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On Character Space of the Algebra of BSE-functions

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Abstract. Suppose that A is a semi-simple and commutative Banach algebra. In this paper we try to characterize the character space of the Banach algebra $C_{\mathrm{BSE}}(\Delta(A))$ consisting of all BSE-functions on $\Delta(A)$ where $\Delta(A)$ denotes the character space of A. Indeed, in the case that $A = C_0(X)$ where X is a non-empty locally compact Hausdroff space, we give a complete characterization of $\Delta(C_{\mathrm{BSE}}(\Delta(A)))$ and in the general case we give a partial answer. Also, using the Fourier algebra, we show that $C_{\mathrm{BSE}}(\Delta(A))$ is not a C^*-algebra in general. Finally for some subsets E of A^*, we define the subspace of BSE-like functions on $\Delta(A) \cup E$ and give a nice application of this space related to Goldstine's theorem.

1. Introduction and Preliminaries

Suppose that A is a semi-simple commutative Banach algebra and $\Delta(A)$ is the character space of A, i.e., the space of all non-zero homomorphisms from A into $\mathbb{C}$.

A bounded continuous function σ on $\Delta(A)$ is called a BSE-function if there exists a constant $C > 0$ such that for each $\phi_1, \ldots, \phi_n \in \Delta(A)$ and complex numbers $c_1, \ldots, c_n$, the inequality

$$\left|\sum_{i=1}^{n} c_i \sigma(\phi_i)\right| \leq C \left\|\sum_{i=1}^{n} c_i \phi_i\right\|_{A^*},$$

holds. For each $\sigma \in C_{\mathrm{BSE}}(\Delta(A))$ we denote by $\|\sigma\|_{\mathrm{BSE}}$ the infimum of such C. Let $C_{\mathrm{BSE}}(\Delta(A))$ be the set of all BSE-functions. We have a good characterization of $C_{\mathrm{BSE}}(\Delta(A))$ as follows:

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Theorem 1.1. *$C_{\mathrm{BSE}}(\Delta(A))$ is equal to the set of all $\sigma \in C_b(\Delta(A))$ for which there exists a bounded net $\{x_\lambda\}$ in A with $\lim_\lambda \phi(x_\lambda) = \sigma(\phi)$ for all $\phi \in \Delta(A)$*

Proof. see [11, Theorem 4 (i)]. □

Using the above characterization one can see that $C_{\mathrm{BSE}}(\Delta(A))$ is unital if and only if A has a bounded weak approximate identity in the sense of Lahr and Jones. We recall that a net $\{x_\alpha\}$ in A is called a bounded weak approximate identity (b.w.a.i) for A if $\{x_\alpha\}$ is bounded in A and

$$\lim_\alpha \phi(x_\alpha a) = \phi(a) \quad (\phi \in \Delta(A), a \in A),$$

or equivalently, $\lim_\alpha \phi(x_\alpha) = 1$ for each $\phi \in \Delta(A)$.

Also, Theorem 1.1, gives the following definition of $\|\cdot\|_{\mathrm{BSE}}$:

$$\|\sigma\|_{\mathrm{BSE}} = \inf\{\beta > 0 : \exists\{x_\lambda\} \text{ in A with } \|x_\lambda\| \le \beta, \\ \lim_\lambda \phi(x_\lambda) = \sigma(\phi)\ (\phi \in \Delta(A))\}.$$

The theory of BSE-algebras for the first time introduced and investigated by Takahasi and Hatori; see [11] and two other notable works [5, 3]. In [3], the authors answered a question raised in [11]. Examples of BSE-algebras are the group algebra $L^1(G)$ of a locally compact abelian group G, the Fourier algebra $A(G)$ of a locally compact amenable group G, all commutative C^*-algebras, the disk algebra, and the Hardy algebra on the open unit disk. We recall that a commutative and without order Banach algebra A is a type I-BSE algebra if

$$\widehat{M(A)} = C_{\mathrm{BSE}}(\Delta(A)) = C_b(\Delta(A)),$$

where $M(A)$ denotes the multiplier algebra of A and $\widehat{M(A)}$ denotes the space of all $\widehat{T}$ which defined by $\widehat{T}(\varphi)\varphi(x) = \widehat{T(x)}(\varphi)$ for all $\varphi \in \Delta(A)$. Note that $x \in A$ should satisfies $\varphi(x) \neq 0$.

