Fuzzy $\epsilon$-regular spaces and strongly $\epsilon$-irresolute mappings

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**Abstract.** The aim of this paper is to introduce fuzzy $(\epsilon, \text{almost})$ $\epsilon^*$-regular spaces in Šostak’s fuzzy topological spaces. Using the $r$-fuzzy $\epsilon$-closed sets, we define $r$-$(r-\theta, r-\epsilon\theta)$ $\epsilon$-cluster points and their properties. Moreover, we investigate the relations among $r$-$(r-\theta, r-\epsilon\theta)$ $\epsilon$-cluster points, $r$-fuzzy $(\epsilon, \text{almost})$ $\epsilon^*$-regular spaces and their functions.

1. Introduction

Kubiak [10] and Šostak [15] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [1], in the sense that not only the objects are fuzzified, but also the axiomatics. In [13, 14], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [2] have redefined the same concept under the name gradation of openness. It has been developed in many directions [2, 11, 12]. Kim et al. [4, 6, 8, 9] investigate $r$-regulars closed sets, several operators and fuzzy (almost) regular spaces in Šostak’s fuzzy topological spaces. In this paper, we introduce $r$-fuzzy $\epsilon$-closed sets in Šostak’s fuzzy topological spaces. We study the notions of $r$-fuzzy $(\epsilon, \text{almost})$ $\epsilon^*$-regular spaces. We investigate some properties. In particular, we define $r$-$(r-\theta, r-\epsilon\theta)$ $\epsilon$-cluster points and their properties. Moreover, we investigate the relations among $r$-$(r-\theta, r-\epsilon\theta)$ $\epsilon$-cluster points, $r$-fuzzy $(\epsilon, \text{almost})$ $\epsilon^*$-regular spaces and their functions.

2010 Mathematics Subject Classification. 54A40, 54C08, 54C10, 54D10.

Key words and phrases. Fuzzy topology, $r$-fuzzy $\epsilon$-open (closed) sets, $r$-$(r-\theta, r-\epsilon\theta)$ $\epsilon$-cluster points, $r$-fuzzy $(\epsilon, \text{almost})$ $\epsilon^*$-regular spaces, (strongly, $\theta$-) $\epsilon$-irresolute mappings.

Received: 29 June 2016, Accepted: 18 October 2017.

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2. Preliminaries

Throughout this paper, let $X$ be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$. A fuzzy set $\lambda$ of $X$ is a mapping $\lambda : X \to I$, and $I^X$ be the family of all fuzzy sets on $X$. The complement of a fuzzy set $\lambda$ is denoted by $\overline{\lambda}$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point $x_t$ is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in $X$. For $\lambda, \mu \in I^X$, $\lambda$ is called quasi coincident with $\mu$, denoted by $\lambda \emph{eq} \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise, we denote $\lambda \neq \mu$. We define $x_t \in \lambda$ if $t \leq \lambda(x)$. All other notations and definitions are standard in the fuzzy set theory.

**Definition 2.1** ([13]). A function $\tau : I^X \to I$ is called a fuzzy topology on $X$ if it satisfies the following conditions:

1. $\tau(\overline{0}) = \tau(\overline{I}) = 1$,
2. $\tau(\bigvee_{i \in I} \mu_i) \geq \bigwedge_{i \in I} \tau(\mu_i)$, for any $\{\mu_i : i \in I\} \subseteq I^X$,
3. $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for all $\mu_1, \mu_2 \in I^X$.

The pair $(X, \tau)$ is called a fuzzy topological space (for short, fts).

**Definition 2.2** ([3]). Let $(X, \tau)$ be a fts, $\lambda, \mu \in I^X$ and $r \in I_0$. We define operators as follows:

$$C_\tau(\lambda, r) = \bigwedge \{ \mu \in I^X : \lambda \leq \mu, \tau(\overline{I} - \mu) \geq r \},$$

$$I_\tau(\lambda, r) = \bigvee \{ \mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r \}.$$

**Definition 2.3** ([3]). Let $(X, \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$, $\lambda$ is called $r$-fuzzy regular open (for short, $r$-fro) (resp. $r$-fuzzy regular closed (for short, $r$-frc)) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$ (resp. $\lambda = C_\tau(I_\tau(\lambda, r), r)$).

**Definition 2.4** ([12]). Let $(X, \tau)$ be a fts. $\lambda, \mu \in I^X$ and $r \in I_0$,

(i) $\delta-I_\tau(\lambda, r) = \bigvee \{ \mu \in I^X : \lambda \leq \mu, \mu \text{ is a } r\text{-fro set} \}$ is called the $r$-fuzzy $\delta$-interior of $\lambda$.

(ii) $\delta-C_\tau(\lambda, r) = \bigwedge \{ \mu \in I^X : \mu \geq \lambda, \mu \text{ is a } r\text{-frc set} \}$ is called the $r$-fuzzy $\delta$-closure of $\lambda$.

**Definition 2.5** ([12]). Let $(X, \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$,

(i) $\lambda$ is called an $r$-fuzzy $\delta$-semiopen (resp. $r$-fuzzy $\delta$-semiopen) set if $\lambda \leq C_\tau(\delta-I_\tau(\lambda, r), r)$ (resp. $I_\tau(\delta-C_\tau(\lambda, r), r) \leq \lambda$).

(ii) $\lambda$ is called an $r$-fuzzy $\delta$-preopen (resp. $r$-fuzzy $\delta$-preopen) set if $\lambda \leq I_\tau(\delta-C_\tau(\lambda, r), r)$ (resp. $C_\tau(\delta-I_\tau(\lambda, r), r) \leq \lambda$).
(iii) \( \lambda \) is called an \( r \)-fuzzy semi \( \delta \)-preopen (resp. \( r \)-fuzzy semi \( \delta \)-preclosed) set if \( \lambda \leq I_r(C_r(\delta-I_r(\lambda, r), r), r) \) (resp. \( C_r(I_r(\delta-C_r(\lambda, r), r), r) \leq \lambda \)).

