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# On the Integral Representations of Generalized Relative Type and Generalized Relative Weak Type of Entire Functions

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ABSTRACT. In this paper we wish to establish the integral representations of generalized relative type and generalized relative weak type as introduced by Datta et al [9]. We also investigate their equivalence relation under some certain conditions.

## 1. INTRODUCTION

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f$ be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire functions which are available in [15]. In the sequel the following two notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), \quad \text{for } k = 1, 2, 3, \ldots,$$
$$\log^{[0]} x = x,$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right), \quad \text{for } k = 1, 2, 3, \ldots,$$
$$\exp^{[0]} x = x.$$

Taking this into account, Sato [13] defined the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function $f$ respectively which are as follows:

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$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r},$$

and

$$\lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r},$$

where $l \geq 1$.

These definitions extended the definitions of order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ which are classical in complex analysis for the integers $l = 2$, since these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

An entire function for which the generalized order and generalized lower order are the same is said to be of regular generalized growth. Functions which are not of regular generalized growth are said to be of irregular generalized growth.

To compare the growth of entire functions having the same generalized order, one may introduce the concepts of generalized type and generalized lower type in the following manner:

**Definition 1.1.** The generalized type $\sigma_f^{[l]}$ and generalized lower type $\overline{\sigma}_f^{[l]}$ of an entire function $f$ are defined as

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}},$$

and

$$\overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f^{[l]} < \infty,$$

where $l \geq 1$. Moreover, when $l = 2$ then $\sigma_f^{[2]}$ and $\overline{\sigma}_f^{[2]}$ are correspondingly denoted as $\sigma_f$ and $\overline{\sigma}_f$ which are respectively known as type and lower type of entire function $f$.

Similarly, extending the notion of weak type as introduced by Datta and Jha [10], one can define the generalized weak type to determine the relative growth of two entire functions having same non zero finite generalized lower order in the following manner:

**Definition 1.2.** The generalized weak type $\tau_f^{[l]}$ for $l \geq 1$ of an entire function $f$ of finite positive generalized lower order $\lambda_f^{[l]}$ is defined by

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}}, \quad 0 < \lambda_f^{[l]} < \infty.$$

Also, one may define the growth indicator $\overline{\tau}_f^{[l]}$ of an entire function $f$ in the following way:

$$\overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}}, \quad 0 < \lambda_f^{[l]} < \infty.$$

For $l = 2$, the above definition reduces to the classical definition as established by Datta and Jha [10]. Also $\tau_f$ and $\overline{\tau}_f$ are stand for $\tau_f^{[2]}$ and $\overline{\tau}_f^{[2]}$.

For any two entire functions $f$ and $g$, Bernal [1, 2], initiated the definition of relative order of $f$ with respect to $g$, indicated by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}, \end{aligned}$$

which keeps away from comparing growth just with $\exp z$ to find out order of entire functions as we see in the earlier and of course this definition corresponds with the classical one [14] for $g = \exp z$.

Analogously, one may define the relative lower order of $f$ with respect to $g$ denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [12] recently introduced the notion of relative type of two entire functions in the following manner:

**Definition 1.3** ([12]). Let $f$ and $g$ be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of $f$ with respect to

$g$ is defined as:

$$\sigma_g(f) = \inf\Big\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\Big\}$$
$$= \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}}.$$

Likewise, one can define the relative lower type of an entire function $f$ with respect to an entire function $g$ denoted by $\overline{\sigma}_g(f)$ as follows:

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}}, \quad 0 < \rho_g(f) < \infty.$$

Analogously, to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, Datta and Biswas [4] introduced the definition of relative weak type of an entire function $f$ with respect to another entire function $g$ of finite positive relative lower order $\lambda_g(f)$ in the following way:

**Definition 1.4** ([4])**.** The relative weak type $\tau_g(f)$ of an entire function $f$ with respect to another entire function $g$ having finite positive relative lower order $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}}.$$

Also, one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function $f$ with respect to an entire function $g$ in the following way:

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}}, \quad 0 < \lambda_g(f) < \infty.$$

For entire functions, the notions of the growth indicators such as order and type (weak type) are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative order, relative type and relative weak type of entire functions as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore, the studies of the growth of entire functions in the light of their relative order, relative type and relative weak type are the prime concern of this paper . In fact some light has already been thrown on such type of works by Datta et al. in [3–9]. Actually in this paper we study some relative growth properties of entire functions with respect to another entire function on the basis of generalized relative type and generalized relative weak type.

