n$ are the critical exponents. Obviously, $p_i(x) \leq p_i^*(x)$ for all $x \in \overline{\Omega}$. In what follows, E will denote the Cartesian product of n Sobolev spaces $W_0^{1,p_1(x)}(\Omega)$, $W_0^{1,p_2(x)}(\Omega)$, $\ldots$ and $W_0^{1,p_n(x)}(\Omega)$ i.e., $E = W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega) \times \cdots \times W_0^{1,p_n(x)}(\Omega)$. X will denote the Sobolev space $W_0^{1,p_1(x)}(\Omega)$.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is an interesting and attractive topic and has been the object of considerable attention in recent years. The reason for such an interest relies on the fact that they model phenomena arising from various fields; we cite, for instance, the motion of electrorheological fluids, which are characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field, the thermo-convective flows of non-Newtonian fluids and the image processing (See [1–5, 11, 13]).

In [7], Mihailescue studied the problem

$$\begin{cases} -\text{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = \lambda f(x,u) & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases}$$

where $\Omega \subset R^N (N \geq 3)$ is a bounded domain with a smooth boundary, $\lambda > 0$ is a real number. He established the existence of at least three weak solutions by using the Ricceri variational principle for $p(x) > N$ [8]. In [12] Honghui Yin, considered the problem

$$\begin{cases} -\Delta_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x)f(x,u) + \mu g(x,u) & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases}$$

He studied the existence of three solutions by using the three critical points theorem due to Ricceri [9]. In [6] , Liu and Shi studied the solutions of the $(p(x), q(x))$-Laplacian equations with Dirichlet boundary conditions on a bounded domain and obtain three solutions under appropriate hypotheses.

In this paper, we have generalized the article of Liu and Shi [6] by using the three critical points theorem obtained by Ricceri [9].

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to [2–4]. Set

$$\begin{aligned} &L^{\infty}_{+}(\Omega) = \left\{ p \in L^{\infty}(\Omega) : \operatorname{ess}\inf_{x\in\Omega} p(x) \geq 1 \right\}, \\ &C_{+}(\overline{\Omega}) = \left\{ h | h \in C(\bar{\Omega}), h(x) > 1 \right\}, \\ &S(\Omega) = \{ u | u \text{ is a measurable real-valued function on } \Omega \}. \end{aligned}$$

For any $p \in C_{+}(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \in S(\Omega) | \int_{\Omega} |u|^{p(x)} dx < \infty \right\},$$

which is equipped with the norm, so-called Luxemburg norm by the formula

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 | \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) | \, |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 (See[4]). (i) *The spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces;*

(ii) *If $q \in C_{+}(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;*

(iii) *There is a constant $C > 0$, such that*

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|_{p(x)}$ in the following discussions.

Proposition 2.2 (See [3]). *Set*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx.$$

For $u, u_k \in L^{p(x)}(\Omega)$, we have

(1) *For* $u \neq 0, |u|_{p(x)} = \lambda \quad \Leftrightarrow \quad \rho(\frac{u}{\lambda}) = 1$;
(2) $|u|_{p(x)} < 1(=1; > 1) \quad \Leftrightarrow \quad \rho(u) < 1(=1; > 1)$;
(3) $|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$;
$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)} \geq \rho(u) \geq |u|^{p^+}_{p(x)}$;
(4) $|u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u) \to 0$;
$|u|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u) \to \infty$.

In this paper, the space E will be endowed with the following equivalent norm:

$$\|u\| = \sum_{i=1}^{n} \|u_i\|_{p_i(x)}, \quad \forall u = (u_1, \ldots, u_n) \in E,$$

here

$$\|u_i\|_{p(x)} = \inf\left\{\lambda > 0 | \int_\Omega \left(\left|\frac{\nabla u_i(x)}{\lambda}\right|^{p_i(x)} + \left|\frac{u_i(x)}{\lambda}\right|^{p_i(x)}\right) dx \leq 1\right\}.$$

Similar to Proposition 2.2, we have the following Proposition.