(iv) \( \lambda \) is called an \( r \)-fuzzy \( e \)-open (resp. \( r \)-fuzzy \( e \)-closed) set if \( \lambda \leq C_r(\delta-I_r(\lambda, r), r) \lor I_r(\delta-C_r(\lambda, r), r) \) (resp. \( C_r(\delta-I_r(\lambda, r), r) \land I_r(\delta-C_r(\lambda, r), r) \) \leq \( \lambda \)).

**Definition 2.6** ([12]). Let \((X, \tau)\) be a fts. \( \lambda, \mu \in I^X \) and \( r \in I_0 \).

(i) \( eI_r(\lambda, r) = \bigvee \{\mu \in I^X : \mu \leq \lambda, \mu \text{ is a r-feo set} \} \) is called the \( r \)-fuzzy \( e \)-interior of \( \lambda \).

(ii) \( eC_r(\lambda, r) = \bigwedge \{\mu \in I^X : \mu \geq \lambda, \mu \text{ is a r-fe set} \} \) is called the \( r \)-fuzzy \( e \)-closure of \( \lambda \).

**Definition 2.7** ([8]). Let \((X, \tau)\) be a fts and \( x_t \in Pt(X) \). We denote

\[
Q_r(x_t, r) = \{\mu \in I^X | x_t \mu \tau(\mu) \geq r\},
\]

\[
R_r(x_t, r) = \{\mu \in I^X | x_t \mu \text{ is r-fro}\}.
\]

**Definition 2.8** ([8]). Let \((X, \tau)\) be a fts, \( \lambda \in I^X, x_t \in Pt(X) \) and \( r \in I_0 \). A fuzzy point \( x_t \) is called:

(i) an \( r \)- (resp. \( r\theta \)-) cluster point of \( \lambda \) if \( \mu q \lambda \) (resp. \( C_r(\mu, r) q \lambda \)) for every \( \mu \in Q_r(x_t, r) \).

(ii) an \( r \)- (resp. \( r\theta \)-) regular cluster point of \( \lambda \) if \( \mu q \lambda \) (resp. \( C_r(\mu, r) q \lambda \)) for every \( \mu \in R_r(x_t, r) \).

Also, we define operators \( RC_r \) and \( RT_r \) with respect to \( r \)-regular cluster and \( r\theta \)-regular cluster points respectively.

**Theorem 2.9** ([8]). Let \((X, \tau)\) be a fts. For each \( \lambda, \mu, \rho \in I^X \) and \( r \in I_0 \) we have the following properties:

1. \( C_r(\lambda, r) = \bigvee \{x_t \in Pt(X) | x_t \text{ is an } r \text{-cluster point of } \lambda\} \),

   \( RC_r(\lambda, r) = \bigwedge \{\mu \in I^X | \mu \leq \lambda, \mu \text{ is r-frc}\} \).

2. \( T_r(\lambda, r) = \bigwedge \{\mu \in I^X | \lambda \leq I_r(\mu, r), \tau(1 - \mu) \geq r\} \),

   \( RT_r(\lambda, r) = \bigwedge \{\mu \in I^X | \mu \leq I_r(\mu, r), \mu \text{ is r-frc}\} \).

3. \( x_t \) is an \( r\theta \)-cluster point of \( \lambda \) iff \( x_t \in T_r(\lambda, r) \),

   \( x_t \) is an \( r\theta \)-regular cluster point of \( \lambda \) iff \( x_t \in RT_r(\lambda, r) \).

**Definition 2.10** ([8]). Let \((X, \tau)\) be a fts. Then \((X, \tau)\) is called an \( r \)-fuzzy regular (resp. \( r \)-fuzzy almost regular) if for each \( \tau(\mu) \geq r \) (resp. \( r \)-regular open \( \mu \)), there exists a family \( \{\nu_i \in I^X | \tau(\nu_i) \geq r\} \) such that \( \mu = \bigvee_{i \in I} \nu_i \) with \( C_r(\nu_i, r) \leq \mu \).

**Definition 2.11.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s, a function \( f : (X, \tau) \rightarrow (Y, \eta) \) is called:

(i) fuzzy continuous ([11]) iff \( \tau(f^{-1}(\mu)) \geq \eta(\mu) \),
(ii) fuzzy open (resp. fuzzy closed) \( f \) iff \( \eta(f(\lambda)) \geq \tau(\lambda) \) (resp. \( \eta(1 - f(\lambda)) \geq \tau(1 - \lambda) \)),

(iii) fuzzy \( \epsilon \)-irresolute \( f \) iff \( f^{-1}(\mu) \) is \( r \)-feo for each \( r \)-feo \( \mu \in I^X \).

(iv) fuzzy \( \epsilon \)-continuous \( F \) (resp. fuzzy weakly \( \epsilon \)-continuous) iff for each \( \mu \in \mathcal{Q}_\eta(f(x)_\mu, r) \), there exists \( \lambda \in e_\tau(x_t, r) \) such that \( f(\lambda) \leq \mu \) (resp. \( f(\lambda) \leq eC_\eta(\mu, r) \)),

(v) \( f \) is called fuzzy \( \delta \)-semiopen \( F \) (resp. fuzzy \( \delta \)-preopen, fuzzy semi \( \delta \)-preopen and fuzzy \( \epsilon \)-open) iff \( f(\lambda) \) is an \( r \)-\( \delta \)-so (resp. \( r \)-\( \delta \)-po, \( r \)-\( fs \)-\( \delta \)-po and \( r \)-feo) set of \( Y \) for each \( \lambda \in I^X, r \in I_0 \) with \( \tau_1(\lambda) \geq r \).

(vi) \( f \) is called fuzzy \( \delta \)-semiclosed \( F \) (resp. fuzzy \( \delta \)-preclosed, fuzzy semi \( \delta \)-preclosed and fuzzy \( \epsilon \)-closed) iff \( f(\lambda) \) is an \( r \)-\( \delta \)-sc (resp. \( r \)-\( \delta \)-pc, \( r \)-\( fs \)-\( \delta \)-pc and \( r \)-\( \gamma \)-c) set of \( Y \) for each \( \lambda \in I^X, r \in I_0 \) with \( \tau_1(1 - \lambda) \geq r \).