In this paper, we give a partial characterization of the character space of $C_{\mathrm{BSE}}(\Delta(A))$ where A is a semi-simple commutative Banach algebra. Indeed, we show that if A has a b.w.a.i and $C_{\mathrm{BSE}}(\Delta(A))$ is an ideal in $C_b(\Delta(A))$, then

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) = \overline{\Delta(A)}^{w^*}.$$

Also, we give a negative answer to this question;
Whether $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is a C^*-algebra? At the final section of this paper we study the space of BSE-like functions on subsets of A^* which containing $\Delta(A)$ and as an application of this space we give a nice relation with Goldstine's theorem.

2. Character Space of $C_{\mathrm{BSE}}(\Delta(A))$

In view of [11, Lemma 1], $C_{\mathrm{BSE}}(\Delta(A))$ is a semi-simple commutative Banach algebra. So, the character space of $C_{\mathrm{BSE}}(\Delta(A))$ should be non-empty and one may ask: Is there a characterization of $\Delta(C_{\mathrm{BSE}}(\Delta(A)))$ for an arbitrary Banach algebra A?

In the sequel of this section, we give a partial answer to this question. Let X be a non-empty locally compact Hausdroff space and put

$$C_{\mathrm{BSE}}(X) := C_{\mathrm{BSE}}(\Delta(C_0(X))).$$

To proceed further we recall some notions. Let X be a non-empty locally compact Hausdroff space. A function algebra (FA) on X is a subalgebra A of $C_b(X)$ that seperates strongly the points of X, that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$ and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. A Banach function algebra (BFA) on X is a function algebra A on X with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra. A topological space X is completely regular if every non-empty closed set and every singleton disjoint from it can be separated by continuous functions.

Theorem 2.1. *Let X be a non-empty locally compact Hausdroff space. Then $C_{\mathrm{BSE}}(X)$ is a unital* BFA *and its character space is homeomorphic to $\overline{X}^{w^*}$, that is, $\Delta(C_{\mathrm{BSE}}(X)) = \overline{\{\phi_x : x \in X\}}^{w^*}$.*

Proof. By [11, Lemma 1], $C_{\mathrm{BSE}}(X)$ is a subalgebra of $C_b(X)$ and $\|\cdot\|_{\mathrm{BSE}}$ is a complete algebra norm. Since $C_0(X)$ has a bounded approximate identity, $C_{\mathrm{BSE}}(X)$ is unital. So, for each $x \in X$, there exists $f \in C_{\mathrm{BSE}}(X)$ with $f(x) \neq 0$. On the other hand, using the Urysohn lemma, for each $x, y \in X$ with $x \neq y$ one can see that there exists $f \in C_{\mathrm{BSE}}(X)$ such that $f(x) \neq f(y)$.

Finally, since X is a locally compact Hausdroff space, it is completely regular by [1, Corollary 2.74]. On the other hand, by [11, Theorem 3], we know that $C_0(X)$ is a type I-BSE algebra. Therefore, $C_{\mathrm{BSE}}(X) = C_b(\Delta(C_0(X))) = C_b(X)$. Also, for every $f \in C_{\mathrm{BSE}}(X)$, by the remark after Theorem 4 of [11], we have $\|f\|_X \leq \|f\|_{\mathrm{BSE}}$. Also, by the Open mapping theorem, there exists a positive constant M such that $\|f\|_{\mathrm{BSE}} \leq M\|f\|_X$. So, $C_{\mathrm{BSE}}(X)$ and $C_b(X)$ are topologically isomorphic, and so $\Delta(C_{\mathrm{BSE}}(X))$ and $\Delta(C_b(X))$ are homeomorphic. Now, by using [4, Theorem 2.4.12], we have

$$\Delta(C_{\mathrm{BSE}}(X)) = \Delta(C_b(X)) = \overline{X}^{w^*} = \overline{\{\phi_x : x \in X\}}^{w^*}.$$

□

Remark 2.2. In general for a commutative Banach algebra A, we have the following conditions concerning the character space of $C_{\mathrm{BSE}}(\Delta(A))$:

(i) If $C_{\mathrm{BSE}}(\Delta(A)) = C_b(\Delta(A))$, then

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) = \overline{\Delta(A)}^{w^*}.$$