Proposition 2.3. *If we define*

$$I(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + |u|^{p(x)}\right) dx,$$

then for $u, u_k \in W^{1,p(x)}(\Omega)$

(1) *For* $u \neq 0, \|u\| = \lambda \quad \Leftrightarrow \quad I(\frac{u}{\lambda}) = 1$;
(2) $\|u\|_{p(x)} < 1(=1; > 1) \quad \Leftrightarrow \quad I(u) < 1(=1; > 1)$;
(3) *If* $\|u\|_{p(x)} > 1$, *then* $\|u\|^{p^-}_{p(x)} \leq I(u) \leq \|u\|^{p^+}_{p(x)}$;
(4) $\|u\|_{p(x)} < 1$, *then* $\|u\|^{p^+}_{p(x)} \leq I(u) \leq \|u\|^{p^-}_{p(x)}$;
(5) $\|u_k\|_{p(x)} \to 0(\to \infty) \quad \Leftrightarrow \quad I(u_k) \to 0(\to \infty)$.

Let

$$G(u) = \int_\Omega \frac{1}{p_1(x)} \left(|\nabla u|^{p_1(x)} + |u|^{p_1(x)}\right) dx, \quad u \in X.$$

We denote $L = G' : X \to X^*$, then

$$(L(u), v) = \int_\Omega \left(|\nabla u|^{p_1(x)-2} \nabla u \nabla v + |u|^{p_1(x)-2} uv\right) dx, \quad \forall u, v \in X.$$

Proposition 2.4 (See [4])**.** (i) $L : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*
(ii) L *is a mapping of type* (S_+), *i.e. if* $u_n \rightharpoonup u$ *in* X *and*

$$\overline{\lim}_{n\to\infty} (L(u_n) - L(u), u_n - u) \leq 0,$$

then $u_n \to u$ *in* X;
(iii) $L : X \to X^*$ *is a homeomorphism.*

Proposition 2.5 (See [8]). *Let X be a reflexive real Banach space, $I \subseteq R$ an interval, $\Phi : X \to R$ a sequentially weakly lower semi-continuous C^1 functional whose derivative admits a continuous inverse on X^* and $J : X \to R$ a C^1 functional with compact derivative. In addition, Φ is bounded on each bounded subset of X. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty, \tag{2.1}$$

for all $\lambda \in I$, and there exists $\rho \in R$ such that

$$\sup_{\lambda \in I} \inf_{\chi \in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{\chi \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda(J(x) + \rho)). \tag{2.2}$$

Then, there exist a nonempty open set $A \subseteq I$ and a positive real number r with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to R$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) + \lambda J'(x) + \mu \Psi'(x) = 0,$$

has at least three solutions in X whose norms are less than r.

Proposition 2.6 (See [10]). *Let X be a nonempty set and Φ, J two real functionals on X. Assume that there are $\gamma > 0, u_0, u_1 \in X$, such that*

$$\Phi(u_0) = J(u_0) = 0, \qquad \Phi(u_1) > \gamma, \tag{2.3}$$

and

$$\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}.$$

Then, for each ρ satisfying

$$\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \rho < \gamma \frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u)) + \lambda(\rho - J(u)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - J(u))).$$

3. Strong Convergence

Definition 3.1. We say that $u = (u_1, u_2, \dots, u_N) \in E$ is a weak solution of problem (1.1) if

$$\begin{aligned}\int_\Omega \sum_{i=1}^n \left(|\nabla u_i|^{p_i(x)-2} \nabla u_i . \nabla \xi_i + |u_i|^{p_i(x)-2} u_i \xi_i \right) dx \\ - \lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1, \dots, u_N) \xi_i dx \\ - \mu \int_\Omega \sum_{i=1}^n \left(|u_i|^{\alpha_i(x)-2} u_i \xi_i + |u_i|^{\beta_i(x)-2} u_i \xi_i \right) dx = 0\end{aligned}$$

for any $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in E$.