**Theorem 2.12** (\( I \)). Let \((X, \tau)\) be a fts and \( r \in I_0 \).

(i) Any union of \( r \)-feo sets is an \( r \)-feo set.

(ii) Any intersection of \( r \)-feo sets is an \( r \)-feo set.

**Definition 2.13** (\( I \)). Let \((X, \tau)\) and \((Y, \eta)\) be fts’s a function \( f : (X, \tau) \to (Y, \eta) \) is called a supercontinuous iff for each \( \mu \in \mathcal{Q}_\eta(f(x)_\mu, r) \), there exists \( \lambda \in \mathcal{R}_\tau(x_t, r) \) such that \( f(\lambda) \leq \mu \).

3. **Fuzzy \( \epsilon \)-regular spaces**

**Definition 3.1.** Let \((X, \tau)\) be a fts and \( x_t \in Pt(X) \). We denote
\[
\mathcal{E}_\tau(x_t, r) = \{ \mu \in I^X | x_t \mu r, \mu \text{ is } r \text{-feo} \}.
\]

**Definition 3.2.** Let \((X, \tau)\) be a fts, \( \lambda \in I^X, x_t \in Pt(X) \) and \( r \in I_0 \). A fuzzy point \( x_t \) is called:

(i) an \( r \)-\( \epsilon \)-\( \theta \)-cluster point of \( \lambda \) if \( eC_\tau(\mu, r)q\lambda \) for every \( \mu \in \mathcal{Q}_\tau(x_t, r) \),

(ii) an \( r \)-\( \epsilon \)-\( \theta \)-\( \theta \)-\( \epsilon \)-cluster point of \( \lambda \) if \( \mu q\lambda \) (resp. \( C_\tau(\mu, r)q\lambda, eC_\tau(\mu, r)q\lambda \) for every \( \mu \in \mathcal{E}_\tau(x_t, r) \),

(iii) an \( r \)-\( \epsilon \)-\( \theta \)-regular cluster point of \( \lambda \) if \( eC_\tau(\mu, r)q\lambda \) for every \( \mu \in \mathcal{R}_\tau(x_t, r) \).

We define operators \( eT_\tau, eeT_\tau : I^X \times I_0 \to I^X \) as follows:
\[
eT_\tau(\lambda, r) = \sqrt{\{x_t \in Pt(X) | x_t \text{ is an } r \text{-}\( \epsilon \)-\( \theta \)-cluster point of } \lambda \}.
\]
\[
eeT_\tau(\lambda, r) = \sqrt{\{x_t \in Pt(X) | x_t \text{ is a } r \text{-}\( \epsilon \)-\( \theta \)-cluster point of } \lambda \}.
\]

Also, we define operators \( ReT_\tau \) and \( CeT_\tau \) with respect to \( r \)-\( \epsilon \)-\( \theta \)-regular cluster and \( r \)-\( \epsilon \)-\( \theta \)-cluster points respectively.

**Theorem 3.3.** Let \((X, \tau)\) be a fts. For \( \lambda, \mu \in I^X, r \in I_0 \), it holds the following properties.
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(1) $eC_\tau(T - \lambda, r) = T - eI_\tau(\lambda, r)$,
(2) $\lambda \leq eC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$,
(3) If $\tau(\lambda) \geq r$ and $\tau(T - \lambda) \geq r$, then $eC_\tau(\lambda, r) = C_\tau(\lambda, r)$,
(4) $eC_\tau(eC_\tau(\lambda, r), r) = eC_\tau(\lambda, r)$.

Proof. (1) For each $\lambda \in I^X, r \in I_0$, we have
$$eI_\tau(T - \lambda, r) = \bigvee \{ \mu \in I^X | \mu \leq T - \lambda, \text{mísr-feo} \} = T - \bigwedge \{ T - \mu | T - \mu \geq \lambda, T - \text{mísr-fec} \} = T - eC_\tau(\lambda, r).$$

(2) Since $\tau(T - \lambda) \geq r$, then $\mu$ is $r$-féc. Thus the result holds.
(3) Suppose $eC_\tau(\lambda, r)(x) < t < C_\tau(\lambda, r)(x)$. There exists an $r$-féc set $\mu$ with $\lambda \leq \mu$ such that
$$eC_\tau(\lambda, r)(x) < \mu(x) < t < C_\tau(\lambda, r)(x).$$
Since $\mu$ is $r$-féc,
$$C_\tau(\delta - I_\tau(\mu, r), r) \wedge I_\tau(\delta - C_\tau(\mu, r), r) \leq \mu.$$ Since $\tau(\lambda) \geq r$ and $\tau(T - \lambda) \geq r$, $I_\tau(\lambda, r) = \lambda$ and $C_\tau(\lambda, r) = \lambda$. So
$$C_\tau(\lambda, r)(x) = C_\tau(\delta - I_\tau(\lambda, r), r)(x) \wedge I_\tau(\delta - C_\tau(\lambda, r), r)(x) \leq C_\tau(\delta - I_\tau(\mu, r), r)(x) \wedge I_\tau(\delta - C_\tau(\mu, r), r)(x) \leq \mu(x) < t.$$
It is a contradiction.
(4) Since $eC_\tau(\lambda, r)$ is $r$-féc from Theorem 3.3 (2), it is trivial.

\[ \square \]

**Theorem 3.4.** Let $(X, \tau)$ be a fts. The following statements hold:

- $r$-$\epsilon$ cluster $\implies$ $r$-$\theta$-$\epsilon$ cluster $\implies$ $r$-$\epsilon$ cluster
- $r$-cluster $\implies$ $r$-$\epsilon$ cluster $\iff$ $r$-$\theta$ cluster
- $r$-regular cluster $\implies$ $r$-$\theta$ regular cluster $\iff$ $r$-$\theta$ regular cluster

Proof. By Theorem 3.3 (3), since $eC_\tau(\mu, r) = C_\tau(\mu, r)$ for $\tau(\mu) \geq r$, $x_t$ is an $r$-$\epsilon$ (resp. $r$-$\epsilon$ regular) cluster point iff $x_t$ is an $r$-$\theta$ (resp. $r$-$\theta$ regular) cluster point. Other implications follow from the definitions.

\[ \square \]

**Theorem 3.5.** Let $(X, \tau)$ be a fts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_0$ we have the following properties:
(1) $R_\tau(x_t, r) \subset Q_\tau(x_t, r) \subset E_\tau(x_t, r)$.