Examples of Banach algebras A satisfying

$$C_{\mathrm{BSE}}(\Delta(A)) = C_b(\Delta(A)),$$

are finite dimensional Banach algebras and commutative C^*-algebras; see the remark on page 609 of [12]. Also, see [11, Lemma 2] for a characterization of Banach algebras A for which satisfying

$$C_{\mathrm{BSE}}(\Delta(A)) = C_b(\Delta(A)).$$

(ii) If A has a b.w.a.i, then $C_{\mathrm{BSE}}(\Delta(A))$ is unital and so

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))),$$

is compact and hence it is w^*-closed. On the other hand, we know that

$$\Delta(A) \subseteq \Delta(C_{\mathrm{BSE}}(\Delta(A))),$$

in the sense that for each $\varphi \in \Delta(A)$, $f_\varphi : C_{\mathrm{BSE}}(\Delta(A)) \longrightarrow \mathbb{C}$ defined by $f_\varphi(\sigma) = \sigma(\varphi)$ is an element of $\Delta(C_{\mathrm{BSE}}(\Delta(A)))$. Note that $f_\varphi \neq 0$, since in this case $C_{\mathrm{BSE}}(\Delta(A))$ is unital and $f_\varphi(1) = 1$. So

$$\overline{\Delta(A)}^{w^*} \subseteq \overline{\Delta(C_{\mathrm{BSE}}(\Delta(A)))}^{w^*} = \Delta(C_{\mathrm{BSE}}(\Delta(A))).$$

(iii) One can see that if $(B, \|\cdot\|_B)$ is a Banach algebra which contains the Banach algebra $(C, \|\cdot\|_C)$ as a two-sided ideal, then every $\varphi \in \Delta(C)$ extends to one $\widetilde{\varphi} \in \Delta(B)$. Now, let $C = C_{\mathrm{BSE}}(\Delta(A))$ and $B = C_b(\Delta(A))$. If C is an ideal in B, then

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) = \Delta(C) \subseteq \Delta(B) = \overline{\Delta(A)}^{w^*}.$$

(iv) Suppose that B is a commutative semi-simple Banach algebra such that $\Delta(B)$ is compact. Then B is unital; see [4, Theorem 3.5.5]. Now, If A has no b.w.a.i, then

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) \neq \overline{\Delta(A)}^{w^*}.$$

Because if

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) = \overline{\Delta(A)}^{w^*},$$

by using the above assertion, $C_{\mathrm{BSE}}(\Delta(A))$ is unital, since

$$\overline{\Delta(A)}^{w^*} = \Delta(C_b(\Delta(A)))$$

is compact and $C_{\mathrm{BSE}}(\Delta(A))$ is a semi-simple commutative Banach algebra. Therefore, A has a b.w.a.i which is impossible.

(v) If A has a b.w.a.i and $C_{\mathrm{BSE}}(\Delta(A))$ is an ideal of $C_b(\Delta(A))$, then using parts (ii) and (iii), we have

$$\Delta(C_{\mathrm{BSE}}(\Delta(A))) = \overline{\Delta(A)}^{w^*}.$$

3. $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ IS NOT A C^*-ALGEBRA

The theory of C^*-algebras is very fruitful and applied. As an advantage of this theory, especially in Harmonic Analysis, one can see the C^*-algebra approach for defining a locally compact quantum group; see [6]. So, verifying the Banach algebras from a C^*-algebraic point of view is very helpful. In this section, using a result due to Kaniuth and Ülger in [5], we show that $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is not a C^*-algebra in general. On the other hand, there is a question which left open that, under what conditions on A, $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is a C^*-algebra?

In the sequel for each locally compact group G, let $A(G)$ denotes the Fourier algebra and $B(G)$ denotes the Fourier-Stieltjes algebra introduced by Eymard; see [9, §19]. Also, let $\widehat{G}$ denotes the dual group of G and $M(G)$ denotes the Measure algebra; see [2, §3.3]. For the convenience of reader, we give the definitions of $A(G)$ and $B(G)$ as follows:

Let G be a locally compact group. Suppose that $A(G)$ denotes the subspace of $C_0(G)$ consisting of functions of the form

$$u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}$$

where $f_i, g_i \in L^2(G)$,

$$\sum_{i=1}^{\infty} ||f_i||_2 ||g_i||_2 < \infty$$

and $\widetilde{f}(x) = \overline{f(x^{-1})}$ for all $x \in G$. The space $A(G)$ with the pointwise operation and the following norm is a Banach algebra,

$$||u||_{A(G)} = \inf \left\{ \sum_{i=1}^{\infty} ||f_i||_2 ||g_i||_2 : u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i} \right\},$$

which we call it the Fourier algebra. It is obvious that for each $u \in A(G)$, $||u|| \leq ||u||_{A(G)}$ where $||u||$ is the norm of u in $C_0(G)$.