Let

$$\begin{aligned}\Phi(u) &= \int_\Omega \sum_{i=1}^n \frac{1}{p_i(x)} \left(|\nabla u_i|^{p_i(x)} + |u_i|^{p_i(x)} \right) dx, \\ J(u) &= -\int_\Omega F(x, u_1, \dots, u_n) dx, \\ \Psi(u) &= -\int_\Omega \sum_{i=1}^n \left(\frac{1}{\alpha_i(x)} |u_i|^{\alpha_i(x)} + \frac{1}{\beta_i(x)} |u_i|^{\beta_i(x)} \right) dx,\end{aligned}$$

for each $u = (u_1, u_2, \dots, u_n)$. The corresponding energy functional of problem (1.1) can be given by

$$H(u) = \Phi(u) + \lambda J(u) + \mu \Psi(u).$$

Then $H(u)$ is C^1 functional and their critical points are weak solutions of problem (1.1).

Theorem 3.2. *Assume that there exist two positive constants C, d and functions $\gamma_i(x) \in C(\overline{\Omega}), 1 \le i \le n$ with $1 < \gamma_i^- < \gamma_i^+ < p_i^-$, such that*

- (H$_0$) $F(x, t_1, \dots, t_n) \ge 0$ *for a.e.* $x \in \Omega$ *and all* $(t_1, \dots, t_n) \in [0, d] \times [0, d] \times \cdots \times [0, d]$;
- (H$_1$) $\exists q_i(x) \in C(\overline{\Omega}), 1 \le i \le n$ *and* $p_i^+ < q_i^- \le q_i(x) < p_i^*(x)$ *such that:*

$$\limsup_{(t_1, \dots, t_n) \to (0,0,\dots,0)} \sup_{x \in \Omega} \frac{F(x, t_1, \dots, t_n)}{|t_1|^{q_1(x)} + |t_2|^{q_2(x)} + \cdots + |t_n|^{q_n(x)}} < +\infty;$$

- (H$_2$) $|F(x, t_1, \dots, t_n)| \le C \left(1 + |t_1|^{\gamma_1(x)} + \cdots + |t_n|^{\gamma_n(x)} \right)$ *for a.e.* $x \in \Omega$ *and all* $(t_1, \dots, t_n) \in \mathbb{R}^n$;
- (H$_3$) $F(x, 0, \dots, 0) = 0$ *for a.e.* $x \in \Omega$;
- (H$_4$) $\alpha_i, \beta_i \in C(\overline{\Omega})$ *and* $\alpha_i^+ < p_i^-, \beta_i^+ < p_i^-$ *for* $1 \le i \le n$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number r such that, for every $\lambda \in \Lambda$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem (1.1) has at least three weak solutions whose norms in E are less than r.

Proof. It is well known that Φ, J and Ψ are well defined and continuously Gateaux differentiable with

$$\Phi'(u)(\xi) = \int_\Omega \sum_{i=1}^n \left(|\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla \xi_i + |u_i|^{p_i(x)-2} u_i \xi_i \right) dx,$$

$$J'(u)(\xi) = -\int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1, \ldots, u_n) \xi_i dx,$$

$$\Psi'(u)(\xi) = -\int_\Omega \sum_{i=1}^n \left(|u_i|^{\alpha_i(x)-2} u_i \xi_i + |u_i|^{\beta_i(x)-2} u_i \xi_i \right) dx,$$

for each $u = (u_1, \ldots, u_n), \xi = (\xi_1, \ldots, \xi_n) \in E$. Hence the weak solutions of problem (1.1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0.$$

Proposition 2.4 ensures that Φ is sequentially weakly lower semicontinuous functional whose the Gateaux derivative admits a continuous inverse on $E^*$and J' is compact. Obviously, Φ is bounded on each bounded subset of E.

From Proposition 2.3, let

$$G(u_1) = \int_\Omega \frac{1}{p_1(x)} \left(|\nabla u_1|^{p_1(x)} + |u_1|^{p_1(x)} \right) dx,$$

just as before, we have: if $\|u_1\|_{p_1(x)} \geq 1$, then

$$\frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^-} \leq G(u_1) \leq \frac{1}{p_1^-} \|u_1\|_{p_1(x)}^{p_1^+}; \tag{3.1}$$

if $\|u_1\|_{p_1(x)} < 1$, then

$$\frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^+} \leq G(u_1) \leq \frac{1}{p_1^-} \|u_1\|_{p_1(x)}^{p_1^-}. \tag{3.2}$$