(2) $eC_\tau(\lambda, r) = \bigvee \{x_t \in Pt(X) \mid x_t \text{ is an } r\text{-cluster point of } \lambda \}$.

(3) $eT_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r), \mu \text{ is } r\text{-feco} \}$.

(4) $eeT_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq eI_\tau(\mu, r), \mu \text{ is } r\text{-feco} \}$,

\[ CeT_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq eI_\tau(\mu, r), \tau(\bar{T} - \mu) \geq r \}, \]

\[ ReT_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq eI_\tau(\mu, r), \mu \text{ is } r\text{-feco} \}. \]

(5) $x_t$ is an $e$-cluster point of $\lambda$ iff $x_t \in eC_\tau(\lambda, r)$, $x_t$ is an $r$-$\theta$- (resp. $r$-$e\theta$-) $e$ cluster point of $\lambda$ iff $x_t \in eT_\tau(\lambda, r)$ (resp. $x_t \in eeT_\tau(\lambda, r)$), $x_t$ is an $r$-$e\theta$-regular cluster point of $\lambda$ iff $x_t \in ReT_\tau(\lambda, r)$.

(6) $CeT_\tau(\lambda, r) = T_\tau(\lambda, r)$ and $ReT_\tau(\lambda, r) = RT_\tau(\lambda, r)$.

(7) $eC_\tau(\lambda, r) \leq eeT_\tau(\lambda, r) \leq eT_\tau(\lambda, r) \leq T_\tau(\lambda, r) \leq RT_\tau(\lambda, r)$.

(8) $eC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \leq RC_\tau(\lambda, r) \leq T_\tau(\lambda, r) \leq RT_\tau(\lambda, r)$.

(9) If $\rho$ is $r$-feco, then

\[ eC_\tau(\rho, r) = eeT_\tau(\rho, r), \]

and

\[ C_\tau(\rho, r) = RC_\tau(\rho, r) = T_\tau(\rho, r) = RT_\tau(\rho, r). \]

(10) If $\tau(\rho) \geq r$, then

\[ eC_\tau(\rho, r) = eeT_\tau(\rho, r) = eT_\tau(\rho, r) = C_\tau(\rho, r) = RC_\tau(\rho, r) = T_\tau(\rho, r) = RT_\tau(\rho, r). \]

Proof. (1) It follows from the definitions.

(2) Put $\rho = \vee\{x_t \in Pt(X) \mid x_t \text{ is an } r\text{-cluster point of } \lambda \}$. Suppose $eC_\tau(\lambda, r) \nless \rho$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $eC_\tau(\lambda, r)(x) > t > \rho(x)$. Then $x_t$ is not an $r$-cluster point of $\lambda$. So, there exists $\mu \in E_\tau(x_t, r)$, $\lambda \leq \bar{T} - \mu$ and $\bar{T} - \mu$ is $r$-feco. By the definition of $eC_\tau$, in Theorem 5.3

\[ eC_\tau(\lambda, r)(x) \leq (\bar{T} - \mu)(x) < t. \]

It is a contradiction. Thus $eC_\tau(\lambda, r) \leq \rho$.

Suppose $eC_\tau(\lambda, r) \nless \rho$. Then there exists an $r$-cluster point $y_s \in Pt(X)$ of $\lambda$ such that $eC_\tau(\lambda, r)(y) < s \leq \rho(y)$. By the
definition of $eC_\tau$, there exists an $r$-fec set $\mu$ with $\lambda \leq \mu$ such that $eC_\tau(\lambda, r)(y) \leq \mu(y) < s < \rho(y)$. Then, $\bar{I} - \mu \in E_\tau(y_s, r)$ and $\lambda \bar{I} - \mu$. Hence, $y_s$ is not an $r$-e cluster point of $\lambda$. It is a contradiction. So $eC_\tau(\lambda, r) \geq \rho$.

(3) Put
$$\delta = \bigwedge \{\mu \in I^X | \lambda \leq I_\tau(\mu, r), \mu \text{is r-fec}\}.$$ 

Suppose $eT_\tau(\lambda, r) \notin \delta$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $eT_\tau(\lambda, r)(x) < t < \delta(x)$. Then $x_t$ is not an $r$-e cluster point of $\lambda$. So, there exists $\mu \in E_\tau(x_t, r)$ and $C_\tau(\mu, r) \leq \bar{I} - \lambda$. Thus $\bar{I} - \mu$ is r-fec and
$$\lambda \leq \bar{I} - C_\tau(\mu, r) = I_\tau(\bar{I} - \mu, r).$$

Hence $\delta(x) \leq (\bar{I} - \mu)(x) < t$. It is a contradiction. Thus $eT_\tau(\lambda, r) \geq \delta$.

Suppose $eT_\tau(\lambda, r) \notin \delta$. Then there exists an $r$-$\theta$-e cluster point $y_s$ of $\lambda$ such that $eT_\tau(\lambda, r)(y) \geq s > \delta(y)$. By the definition of $\delta$, there exists $\mu$ with $\lambda \leq I_\tau(\mu, r)$ and $\mu$ is r-fec such that
$$eT_\tau(\lambda, r)(y) \geq s > \mu(y) \geq \delta(y).$$

Then, $\mu$ is r-fec and $\bar{I} - \mu \in E_\tau(y_s, r)$. So
$$\lambda \leq I_\tau(\mu, r) = \bar{I} - C_\tau(\bar{I} - \mu, r),$$

implies $\lambda \bar{I} C_\tau(\bar{I} - \mu, r)$. Hence, $y_s$ is not an $r$-$\theta$-e cluster point of $\lambda$. It is a contradiction. Thus $eT_\tau(\lambda, r) \leq \delta$.