Now, let Σ denotes the equivalence class of all irreducible representations of G. Then $B(G)$ consisting of all functions ϕ of the form $\phi(x) =< \pi(x)\xi, \eta >$ where $\pi \in \Sigma$ and ξ, η are elements of H_π, the Hilbert space associated to the representation π. It is well-known that $A(G)$ is a closed ideal of $B(G)$.

Also, recall that an involutive Banach algebra A is called a C^*-algebra if its norm satisfies $\|aa^*\| = \|a\|^2$ for each $a \in A$. We refer the reader to [8] to see a complete description of C^*-algebras.

In the following remark we give the main result of this section.

Remark 3.1. In general $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is not a C^*-algebra, that is, there is not any involution "*" on $C_{\mathrm{BSE}}(\Delta(A))$ such that

$$\|\sigma^*\sigma\|_{\mathrm{BSE}} = \|\sigma\|_{\mathrm{BSE}}^2, \quad \forall \sigma \in C_{\mathrm{BSE}}(\Delta(A)).$$

Because we know that every commutative C^*-algebra is a BSE-algebra. For a non-compact locally compact Abelian group G take $A = A(G)$. By [5, Theorem 5.1], we know that $C_{\mathrm{BSE}}(\Delta(A)) = B(G)$ and for each $u \in B(G)$, $\|u\|_{B(G)} = \|u\|_{\mathrm{BSE}}$. But $B(G) = M(\widehat{G})$ and it is shown in [11] that $M(\widehat{G})$ and hence $B(G)$ is not a BSE-algebra. Therefore, $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is not a C^*-algebra.

As the second example, let G be a locally compact Abelian group. It is well-known that $C_{\mathrm{BSE}}(\Delta(L^1(G)))$ is isometrically isomorphic to $M(G)$, where $L^1(G)$ denotes the group algebra; see the last remark on page 151 of [11]. On the other hand, by the Gelfand-Nimark theorem we know that every commutative C^*-algebra should be symmetric. But in general $M(G)$ is not symmetric, i.e., the formula $\widehat{\mu^*}(\xi) = \overline{\widehat{\mu}}(\xi)$ does not hold for every $\xi \in \Delta(M(G))$. For example if G is non-discrete then by [10, Theorem 5.3.4], $M(G)$ is not symmetric and hence fails to be a C^*-algebra.

It is a good question to characterize Banach algebras A for which $(C_{\mathrm{BSE}}(\Delta(A)), \|\cdot\|_{\mathrm{BSE}})$ is a C^*-algebra.

4. BSE-like Functions on Subsets of A^*

Suppose that A is a Banach algebra and $E \subseteq A^* \setminus \Delta(A)$. A complex-valued bounded continuous function σ on $\Delta(A) \cup E$ is called a BSE-like function if there exists an $M > 0$ such that for each $f_1, f_2, f_3, \ldots f_n \in \Delta(A) \cup E$ and complex numbers $c_1, c_2, c_3, \ldots, c_n$,

$$\left|\sum_{i=1}^{n} c_i \sigma(f_i)\right| \leq M \left\|\sum_{i=1}^{n} c_i f_i\right\|_{A^*}. \tag{4.1}$$

We show the set of all the BSE-like functions on $\Delta(A) \cup E$ by $C_{\mathrm{BSE}}(\Delta(A), E)$ and let $\|\sigma\|_{\mathrm{BSE}}$ be the infimum of all M satisfying relation 4.1. Obviously, $C_{\mathrm{BSE}}(\Delta(A), E)$ is a linear subspace of $C_b(\Delta(A) \cup E)$ and we have

$$\left\{\sigma_{|\Delta(A)} : \sigma \in C_{\mathrm{BSE}}(\Delta(A), E)\right\} \subseteq C_{\mathrm{BSE}}(\Delta(A)).$$

Clearly, $\iota_A(A) \subseteq C_{\mathrm{BSE}}(\Delta(A), E)$ where $\iota_A : A \longrightarrow A^{**}$ is the natural embedding. For $a \in A$, we let $\widehat{a} = \iota_A(a)$ and $\widehat{A} = \iota_A(A)$.

To proceed further, we recall the Helly theorem.