In fact, when $\|u_1\|_{p_1(x)} < 1$ we can set

$$C_0 \geq \frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^-} - \frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^+} \geq 0,$$

then we get

$$G(u_1) = \int_\Omega \frac{1}{p_1(x)} \left(|\nabla u_1|^{p_1(x)} + |u_1|^{p_1(x)} \right) dx \geq \frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^-} - C_0.$$

It follows that

$$G(u_1) = \int_\Omega \frac{1}{p_1(x)} \left(|\nabla u_1|^{p_1(x)} + |u_1|^{p_1(x)} \right) dx \geq \frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^-} - C_0,$$

for all $u_1 \in X$.

So there exists a constant $C_1 > 0$, such that

$$\begin{aligned} \Phi(u) &= \int_\Omega \sum_{i=1}^n \frac{1}{p_i(x)} \left(|\nabla u_i|^{p_i(x)} + |u_i|^{p_i(x)} \right) dx \\ &\geq \sum_{i=1}^n \frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^-} - C_1, \end{aligned}$$

holds for each $u = (u_1, \dots, u_n) \in E$. Also,

$$\begin{aligned} \lambda J(u) &= -\lambda \int_\Omega F(x, u_1, \dots, u_n) dx \\ &\geq -\lambda \int_\Omega C \left(1 + |u_1|^{\gamma_1(x)} + \cdots + |u_n|^{\gamma_n(x)} \right) dx \\ &\geq -\lambda C \left(|\Omega| + |u_1|_{\gamma_1(x)}^{\gamma_1^+} + |u_1|_{\gamma_1(x)}^{\gamma_1^-} + \cdots + |u_n|_{\gamma_n(x)}^{\gamma_n^+} + |u_n|_{\gamma_n(x)}^{\gamma_n^-} \right) \\ &\geq -C_2 \left(1 + |u_1|_{\gamma_1(x)}^{\gamma_1^+} + \cdots + |u_n|_{\gamma_n(x)}^{\gamma_n^+} \right) \\ &\geq -C_3 \left(1 + \|u_1\|_{p_1(x)}^{\gamma_1^+} + \cdots + \|u_n\|_{p_n(x)}^{\gamma_n^+} \right) \end{aligned}$$

holds for any $u = (u_1, \dots, u_n) \in E$ where constants $C_2 \geq 0, C_3 \geq 0$. Here we used condition (H_3) and (ii) of Proposition 2.1. Combining the two inequalities above, we get

$$\Phi(u) + \lambda J(u) \geq \sum_{i=1}^n \frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^-} - C_3 \left(1 + \|u_1\|_{p_1(x)}^{\gamma_1^+} + \cdots + \|u_n\|_{p_n(x)}^{\gamma_n^+} \right) - C_1.$$

Because of $\gamma_i^+ < p_i^-$, it follows that

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty, \quad \forall u = (u_1, \dots, u_n) \in E, \quad \lambda \in [0, +\infty).$$

Then assumption (2.1) of Proposition 2.5 is satisfied. Next, we will prove that assumption (2.2) is also satisfied. It suffices to verify the conditions of Proposition 2.6. Let $u_0 = (u_1, \dots, u_n) = (0, \dots, 0)$, we easily have

$$\Phi(u_0) = J(u_0) = 0.$$

Now we claim that there exist $\gamma > 0$ and $v = (v_1, \dots, v_n) \in E$ such that $\Phi(v) > \gamma$ and (2.3) is satisfied.

There is a point $x^0 \in \Omega$, since Ω is a nonempty bounded open set. Let $r_2 > r_1 > 0$ and put

$$w(x) = \begin{cases} 0, & x \in \bar{\Omega} \backslash B(x^0, r_2), \\ \frac{d}{r_2 - r_1} \left(r_2 - \sqrt{\sum\limits_{i=1}^{n} (x_i - x_i^0)^2} \right), & x \in B(x^0, r_2) \backslash B(x^0, r_1), \\ d, & x \in B(x^0, r_1). \end{cases}$$

Here, $B(x, r)$ stands for the open ball in $\mathbb{R}^N$ of radius r centered at x.