(4) Put
$$\gamma = \bigwedge \{\mu \in I^X | \lambda \leq eI_\tau(\mu, r), C_\tau(I_\tau(\mu, r), r) = \mu\}.$$ 

Suppose $ReT_\tau(\lambda, r) \notin \gamma$. There exist $x \in X$ and $t \in (0, 1)$ such that $ReT_\tau(\lambda, r)(x) < t < \gamma(x)$. Then $x_t$ is not an $r$-$\theta$ regular cluster point of $\lambda$. So, there exists $\mu \in R_\tau(x_t, r)$, $eC_\tau(\mu, r) \leq \bar{I} - \lambda$. Thus
$$\lambda \leq \bar{I} - eC_\tau(\mu, r)$$
$$= eI_\tau(\bar{I} - \mu, r), C_\tau(I_\tau(\mu, r), r)$$
$$= \bar{I} - \mu.$$ 

Hence $\gamma(x) \leq (\bar{I} - \mu)(x) < t$. It is a contradiction. Thus $ReT_\tau(\lambda, r) \geq \gamma$.

Suppose $ReT_\tau(\lambda, r) \notin \gamma$. Then there exists an $r$-$\theta$ regular cluster point $y_s$ of $\lambda$ such that $ReT_\tau(\lambda, r)(y) \geq s > \gamma(y)$. By the definition of $\gamma$, there exists $\mu$ with $\lambda \leq eI_\tau(\mu, r)$, $C_\tau(I_\tau(\mu, r), r) = \mu$ such that $RT_\tau(\lambda, r)(y) \geq s > \mu(y) \geq \gamma(y)$. Then, $\mu$ is r-fec and $\bar{I} - \mu \in R_\tau(y_s, r)$. Furthermore, $\lambda \leq eI_\tau(\mu, r)$ =
\[ 1 - eC_\tau(T - \mu, r) \text{ implies } \lambda \notin eC_\tau(T - \mu, r). \] Hence, \( y_\lambda \) is not an \( r \)-\( e \)-regular cluster point of \( \lambda \). It is a contradiction. Thus \( ReT_\tau(\lambda, r) \leq \gamma \). Other cases are similarly proved.

(5) We show that \( x_t \) is an \( r \)-\( e \)-\( e \)-cluster point of \( \lambda \) iff \( x_t \in eeT_\tau(\lambda, r) \).

\( \Rightarrow \) It is trivial.

\( \Leftarrow \) Suppose that \( x_t \) is not an \( r \)-\( e \)-\( e \)-cluster point of \( \lambda \). Then there exists \( \mu \in E_\tau(x_t, r) \) such that \( eC_\tau(\mu, r) \leq T - \lambda \). Thus,

\[ \lambda \leq T - eC_\tau(\mu, r) = eI_\tau(\mu, r). \]

By (3), we have \( eeT_\tau(\lambda, r)(x) \leq (T - \mu)(x) < t \). Hence \( x_t \notin eeT_\tau(\lambda, r) \). Other cases are similarly proved.

(6)-(8) Are easily proved from Theorem 5.31.

(9) For each \( r \)-feo set \( \rho \), we will show that \( eC_\tau(\rho, r) = eeT_\tau(\rho, r) \).

Then there exist \( x \in X \) and \( t \in I_0 \) such that

\[ eC_\tau(\rho, r)(x) < t < eeT_\tau(\rho, r)(x). \]

Thus, \( x_t \) is not an \( r \)-\( e \) cluster point of \( \rho \). So, there exists \( \lambda \in E_\tau(x_t, r) \) such that \( \lambda \leq T - \rho \). It implies \( eC_\tau(\lambda, r) \leq T - \rho \). Thus, \( x_t \) is not an \( r \)-\( e \)-\( e \)-cluster point of \( \rho \). Hence \( eC_\tau(\rho, r) \leq eeT_\tau(\rho, r) \) and \( C_\tau(\rho, r) \) is \( r \)-frc. By (3), \( eeT_\tau(\rho, r) \) be given. Since \( C_\tau(\rho, r) \) is \( r \)-frc, \( eC_\tau(\rho, r) \leq C_\tau(\rho, r) \). Moreover, since

\[ C_\tau(\rho, r) \leq C_\tau(I_\tau(C_\tau(\rho, r), r), r) \]

\[ \leq C_\tau(\rho, r), \]

then \( C_\tau(\rho, r) \) is \( r \)-frc. Since \( \rho \leq I_\tau(C_\tau(\rho, r), r) \) and \( C_\tau(\rho, r) \) is \( r \)-frc, by (3), \( RT_\tau(\rho, r) = C_\tau(\rho, r) \). From (8), we have

\[ C_\tau(\rho, r) = RC_\tau(\rho, r) = T_\tau(\rho, r) = RT_\tau(\rho, r). \]

(10) There exist \( \rho \in IX \) with \( \tau(\rho) \geq r \) such that

\[ eC_\tau(\rho, r) \notin eT_\tau(\rho, r). \]

Then there exists \( x \in X \) and \( t \in I \) such that

\[ eC_\tau(\rho, r)(x) < t < eT_\tau(\rho, r)(x). \]

Thus, \( x_t \) is not an \( r \)-\( e \) cluster point of \( \rho \). So, there exists \( \lambda \in E_\tau(x_t, r) \) such that \( \lambda \leq T - \rho \). It implies \( C_\tau(\lambda, r) \leq T - \rho \). Thus, \( x_t \) is not an \( r \)-\( \theta \)-\( e \)-cluster point of \( \rho \). Hence

\[ eC_\tau(\rho, r) = eeT_\tau(\rho, r) = eT_\tau(\rho, r). \]
By (7-9), we have
\[
\begin{align*}
e C_\tau(\rho, r) &= C_\tau(\rho, r) \\
&= RC_\tau(\rho, r) \\
&= T_\tau(\rho, r) \\
&= RT_\tau(\rho, r).
\end{align*}
\]
\[\square\]