Lahiri and Banerjee [11] gave a more generalized concept of relative order in the following way:

**Definition 1.5** ([11])**.** If $l \geq 1$ is a positive integer, then the $l$- th generalized relative order of $f$ with respect to $g$, denoted by $\rho_g^{[l]}(f)$ is defined by

$$\rho_g^{[l]}(f) = \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } r > r_0(\mu) > 0\right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

Clearly $\rho_g^{[1]}(f) = \rho_g(f)$ and $\rho_{\exp z}^{[1]}(f) = \rho_f$.

Likewise, one can define the generalized relative lower order of $f$ with respect to $g$ denoted by $\lambda_g^{[l]}(f)$ as

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

Further to compare the relative growth of two entire functions having same non zero finite generalized relative orders with respect to another entire function, Datta et al [9] introduced the definition of generalized relative type of an entire function with respect to another entire function which is as follows:

**Definition 1.6** ([9])**.** The generalized relative type $\sigma_f^{[l]}$ and generalized relative lower type $\overline{\sigma}_f^{[l]}$ of an entire function $f$ are defined as

$$\sigma_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}}, \quad 0 < \rho_g^{[l]}(f) < \infty,$$

and

$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}}, \quad 0 < \rho_g^{[l]}(f) < \infty.$$

For $l = 2$, Definition 1.6 reduces to Definition 1.3.

The above definition can alternatively defined in the following manner:

**Definition 1.7.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized order $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ where $l$ is any positive integer. Then the relative generalized type $\sigma_g^{[l]}(f)$ of entire function $f$ with respect to the entire function $g$ is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \sigma_g^{[l]}(f)$ and diverges for $k < \sigma_g^{[l]}(f)$.

Similarly, to determine the relative growth of two entire functions having same non zero finite generalized relative lower orders with respect to another entire function, Datta et al [9] also introduced the concepts of generalized relative weak type of an entire function with respect to another entire function in the following manner:

**Definition 1.8** ([9])**.** The generalized relative weak type $\tau_g^{[l]}(f)$ of an entire function $f$ with respect to another entire function $g$ having finite positive generalized relative lower order $\lambda_g^{[l]}(f)$ is defined as:

$$\tau_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}}, \quad 0 < \lambda_g^{[l]}(f) < \infty.$$

Definition 1.8 also reduces to Definition 1.4 for particular $l = 2$.

The above definition can also alternatively defined as:

**Definition 1.9.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized lower order $\lambda_g^{[l]}(f)$ $\left(0 < \lambda_g^{[l]}(f) < \infty\right)$ where $l$ is any positive integer. Then the relative generalized weak type $\tau_g^{[l]}(f)$ of the entire function $f$ with respect to the entire function $g$ is define as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \tau_g^{[l]}(f)$ and diverges for $k < \tau_g^{[l]}(f)$.

Now, a question may arise about the equivalence of the definitions of relative generalized type and relative generalized weak type with their integral representations. In this paper, we would like to establish such equivalence of Definition 1.6 and Definition 1.7, and Definition 1.8 and Definition 1.9 and also investigate some growth properties related to relative generalized type and relative generalized weak type of entire functions with respect to another entire function.

## 2. Lemmas

In this section, we present a lemma which will be needed in sequel.