Let $v(x) = (v_1(x), \ldots, v_n(x)) = (w(x), \ldots, w(x))$, then, thanks to (H_0) we obtain that

$$-J(v_1, \ldots, v_n) = -J(w, \ldots, w) = \int_\Omega F(x, w, \ldots, w) > 0.$$

From $(H_1), \exists \eta \in [0, 1]$ and $C_1 > 0$, such that

$$\begin{aligned} F(x, t_1, \ldots, t_n) &< C_1 \left(|t_1|^{p_1(x)} + \cdots + |t_n|^{p_n(x)} \right) dx \\ &< C_1 \left(|t_1|^{p_1^-} + \cdots + |t_n|^{p_n^-} \right), \end{aligned}$$

for all $(t_1, \ldots, t_n) \in [-\eta, \eta] \times \cdots \times [-\eta, \eta]$ a.e. $x \in \Omega$. From (H_2) there are 3^n positive real numbers $M_i (i = 1, \ldots, 3^n)$ according to $|t_i|$ larger or smaller than η and 1. For example, when $|t_1| > 1, |t_2| > 1, \ldots, \eta < |t_n| < 1$ some

$$M_i = \sup_{|t_1|>1, |t_2|>1, \ldots, |t_n|<\eta} \frac{C\left(1 + |t_1|^{\gamma_1^+} + |t_2|^{\gamma_2^+} + \cdots + |t_n|^{\gamma_n^-}\right)}{|t_1|^{q_1^-} + |t_2|^{q_2^-} + \cdots + |t_n|^{q_n^-}}.$$

Let $M = \max\{C_1, M_1, \ldots, M_{3^n}\}$, then

$$F(x, t_1, \ldots, t_n) < M(|t_1|^{q_1^-} + \cdots + |t_n|^{q_n^-}), \quad \forall (t_1, \ldots, t_n) \in R^N \text{ a.e. } x \in \Omega.$$

Consequently, fix γ such that $0 < \gamma < 1$. And when

$$\frac{1}{p_1^+} \|u_1\|_{p_1(x)}^{p_1^+} + \cdots + \frac{1}{p_n^+} \|u_n\|_{p_n(x)}^{p_n^+} \le \gamma < 1,$$

by the Sobolev embedding theorem ($X \hookrightarrow L^{q_1^-}$ is continuous), we have (for suitable positive constants C_2, C_3)

$$\begin{aligned} -J(u_1,\dots,u_n) &= \int_\Omega F(x,u_1,\dots,u_n)dx \\ &< M\int_\Omega \left(|u_1|^{q_1^-}+\cdots+|u_n|^{q_n^-}\right)dx \\ &\le C_2\left(\|u_1\|_{p_1(x)}^{q_1^-}+\cdots+\|u_n\|_{p_n(x)}^{q_n^-}\right) \\ &\le C_3\left(\gamma^{\frac{q_1^-}{p_1^+}}+\cdots+\gamma^{\frac{q_n^-}{p_n^+}}\right). \end{aligned}$$

Since $q_1^- > p_1^+, \dots, q_n^- > p_n^+$, we have

$$\lim_{\gamma\to 0^+} \frac{\sup_{\frac{1}{p_1^+}\|u_1\|_{p_1(x)}^{p_1^+}+\cdots+\frac{1}{p_n^+}\|u_n\|_{p_n(x)}^{p_n^+}\le\gamma} -J(u_1,\dots,u_n)}{\gamma} = 0. \tag{3.3}$$

We choose $w(x) \in E$ as above such that $-J(w,\dots,w) > 0$. Fix γ_0 such that

$$\begin{aligned} &0<\gamma<\gamma_0 \\ &\quad< \min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\} \\ &\quad\cdot \min\left\{\|w\|_{p_1(x)}^{p_1^+}+\cdots+\|w\|_{p_n(x)}^{p_n^+}, \|w\|_{p_1(x)}^{p_1^-}+\cdots+\|w\|_{p_n(x)}^{p_n^-},1\right\}\le 1. \end{aligned}$$

Then, we divide the proof into two cases.