**Example 3.6.** Let \(X = \{a, b, c\}\), \(\alpha, \beta, \gamma, \delta \in I^X\) are defined as
\[
\begin{align*}
\alpha(a) &= 0.3, & \beta(a) &= 0.6, & \gamma(a) &= 0.6, & \delta(a) &= 0.3, \\
\alpha(b) &= 0.4, & \beta(b) &= 0.5, & \gamma(b) &= 0.5, & \delta(b) &= 0.4, \\
\alpha(c) &= 0.5, & \beta(c) &= 0.5, & \gamma(c) &= 0.4, & \delta(c) &= 0.4.
\end{align*}
\]

We define the smooth topology \(\tau : I^X \to I\) as follows:
\[
\tau(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = \alpha, \\
\frac{1}{3} & \text{if } \lambda = \beta, \\
\frac{1}{4} & \text{if } \lambda = \gamma, \\
\frac{1}{5} & \text{if } \lambda = \delta, \\
0 & \text{otherwise}.
\end{cases}
\]

For \(r = \frac{1}{2}\), then the fuzzy sets \(\alpha, \beta, \gamma, \delta\) are \(r\)-feo sets, \(\overline{I} - \alpha, \overline{I} - \beta, \overline{I} - \gamma, \overline{I} - \delta\) are \(r\)-fec sets. Let \(\lambda(a) = 0.4, \lambda(b) = 0.5, \lambda(c) = 0.5, e C_\tau(\lambda, r) = \lambda\), and clearly, \(e e T_\tau(\lambda, r) \leq e T_\tau(\lambda, r) = T_\tau(\lambda, r) = RT_\tau(\lambda, r)\).

**Definition 3.7.** Let \((X, \tau)\) be a fts. Then \((X, \tau)\) is called:

1. \(r\)-fuzzy \(e\)-regular if for each \(r\)-feo \(\mu\) there exists a family \(\{\nu_i \in I^X \mid C_\tau(\nu_i, r) \geq r\}\) such that \(\mu = \bigvee_{i \in \Gamma} \nu_i\) with \(e C_\tau(\nu_i, r) \leq \mu\).
2. \(r\)-fuzzy \(e^*\)-regular (resp. \(r\)-fuzzy \(ee^*\)-regular, \(r\)-fuzzy almost \(e^*\)-regular) if for each \(\tau(\mu) \geq r\) (resp. \(r\)-fuzzy \(e\)-regular, \(r\)-fuzzy \(e\)-regular) there exists a family \(\{\nu_i \in I^X \mid C_\tau(\nu_i, r) \geq r\}\) such that \(\mu = \bigvee_{i \in \Gamma} \nu_i\) with \(e C_\tau(\nu_i, r) \leq \mu\).
3. fuzzy (\(e\), almost) \((e^*)\)-regular if \((X, \tau)\) is \(r\)-fuzzy \((e\), almost)\((e^*)\)-regular, for each \(r \in I_0\).

We easily prove the following Lemma.

**Lemma 3.8.** For \(\lambda, \lambda_i, \mu \in I^X\) and \(x_t \in Pt(X)\), we have

1. \(\lambda \leq \mu\) iff \(x_t q \lambda\) implies \(x_t q \mu\).
2. \(x_t q \bigvee_{i \in A} \lambda_i\) iff there exists \(i \in A\) such that \(x_t q \lambda_i\).

**Theorem 3.9.** Let \((X, \tau)\) be a fts and \(r \in I_0\). Then the following statements are equivalent:

1. \((X, \tau)\) is \(r\)-fuzzy almost \(e^*\)-regular.
(2) For all \( \mu \in R_\tau(x_t, r) \), there exists \( \nu \in \mathcal{E}_\tau(x_t, r) \) with \( eC_\tau(\nu, r) \leq \mu \).

(3) For each \( x_t \in Pt(X) \) and each \( r \)-frc \( \lambda \in I^X \) with \( x_t \notin \lambda \), there exist \( \nu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-feo \( \mu \in I^X \) such that \( \lambda \leq \mu \) and \( \mu \rho_\tau \nu \).

(4) For each \( r \)-frc \( \lambda \in I^X \), \( \lambda = \bigwedge \{ eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r \text{-feo} \} \).

(5) For each \( r \)-frc \( \lambda \in I^X \) with \( \rho \not\subseteq \lambda \), there exist \( \nu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-feo \( \mu \) such that \( \lambda \leq \mu \), \( \rho_\tau \nu \) and \( \mu \rho_\tau \nu \).

**Proof.**

(1) \( \Rightarrow \) (2): Let \( \mu \in \mathcal{R}_\tau(x_t, r) \) be given. Since \((X, \tau)\) is \( r \)-fuzzy almost \( e^* \)-regular, there exists a family \( \{ \nu_i | \nu_i \text{ is } r \text{-feo} \} \) such that \( \mu = \bigvee_{i \in \Gamma} \nu_i \) with \( eC_\tau(\nu_i, r) \leq \mu \). Since \( x_t \rho (\mu = \bigvee_{i \in \Gamma} \nu_i) \), by Lemma 6.3 (2), there exists \( i \in \Gamma \) such that \( \nu_i \in \mathcal{E}_\tau(x_t, r) \) with \( eC_\tau(\nu_i, r) \leq \mu \).

(2) \( \Rightarrow \) (1): For each \( \mu \in \mathcal{R}_\tau(x_t, r) \), there exists \( \nu_i \in \mathcal{E}_\tau(x_t, r) \) such that \( eC_\tau(\nu_i, r) \leq \mu \). Let \( \{ \nu_i \in \mathcal{E}_\tau(x_t, r) | i \in \Lambda, eC_\tau(\nu_i, r) \leq \mu \} \) be the family satisfying the above condition. Trivially, \( \bigvee_{i \in \Lambda} \nu_i \leq \mu \).

(2) \( \Rightarrow \) (3): Let \( x_t \notin \lambda \) with \( r \)-frc \( \lambda \). Then \( I - \lambda \in \mathcal{R}_\tau(x_t, r) \). By (2), there exists \( \nu \in \mathcal{E}_\tau(x_t, r) \) such that \( eC_\tau(\nu, r) \leq I - \lambda \). Put \( \mu = I - eC_\tau(\nu, r) \). By Theorem 2.2 (1), \( r \)-feo such that \( \lambda \leq \mu \) and \( \mu \rho_\tau \nu \).