**Lemma 2.1.** *Let the integral*

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{A}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

*converges where* $0 < A < \infty$. *Then*

$$\lim_{r\to\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^k} = 0.$$

*Proof.* Since the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges, then

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^{k+1}} dr < \varepsilon, \quad \text{if } r_0 > R(\varepsilon).$$

Therefore,

$$\int_{r_0}^{\exp\left(r_0^A\right)+r_0} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^{k+1}} dr < \varepsilon.$$

Since $\log^{[l-2]} M_g^{-1} M_f(r)$ increases with $r$, so

$$\int_{r_0}^{\exp\left(r_0^A\right)+r_0} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^{k+1}} dr \geq \frac{\log^{[l-2]} M_g^{-1} M_f(r_0)}{\left[\exp\left(r_0^A\right)\right]^{k+1}} \cdot \left[\exp\left(r_0^A\right)\right],$$

i.e., for all large values of $r$,

$$\int_{r_0}^{\exp\left(r_0^A\right)+r_0} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^{k+1}} dr \geq \frac{\log^{[l-2]} M_g^{-1} M_f(r_0)}{\left[\exp\left(r_0^A\right)\right]^{k}},$$

so that

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r_0)}{\left[\exp\left(r_0^A\right)\right]^{k}} < \varepsilon \text{ if } r_0 > R(\varepsilon),$$

i.e.,

$$\lim_{r\to\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^A\right)\right]^k} = 0.$$

This proves the lemma. □

## 3. Main Results

In this section, we state the main results of this chapter.

**Theorem 3.1.** *Let* $f$ *and* $g$ *be any two entire functions having finite positive relative generalized order* $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ *and relative generalized type* $\sigma_g^{[l]}(f)$ *where* $l$ *is any positive integer. Then Definition 1.6 and Definition 1.7 are equivalent.*

*Proof.* Let us consider $f$ and $g$ be any two entire functions such that $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ exists for any positive integer $l$.

**Case I.** $\sigma_g^{[l]}(f) = \infty$.

**Definition 1.6 $\Rightarrow$ Definition 1.7.**

As $\sigma_g^{[l]}(f) = \infty$, from Definition 1.6 we have for an arbitrary positive $G$ and a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} M_g^{-1} M_f(r) > G \cdot r^{\rho_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^G . \tag{3.1}$$

If possible let the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{G+1}} dr, \quad (r_0 > 0),$$

be converge.

Then by Lemma 2.1,

$$\limsup_{r\to\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{G}} = 0.$$

So for all sufficiently large values of $r$,

$$\log^{[l-2]} M_g^{-1} M_f(r) < \left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^G . \tag{3.2}$$

Therefore, from (3.1) and (3.2) we arrive at a contradiction.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{G+1}} dr, \quad (r_0 > 0),$$

diverges whenever $G$ is finite, which is the Definition 1.7.

**Definition 1.7 $\Rightarrow$ Definition 1.6.**

Let $G$ be any positive number. Since $\sigma_g^{[l]}(f) = \infty$, from Definition 1.7, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{G+1}} dr, \quad (r_0 > 0),$$

gives an arbitrary positive $\varepsilon$ and for a sequence of values of $r$ tending to infinity

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{G-\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1} M_f(r) > (G-\varepsilon)\, r^{\rho_g^{[l]}(f)},$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \geq G-\varepsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} = \infty.$$

Thus Definition 1.6 follows.

**Case II.** $0 \leq \sigma_g^{[l]}(f) < \infty$.

**Definition 1.6 $\Rightarrow$ Definition 1.7.**

**Subcase (A).** $0 < \sigma_g^{[l]}(f) < \infty$.

Let $f$ and $g$ be any two entire functions such that $0 < \sigma_g^{[l]}(f) < \infty$ exists for any positive integer $l$. Then according to Definition 1.6, for any arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$, we obtain that

$$\log^{[l-1]} M_g^{-1} M_f(r) < \left(\sigma_g^{[l]}(f)+\varepsilon\right) r^{\rho_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) < \left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon},$$

i.e.,

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k}} < \frac{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon}}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k}},$$

i.e.,

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k}} < \frac{1}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k-\left(\sigma_g^{[[l]]}(f)+\varepsilon\right)}}.$$

Therefore

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \sigma_g^{[l]}(f)$.