Theorem 4.1. *(*Helly) Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{C}$ and let $M > 0$. Suppose that $x_1^*, \dots, x_n^*$ are in X^* and $c_1^*, \dots, c_n^*$ are in $\mathbb{C}$. Then the following are equivalent:

(i) for all $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $\|x_\epsilon\| \leq M + \epsilon$ and $x_k^*(x_\epsilon) = c_k$ for $k = 1, \dots, n$.

(ii) for all $a_1, \dots, a_n \in \mathbb{C}$,

$$\left|\sum_{i=1}^{n} a_i c_i\right| \leq M \left\|\sum_{i=1}^{n} a_i x_i^*\right\|_{X^*}.$$

Proof. See [7, Theorem 4.10.1]. □

As an application of Helly's theorem, we give the following characterization which is similar to [11, Theorem 4 (i)].

Theorem 4.2. $C_{\mathrm{BSE}}(\Delta(A), E)$ *is equal to the set of all* $\sigma \in C_b(\Delta(A) \cup E)$ *for which there exists a bounded net* $\{x_\alpha\}$ *in* A *with* $\lim_\alpha f(x_\alpha) = \sigma(f)$ *for all* $f \in \Delta(A) \cup E$.

Proof. Suppose that $\sigma \in C_b(\Delta(A) \cup E)$ is such that there exists $\beta < \infty$ and a net $\{x_\alpha\} \subseteq X$ with $\|x_\alpha\| < \beta$ for all α and $\lim_\alpha f(x_\alpha) = \sigma(f)$ for all $f \in \Delta(A) \cup E$. Let $f_1, \dots, f_n$ be in $\Delta(A) \cup E$ and $c_1, \dots, c_n$ be complex numbers. Then we have

$$\begin{aligned}\left|\sum_{i=1}^{n} c_i \sigma(f_i)\right| &\leq \left|\sum_{i=1}^{n} c_i f_i(x_\alpha)\right| + \left|\sum_{i=1}^{n} c_i (f_i(x_\alpha) - \sigma(f_i))\right| \\ &\leq \beta \left\|\sum_{i=1}^{n} c_i f_i\right\| + \sum_{i=1}^{n} |c_i||f_i(x_\alpha) - \sigma(f_i)|\end{aligned}$$

Taking the limit with respect to α, we conclude that $\sigma \in C_{\mathrm{BSE}}(\Delta(A), E)$.

Conversely, let $\sigma \in C_{\mathrm{BSE}}(\Delta(A), E)$. Suppose that Λ is the net consisting of all finite subsets of $\Delta(A) \cup E$. By Helly's theorem, for each $\epsilon > 0$ and $\lambda \in \Lambda$, there exists $x_{(\lambda,\epsilon)} \in A$ with $\|x_{(\lambda,\epsilon)}\| \leq \|\sigma\|_{\mathrm{BSE}} + \epsilon$ and $f(x_{(\lambda,\epsilon)}) = \sigma(f)$ for all $f \in \lambda$. Clearly, $\{(\lambda, \epsilon) : \lambda \in \Lambda, \epsilon > 0\}$ is a directed set with $(\lambda_1, \epsilon_1) \preceq (\lambda_2, \epsilon_2)$ iff $\lambda_1 \subseteq \lambda_2$ and $\epsilon_1 \leq \epsilon_2$. Therefore, we have

$$\lim_{(\lambda,\epsilon)} f(x_{(\lambda,\epsilon)}) = \sigma(f), \quad (f \in \Delta(A) \cup E).$$

□

Remark 4.3. As an application of Theorem 4.2, if $E = A^* \setminus \Delta(A)$, then one can see that $\overline{\widehat{A}}^{w^*} = A^{**}$, i.e., we conclude Goldstine's theorem. That is, $\widehat{A}$ with the w^*-topology of A^{**} is dense in A^{**}.

Remark 4.4. We say that A has a b.w.a.i respect to E if there exists a bounded net $\{x_\alpha\}$ in A with

$$\lim_\alpha f(x_\alpha) = 1, \quad (f \in \Delta(A) \cup E).$$

Using Theorem 4.2, one can check that $1 \in C_{\mathrm{BSE}}(\Delta(A), E)$ if and only if A has a b.w.a.i respect to E.

We conclude this section with the following question.

Question. Is $C_{\mathrm{BSE}}(\Delta(A), E)$ a commutative and semi-simple Banach algebra? If it is what is its character space?

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