(3) \( \Rightarrow \) (4): Suppose there exists \( r \)-frc \( \lambda \in I^X \) such that

\[
\lambda \not\subseteq \bigwedge \{ eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r \text{-feo} \}.
\]

Then there exist \( x \in X \) and \( t \in I_0 \) such that

\[
\lambda(x) < t < \bigwedge \{ eC_\tau(\nu, r)(x) | \lambda \leq \nu, \nu \text{ is } r \text{-feo} \}.
\]

Since \( x_t \notin \lambda \), by (4), there exist \( \mu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-feo \( \nu \) such that \( \lambda \leq \nu \) and \( \mu \rho_\tau \nu \). Since \( \nu \) is \( r \)-feo, \( \lambda \leq eI_\tau(\nu, r) \) and \( eI_\tau(\nu, r) \) is \( r \)-feo. Hence

\[
\lambda(x) < t < eC_\tau(eI_\tau(\nu, r), r)(x).
\]

By the definition of \( eC_\tau \), we have

\[
eC_\tau(eI_\tau(\nu, r), r)(x) \leq eC_\tau(\nu, r)(x)
\]

\[
\leq I - \mu(x) < t.
\]

It is contradiction for (3.1). Thus

\[
\lambda = \bigwedge \{ eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r \text{-feo} \}.
\]

(4) \( \Rightarrow \) (5): Let \( \lambda \in I^X \) be \( r \)-frc with \( \rho \not\subseteq \lambda \). Then \( x_t \in Pt(X) \) such that \( x_t \in \rho \) and \( t > \lambda(x) \). By (4), there exist \( r \)-feo \( \mu \) such that \( \lambda \leq \mu \).
and \( eC_\tau(\mu, r)(x) < t \). Put \( \nu = \mathcal{T} - eC_\tau(\mu, r) \). By Theorem 3.12, (1), \( \nu \) is \( r \)-feo, that is, \( \nu \in \mathcal{E}_\tau(x_t, r) \) such that \( \lambda \leq \mu, \rho \mu \nu \) and \( \mu \overline{\nu} \).

(5) \( \Rightarrow \) (2): For all \( \mu \in \mathcal{R}_\tau(x_t, r), t > \mathcal{T} - \mu(x) \). So, \( x_t \notin \mathcal{T} - \mu \) and \( \mathcal{T} - \mu \) is \( r \)-fec, by (5), there exist \( \nu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-fec \( \rho \) such that \( \mathcal{T} - \mu \leq \rho \) and \( \rho \overline{\nu} \). Thus, \( \nu \leq \mathcal{T} - \rho \leq \mu \). Since \( \mathcal{T} - \rho \) is \( r \)-fec and \( \mu \) is \( r \)-fro, \( eC_\tau(\mathcal{T} - \rho, r) \leq \mu \). It implies \( \nu \in \mathcal{E}_\tau(x_t, r) \) such that \( eC_\tau(\nu, r) \leq \mu \).

\[ \square \]

**Corollary 3.10.** Let \( (X, \tau) \) be a fts and \( r \in I_0 \). Then the following statements are equivalent:

(i) \( (X, \tau) \) is \( r \)-fuzzy \( ee^* \)-regular (resp. \( r \)-fuzzy \( e^* \)-regular).

(ii) For all \( \mu \in \mathcal{E}_\tau(x_t, r) \) (resp. \( \mu \in \mathcal{Q}_\tau(x_t, r) \)), there exists \( \nu \in \mathcal{E}_\tau(x_t, r) \) with \( eC_\tau(\nu, r) \leq \mu \).

(iii) For each \( x_t \in \mathcal{P}(X) \) and each \( r \)-fec \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r \)) with \( x_t \notin \lambda \), there exist \( \nu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-feco \( \rho \) such that \( \lambda \leq \mu \) and \( \mu \overline{\nu} \).

(iv) For each \( r \)-fec \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r \)),

\[ \lambda = \bigwedge \{ eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r \text{-feo} \}. \]

(v) For each \( r \)-feco \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r \)) with \( \rho \notin \lambda \), there exist \( \nu \in \mathcal{E}_\tau(x_t, r) \) and \( r \)-feco \( \mu \) such that \( \lambda \leq \mu \) and \( \mu \overline{\nu} \).

**Corollary 3.11.** Let \( (X, \tau) \) be a fts and \( r \in I_0 \). Then the following statements are equivalent:

(i) \( (X, \tau) \) is \( r \)-fuzzy \( e \)-regular (resp. \( r \)-fuzzy regular, \( r \)-fuzzy almost regular).

(ii) For all \( \mu \in \mathcal{E}_\tau(x_t, r) \) (resp. \( \mu \in \mathcal{Q}_\tau(x_t, r), \mu \in \mathcal{R}_\tau(x_t, r) \)), there exists \( \nu \in \mathcal{Q}_\tau(x_t, r) \) with \( C_\tau(\nu, r) \leq \mu \).

(iii) For each \( x_t \in \mathcal{P}(X) \) and each \( r \)-fec \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r, r \)-feco) with \( x_t \notin \lambda \), there exist \( \nu \in \mathcal{Q}_\tau(x_t, r) \) and \( \tau(\mu) \geq r \) such that \( \lambda \leq \mu \) and \( \mu \overline{\nu} \).

(iv) For each \( r \)-fec \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r, r \)-feco),

\[ \lambda = \bigwedge \{ C_\tau(\nu, r) | \lambda \leq \nu, \tau(\nu) \geq r \}. \]

(v) For each \( r \)-feco \( \lambda \in I^X \) (resp. \( \tau(\mathcal{T} - \lambda) \geq r, r \)-feco) with \( \rho \notin \lambda \), there exist \( \nu \in \mathcal{Q}_\tau(x_t, r) \) and \( \tau(\mu) \geq r \) such that \( \lambda \leq \mu \) and \( \mu \overline{\nu} \).