Again by Definition 1.6, we obtain for a sequence of values of $r$ tending to infinity that

$$\log^{[l-1]} M_g^{-1} M_f(r) > \left(\sigma_g^{[l]}(f)-\varepsilon\right) r^{\rho_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)-\varepsilon}. \tag{3.3}$$

So for $k < \sigma_g^{[l]}(f)$, we get from (3.3) that

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k}} > \frac{1}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k-\left(\sigma_g^{[l]}(f)-\varepsilon\right)}}.$$

Therefore

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

diverges for $k < \sigma_g^{[l]}(f)$.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \sigma_g^{[l]}(f)$ and diverges for $k < \sigma_g^{[l]}(f)$.

**Subcase (B).** $\sigma_g^{[l]}(f) = 0$.

When $\sigma_g^{[l]}(f) = 0$ for any positive integer $l$ , Definition 1.6 gives for all sufficiently large values of $r$ that

$$\frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} < \varepsilon.$$

Then, as before we obtain that

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > 0$ and diverges for $k < 0$.

Thus combining subcase $(A)$ and subcase $(B)$, Definition 1.7 follows.

**Definition 1.7 $\Rightarrow$ Definition 1.6.**

From Definition 1.7 and for any arbitrary positive $\varepsilon$, the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon+1}} dr, \quad (r_0 > 0),$$

converges. Then by Lemma 2.1 we get that

$$\limsup_{r\to\infty}\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon}}=0.$$

So, we obtain for all sufficiently large values of $r$ that

$$\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon}}<\varepsilon,$$

i.e.,

$$\log^{[l-2]} M_g^{-1}M_f(r)<\varepsilon\cdot\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)+\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1}M_f(r)<\log\varepsilon+\left(\sigma_g^{[l]}(f)+\varepsilon\right)r^{\rho_g^{[l]}(f)},$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[l-1]} M_g^{-1}M_f(r)}{r^{\rho_g^{[l]}(f)}}\le\sigma_g^{[l]}(f)+\varepsilon.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[l-1]} M_g^{-1}M_f(r)}{r^{\rho_g^{[l]}(f)}}\le\sigma_g^{[l]}(f)\,. \tag{3.4}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)-\varepsilon+1}}dr,\quad(r_0>0)\,,$$

implies that there exists a sequence of values of $r$ tending to infinity such that

$$\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)-\varepsilon+1}}>\frac{1}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{1+\varepsilon}},$$

i.e.,

$$\log^{[l-2]} M_g^{-1}M_f(r)>\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{\sigma_g^{[l]}(f)-2\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1}M_f(r)>\left(\sigma_g^{[l]}(f)-2\varepsilon\right)\left(r^{\rho_g^{[l]}(f)}\right),$$

i.e.,

$$\frac{\log^{[l-1]} M_g^{-1}M_f(r)}{r^{\rho_g^{[l]}(f)}}>\left(\sigma_g^{[l]}(f)-2\varepsilon\right).$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[l-1]} M_g^{-1}M_f(r)}{r^{\rho_g^{[l]}(f)}} \geq \sigma_g^{[l]}(f)\,. \tag{3.5}$$

So, from (3.4) and (3.5) we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[l-1]} M_g^{-1}M_f(r)}{r^{\rho_g^{[l]}(f)}} = \sigma_g^{[l]}(f)\,.$$

This proves the theorem. □

**Theorem 3.2.** *Let $f$ and $g$ be any two entire functions having finite positive relative generalized lower order $\lambda_g^{[l]}(f)$ $\left(0<\lambda_g^{[l]}(f)<\infty\right)$ and relative generalized weak type $\tau_g^{[l]}(f)$ where $l$ is any positive integer. Then Definition 1.8 and Definition 1.9 are equivalent.*

*Proof.* Let us consider $f$ and $g$ be any two entire functions such that $\lambda_g^{[l]}(f)$ $\left(0<\lambda_g^{[l]}(f)<\infty\right)$ exists for any positive integer $l$.