(i) When $\|w\|<1$, from (3.2) we have

$$\begin{aligned} \Phi(v_1,\dots,v_n) &= \Phi(w,\dots,w) \\ &= \int_\Omega \sum_{i=1}^n \frac{1}{p_i(x)}\left(|\nabla w|^{p_i(x)}+|w|^{p_i(x)}\right)dx \\ &\ge \min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}\cdot\int_\Omega\sum_{i=1}^n\left(|\nabla w|^{p_i(x)}+|w|^{p_i(x)}\right)dx \\ &\ge \min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}\cdot\left(\|w\|_{p_1(x)}^{p_1^+}+\cdots+\|w\|_{p_n(x)}^{p_n^+}\right) \\ &\ge \gamma_0>\gamma. \end{aligned}$$

From (3.3), we know that

$$\begin{aligned}
&\sup_{\frac{1}{p_1^+}\|u_1\|_{p_1(x)}^{p_1^+}+\cdots+\frac{1}{p_n^+}\|u_n\|_{p_n(x)}^{p_n^+}\le\gamma} -J(u_1,\dots,u_n)\\
&\le \frac{\gamma}{2}\cdot\frac{-J(v_1,\dots,v_n)}{\max\left\{\frac{1}{p_1^-},\dots,\frac{1}{p_n^-}\right\}\cdot\left(\|w\|_{p_1(x)}^{p_1^-}+\cdots+\|w\|_{p_n(x)}^{p_n^-}\right)}\\
&\le \frac{\gamma}{2}\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}\\
&< \gamma\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}.
\end{aligned}$$

(ii) When $\|w\|\ge 1$, then from (3.1) we have

$$\begin{aligned}
\Phi(v_1,\dots,v_n) &= \Phi(w,\dots,w)\\
&= \int_\Omega \sum_{i=1}^n \frac{1}{p_i(x)}\left(|\nabla w|^{p_i(x)}+|w|^{p_i(x)}\right)dx\\
&\ge \min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}\cdot\int_\Omega\sum_{i=1}^n\left(|\nabla w|^{p_i(x)}+|w|^{p_i(x)}\right)dx\\
&\ge \min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}\cdot\left(\|w\|_{p_1(x)}^{p_1^-}+\cdots+\|w\|_{p_n(x)}^{p_n^-}\right)\\
&\ge \gamma_0>\gamma.
\end{aligned}$$

From (3.3), we know that

$$\begin{aligned}
&\sup_{\frac{1}{p_1^+}\|u_1\|_{p_1(x)}^{p_1^+}+\cdots+\frac{1}{p_n^+}\|u_n\|_{p_n(x)}^{p_n^+}\le\gamma} -J(u_1,\dots,u_n)\\
&\le \frac{\gamma}{2}\cdot\frac{-J(v_1,\dots,v_n)}{\max\left\{\frac{1}{p_1^-},\dots,\frac{1}{p_n^-}\right\}\cdot\left(\|w\|_{p_1(x)}^{p_1^+}+\cdots+\|w\|_{p_n(x)}^{p_n^+}\right)}\\
&\le \frac{\gamma}{2}\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}\\
&< \gamma\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}.
\end{aligned}$$

For any $(u_1,\dots,u_n)\in\Phi^{-1}((-\infty,\gamma])$, we get $\Phi(u_1,\dots,u_n)\le\gamma$, i.e.

$$\int_\Omega\sum_{i=1}^n\frac{1}{p_i(x)}\left(|\nabla u_i|^{p_i(x)}+|u_i|^{p_i(x)}\right)dx\le\gamma.$$

Then, we get

$$\min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}\cdot\int_\Omega\sum_{i=1}^n\frac{1}{p_i(x)}\left(|\nabla u_i|^{p_i(x)}+|u_i|^{p_i(x)}\right)dx\le\gamma.$$

So,

$$\begin{aligned}\int_\Omega\sum_{i=1}^n\frac{1}{p_i(x)}\left(|\nabla u_i|^{p_i(x)}+|u_i|^{p_i(x)}\right)dx&\le\gamma\cdot\frac{1}{\min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}}\\&<\gamma_0\cdot\frac{1}{\min\left\{\frac{1}{p_1^+},\dots,\frac{1}{p_n^+}\right\}}\\&<1.\end{aligned}$$