**Case I.** $\tau_g^{[l]}(f)=\infty$.

**Definition 1.8 $\Rightarrow$ Definition 1.9.**

As $\tau_g^{[l]}(f)=\infty$, from Definition 1.8 we obtain for any arbitrary positive $G$ and for all sufficiently large values of $r$ that

$$\log^{[l-1]} M_g^{-1}M_f(r) > G\cdot r^{\lambda_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1}M_f(r) > \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^G. \tag{3.6}$$

Now, if possible, let the integral

$$\int_{r_0}^{\infty}\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{G+1}}dr, \quad (r_0>0)\,,$$

be converge.

Then by Lemma 2.1,

$$\liminf_{r\to\infty}\frac{\log^{[l-2]} M_g^{-1}M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{G}} = 0.$$

So, for a sequence of values of $r$ tending to infinity we get that

$$\log^{[l-2]} M_g^{-1}M_f(r) < \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^G. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we arrive at a contradiction.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{G+1}} dr, \quad (r_0 > 0),$$

diverges whenever $G$ is finite, which is the Definition 1.9.

**Definition 1.9 ⇒ Definition 1.8.**

Let $G$ be any positive number. Since $\tau_g^{[l]}(f) = \infty$, from Definition 1.9, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{G+1}} dr, \quad (r_0 > 0),$$

gives for any arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{G-\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1} M_f(r) > (G - \varepsilon) r^{\lambda_g^{[l]}(f)},$$

which implies that

$$\liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \geq G - \varepsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} = \infty.$$

Thus Definition 1.8 follows.

**Case II.** $0 \leq \tau_g^{[l]}(f) < \infty$.

**Definition 1.8 ⇒ Definition 1.9.**

**Subcase (C).** $0 < \tau_g^{[l]}(f) < \infty$.

Let $f$ and $g$ be any two entire functions such that $0 < \tau_g^{[l]}(f) < \infty$ exists for any positive integer $l$. Then, according to Definition 1.8, for a sequence of values of $r$ tending to infinity we get that

$$\log^{[l-1]} M_g^{-1} M_f(r) < \left(\tau_g^{[l]}(f) + \varepsilon\right) r^{\lambda_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) < \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)+\varepsilon},$$

i.e.,

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^k} < \frac{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)+\varepsilon}}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^k},$$

$$i.e., \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^k} < \frac{1}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k-\left(\tau_g^{[l]}(f)+\varepsilon\right)}}.$$

Therefore

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \tau_g^{[l]}(f)$.

Again, by Definition 1.8, we obtain for all sufficiently large values of $r$ that

$$\log^{[l-1]} M_g^{-1} M_f(r) > \left(\tau_g^{[l]}(f) - \varepsilon\right) r^{\lambda_g^{[l]}(f)},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)-\varepsilon}. \tag{3.8}$$

So, for $k < \tau_g^{[l]}(f)$, we get from (3.8) that

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^k} > \frac{1}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k-\left(\tau_g^{[l]}(f)-\varepsilon\right)}}.$$

Therefore

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

diverges for $k < \tau_g^{[l]}(f)$.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \tau_g^{[l]}(f)$ and diverges for $k < \tau_g^{[l]}(f)$.

**Subcase (D).** $\tau_g^{[l]}(f) = 0$.

When $\tau_g^{[l]}(f) = 0$ for any positive integer $l$, Definition 1.8 gives for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} < \varepsilon.$$

Then, as before we obtain that

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > 0$ and diverges for $k < 0$.

Thus, combining Subcase $(C)$ and Subcase $(D)$, Definition 1.9 follows.