This inequality implies

$$\int_\Omega\left(|\nabla u_i|^{p_i(x)}+|u_i|^{p_i(x)}\right)dx<1,\quad 1\le i\le n,$$

i.e.

$$\|u_i\|_{p_i(x)}<1,\quad 1\le i\le n.$$

It follows that

$$\sum_{i=1}^n\frac{1}{p_i^+}\|u_i\|_{p_i(x)}^{p_i^+}<\int_\Omega\sum_{i=1}^n\frac{1}{p_i(x)}\left(|\nabla u_i|^{p_i(x)}+|u_i|^{p_i(x)}\right)dx\le\gamma.$$

So, we get that

$$\left\{\Phi^{-1}((-\infty,\gamma])\subset\{(u_1,\dots,u_n):(u_1,\dots,u_n)\in E,\sum_{i=1}^n\frac{1}{p_i^+}\|u_i\|_{p_i(x)}^{p_i^+}<\gamma\right\}.$$

Then

$$\begin{aligned}&\sup_{(u_1,\dots,u_n)\in\Phi^{-1}((-\infty,\gamma])}-J(u_1,\dots,u_n)\\&\quad\le\sup_{\sum_{i=1}^n\frac{1}{p_i^+}\|u_i\|_{p_i(x)}^{p_i^+}<\gamma}-J(u_1,\dots,u_n)\le\gamma\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)},\end{aligned}$$

that is

$$\sup_{(u_1,\dots,u_n)\in\Phi^{-1}((-\infty,\gamma])}-J(u_1,\dots,u_n)\le\gamma\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}.$$

So, we find $\gamma>0, v_1=\cdots=v_n=w$ and $\Phi(w,\dots,w)\le\gamma$ satisfying (2.3). Also, we can find ρ satisfying

$$\sup_{(u_1,\dots,u_n)\in\Phi^{-1}((-\infty,\gamma])}-J(u_1,\dots,u_n)\le\rho<\gamma\cdot\frac{-J(v_1,\dots,v_n)}{\Phi(v_1,\dots,v_n)}.$$

Let $I=[0,+\infty), \Phi(u_1,\ldots,u_n)$ and $J(u_1,\ldots,u_n)$ satisfy the assumptions of Proposition 2.6, so using Proposition 2.6 we can easily obtain that (2.2) is satisfied. Moreover, the functional

$$\Psi(u)=-\int_\Omega\sum_{i=1}^{n}\left(\frac{1}{\alpha_i(x)}|u_i|^{\alpha_i(x)}+\frac{1}{\beta_i(x)}|u_i|^{\beta_i(x)}\right)dx,$$

is continuously Gateaux differentiable on E, with compact derivative.

Thus, Φ, J and Ψ satisfy all the assumptions of Proposition 2.5, and the proof is complete.

□

References

1. E. Acerbi and G. Mingione, *Regularity results for stationary electrorheological fluids*, Arch. Ration. Mech. Anal., 156 (2001), pp. 121-140.
2. X.L. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., 262 (2001), pp. 749-760.
3. X.L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$*, J. Math. Anal. Appl., 263 (2001), pp. 424-446.
4. X.L. Fan and Q.H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal., 52 (2003), pp. 1843-1852.
5. A.El. Hamidi, *Existence result to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl., 300 (2004), pp. 30-42.
6. J.J. Liu and X.Y. Shi, *Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x),q(x))$-Laplacian*, Nonlinear Anal., 71 (2009), pp. 550-557.
7. M. Mihăilescue, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal., 67 (2007), pp. 1419-1425.
8. B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal., 70 (2009), pp. 3084-3089.
9. B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math., 113 (2000), pp. 401-410.
10. B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problem*, Math. Comput. Modelling., 32 (2000), pp. 1485-1494.
11. M. Rûzicka, *Electrorheological Fluids : Modeling and Mathematical Theory*, Lecture Notes in Math, Springer-Verlag, Berlin, Vol. 1784, 2000.
12. H.H. Yin, *Existence of three solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Math. Meth. Appl. Sci., 35 (2012), pp. 307-313.

13. V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv., 29 (1987), pp. 33-36.

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