**Definition 1.9 $\Rightarrow$ Definition 1.8.**

From Definition **1.9** and for any arbitrary positive $\varepsilon$, the integral $\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)+\varepsilon+1}} dr \; (r_0 > 0)$ converges. Then by Lemma 2.1 we get that

$$\liminf_{r\to\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)+\varepsilon}} = 0.$$

So, we get for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{l}(f)}\right)\right]^{\tau_g^{(p,q)}(f)+\varepsilon}} < \varepsilon,$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) < \varepsilon \cdot \left[\exp\left(r^{\lambda_g^{l}(f)}\right)\right]^{\tau_g^{[l]}(f)+\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1} M_f(r) < \log \varepsilon + \left(\tau_g^{[l]}(f) + \varepsilon\right) r^{\lambda_g^{[l]}(f)},$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \le \tau_g^{[l]}(f) + \varepsilon.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \le \tau_g^{[l]}(f). \tag{3.9}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)-\varepsilon+1}} dr, \quad (r_0 > 0),$$

implies for all sufficiently large values of $r$ that

$$\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)-\varepsilon+1}} > \frac{1}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{1+\varepsilon}},$$

i.e.,

$$\log^{[l-2]} M_g^{-1} M_f(r) > \left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{\tau_g^{[l]}(f)-2\varepsilon},$$

i.e.,

$$\log^{[l-1]} M_g^{-1} M_f(r) > \left(\tau_g^{[l]}(f) - 2\varepsilon\right)\left(r^{\lambda_g^{[l]}(f)}\right),$$

i.e.,

$$\frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} > \left(\tau_g^{[l]}(f) - 2\varepsilon\right).$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \geq \tau_g^{[l]}(f). \tag{3.10}$$

So from (3.9) and (3.10), we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} = \tau_g^{[l]}(f).$$

This proves the theorem. □

Datta et al [9] also introduced the following two relative growth indicators which will also enable to help our subsequent study.

**Definition 3.3** ([9])**.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized order $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ where $l$ is any positive integer. Then, the generalized relative lower type $\overline{\sigma}_f^{[l]}$ of the entire function $f$ with respect to the entire function $g$ is defined as:

$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}}.$$

**Definition 3.4.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized lower order $\lambda_g^{[l]}(f)$ $\left(0 < \lambda_g^{[l]}(f) < \infty\right)$. Then the growth indicator $\overline{\tau}_g^{[l]}(f)$ of the entire function $f$ with respect to the entire function $g$ is defined as:

$$\overline{\tau}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}}.$$

The above two definitions can alternatively defined in the following manner:

**Definition 3.5.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized order $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ where $l$ is any positive integer. Then the relative generalized lower type $\overline{\sigma}_g^{[l]}(f)$ of entire the function $f$ with respect to the entire function $g$ is define as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \overline{\sigma}_g^{[l]}(f)$ and diverges for $k < \overline{\sigma}_g^{[l]}(f)$.

**Definition 3.6.** Let $f$ and $g$ be any two entire functions having finite positive relative generalized lower order $\lambda_g^{[l]}(f)$ $\left(0 < \lambda_g^{[l]}(f) < \infty\right)$ where $l$ is any positive integer. Then the growth indicator $\overline{\tau}_g^{[l]}(f)$ of the entire function $f$ with respect to the entire function $g$ is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \overline{\tau}_g^{[l]}(f)$ and diverges for $k < \overline{\tau}_g^{[l]}(f)$.

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 2.1 and in the line of Theorem 3.1 and Theorem 3.2 respectively.

**Theorem 3.7.** *Let $f$ and $g$ be any two entire functions having finite positive relative generalized order $\rho_g^{[l]}(f)$ $\left(0 < \rho_g^{[l]}(f) < \infty\right)$ and relative generalized lower type $\overline{\sigma}_g^{[l]}(f)$ where $l$ is any positive integer. Then Definition 3.3 and Definition 3.5 are equivalent.*

**Theorem 3.8.** *Let $f$ and $g$ be any two entire functions having finite positive relative generalized lower order $\lambda_g^{[l]}(f)$ $\left(0 < \lambda_g^{[l]}(f) < \infty\right)$ and the* *growth indicator* $\overline{\tau}_g^{[l]}(f)$ *where* $l$ *is any positive integer. Then Definition 3.4 and Definition 3.6 are equivalent.*

In the following theorem, we obtain a relationship between $\sigma_g^{[l]}(f)$, $\overline{\sigma}_g^{[l]}(f)$, $\overline{\tau}_g^{[l]}(f)$ and $\tau_g^{[l]}(f)$.

**Theorem 3.9.** *Let* $f$ *and* $g$ *be any two entire functions such that* $f$ *is of regular relative generalized growth with respect to* $g$, *i.e.,* $\rho_g^{[l]}(f) = \lambda_g^{[l]}(f)$ $\left(0 < \lambda_g^{[l]}(f) = \rho_g^{[l]}(f) < \infty\right)$ *where* $l$ *is any positive integer, then the following quantities*

$$(i)\, \sigma_g^{[l]}(f), \qquad (ii)\, \tau_g^{[l]}(f), \qquad (iii)\, \overline{\sigma}_g^{[l]}(f), \qquad (iv)\, \overline{\tau}_g^{[l]}(f),$$

*are all equivalent.*

*Proof.* From Definition 1.9, it follows that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

is converges for $k > \tau_g^{[l]}(f)$ and diverges for $k < \tau_g^{[l]}(f)$. On the other hand, Definition 1.7 implies that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad (r_0 > 0),$$

converges for $k > \sigma_g^{[l]}(f)$ and diverges for $k < \sigma_g^{[l]}(f)$.

$(i) \Rightarrow (ii)$**.**

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} - \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}}\right],$$

are of non negative type. So

$$\int_{r_0}^{\infty} \left[\frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} - \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}}\right] dr \geq 0, \quad (r_0 > 0),$$

i.e.,

$$\int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\lambda_g^{[l]}(f)}\right)\right]^{k+1}} dr \geq \int_{r_0}^{\infty} \frac{\log^{[l-2]} M_g^{-1} M_f(r)}{\left[\exp\left(r^{\rho_g^{[l]}(f)}\right)\right]^{k+1}} dr, \quad r_0 > 0.$$

i.e.,

$$\tau_g^{[l]}(f) \geq \sigma_g^{[l]}(f). \tag{3.11}$$

Further, $f$ is of regular relative generalized growth with respect to $g$. Therefore we get that

$$\begin{aligned} \sigma_g^{[l]}(f) &= \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \\ &\geq \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \\ &= \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \\ &= \tau_g^{[l]}(f). \end{aligned} \tag{3.12}$$

Hence from (3.11) and (3.12) we obtain that

$$\sigma_g^{[l]}(f) = \tau_g^{[l]}(f). \tag{3.13}$$

$(ii) \Rightarrow (iii)$.

Since $f$ is of regular relative generalized growth with respect to $g$, i.e., $\rho_g^{[l]}(f) = \lambda_g^{[l]}(f)$, we get that

$$\begin{aligned} \tau_g^{[l]}(f) &= \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} \\ &= \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \\ &= \overline{\sigma}_g^{[l]}(f). \end{aligned}$$

$(iii) \Rightarrow (iv)$.

In view of (3.13) and the condition $\rho_g^{[l]}(f) = \lambda_g^{[l]}(f)$, it follows that

$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}},$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[;]}(f)}},$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \tau_g^{[l]}(f),$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \sigma_g^{[[l]]}(f),$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}},$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}},$$

i.e.,

$$\overline{\sigma}_g^{[l]}(f) = \overline{\tau}_g^{[l]}(f).$$

$(iv) \Rightarrow (i)$.

As $f$ is of regular relative generalized growth with respect to $g$, i.e., $\rho_g^{[l]}(f) = \lambda_g^{[l]}(f)$, we obtain that

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} &= \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \\ &= \sigma_g^{[l]}(f).\end{aligned}$$

Thus the theorem follows. □

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