**Lemma 3.12.** Let \( (X, \tau) \) be a fts.

(i) For each \( x_t \in \mathcal{Q}_\tau(x_t, r) \), there exists \( \mu \in \mathcal{Q}_\tau(x_t, r) \) such that

\[ C_\tau(\mu, r) \leq \lambda \text{ iff } \mathcal{T} - \lambda = T_\tau(\mathcal{T} - \lambda, r). \]
(ii) For each \( x_tq\lambda \), there exists \( \mu \in \mathcal{E}_\tau(x_t, r) \) such that
\[
eC_\tau(\mu, r) \leq \lambda \quad \text{iff} \quad \bar{\lambda} = eeT_\tau(\bar{\lambda}, r).
\]

Proof. 
(i) It is similarly proved as the following (ii).
(ii) \((\Rightarrow)\) We only show that \( 1 = eeT_\tau(1, r) \). Let \( x_t \notin \bar{\lambda} = eeT_\tau(\bar{\lambda}, r) \). Then \( x_tq\lambda \). By hypothesis, there exists \( \mu \in \mathcal{E}_\tau(x_t, r) \) such that
\[
eC_\tau(\mu, r) \leq \lambda \quad \text{iff} \quad \bar{\lambda} = eeT_\tau(\bar{\lambda}, r).
\]

Theorem 3.13. Let \((X, \tau)\) be a fts and \( r \in I_0 \). The following statements are equivalent:

1. \((X, \tau)\) is \( r \)-fuzzy \( ee^* \)-regular (resp. \( r \)-fuzzy \( e^* \)-regular, \( r \)-fuzzy almost \( e^* \)-regular).
2. For each \( r \)-feo \( \mu \) (resp. \( \tau(\mu) \geq r, r \)-fro \( \mu \)), \( \bar{\lambda} = eeT_\tau(\bar{\lambda} - \mu, r) \).
3. For each \( \lambda \in I^X \), \( eC_\tau(\lambda, r) = eeT_\tau(\lambda, r) \) (resp. \( C_\tau(\lambda, r) = eeT_\tau(\lambda, r) \), \( RC_\tau(\lambda, r) = eeT_\tau(\lambda, r) \)).

Proof. 
(1) \( \Leftrightarrow \) (2) It is easy from Lemma 3.12. (2).
(2) \( \Rightarrow \) (3) Suppose there exists \( \lambda \in I^X \) with \( eC_\tau(\lambda, r) \not\subseteq eeT_\tau(\lambda, r) \). Then there exist \( x \in X \) and \( t \in I_0 \) such that
\[
eC_\tau(\lambda, r)(x) < t < eeT_\tau(\lambda, r)(x).
\]

By the definition of \( eC_\tau \), there exists \( r \)-feo set \( \rho \in I^X \) with \( \lambda \leq \rho \) such that
\[
eC_\tau(\lambda, r)(x) \leq \rho(x) < t < eeT_\tau(\lambda, r)(x).
\]

By (2), since \( eeT_\tau(\rho, r) = \rho \), we have
\[
D_\tau(\lambda, r)(x) \leq eeT_\tau(\rho, r)(x) = \rho(x) < t.
\]

It is a contradiction.
(3) \( \Rightarrow \) (2) It is easy.

Corollary 3.14. Let \((X, \tau)\) be a fts and \( r \in I_0 \). The following statements are equivalent:

(i) \((X, \tau)\) is \( r \)-fuzzy regular (resp. \( r \)-fuzzy \( e \)-regular, \( r \)-fuzzy almost regular)
(ii) For each \( \tau(\mu) \geq r \) (resp. \( r \)-feo \( \mu \), \( r \)-fro \( \mu \)), \( \bar{\lambda} - \mu = T(\bar{\lambda} - \mu, r) \).
(iii) For each \( \lambda \in I^X \), \( C_\tau(\lambda, r) = T_\tau(\lambda, r) \) (resp. \( eC_\tau(\lambda, r) = T_\tau(\lambda, r) \), \( RC_\tau(\lambda, r) = T_\tau(\lambda, r) \)).

Remark 3.15. Let \((X, \tau)\) be a fts. We have:
Let $r$-fuzzy $e$-regular $\Rightarrow r$-fuzzy regular $\Rightarrow r$-fuzzy almost regular

$r$-fuzzy $e^*$-regular $\Rightarrow r$-fuzzy $e^*$-regular $\Rightarrow r$-fuzzy almost $e^*$-regular.

Example 3.16. Let $X = \{a, b, c\}$ be a set and $a_{0.6} \in Pt(X)$. We define the fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b, c\}}\}, \\ \frac{1}{2} & \text{if } \lambda \in \{a_{0.6}, a_{0.6} \vee \chi_{\{b, c\}}\}, \\ 0 & \text{otherwise}. \end{cases}$$

(1) For $0 < r \leq \frac{1}{2}$, since $\chi_{\{a\}}$ and $\chi_{\{b, c\}}$ are $r$-fro and $r$-frc sets, $C_r(\chi_{\{a\}}, r) = \chi_{\{a\}}$ and $C_r(\chi_{\{b, c\}}, r) = \chi_{\{a, c\}}$, then $(X, \tau)$ is $r$-fuzzy almost regular.

(2) For $a_{0.6} \in Q_{\tau}(a_{0.7}, 1/2)$, for all $\mu \in Q_{\tau}(a_{0.7}, 1/2)$ we have $C_{\tau}(\mu, 1/2) \neq a_{0.6}$. So, $(X, \tau)$ is not a 1/2-fuzzy regular. Moreover, for $a_{0.9} \in E_{\tau}(a_{0.2}, 1/2)$ and for all $\mu \in E_{\tau}(a_{0.2}, 1/2)$, we have $eC_{\tau}(\mu, 1/2) \neq a_{0.9}$. So, $(X, \tau)$ is not a 1/2-fuzzy $e^*$-regular.

(3) For $0 < r \leq 1/2$, we have the following (a) and (b).

(a) If $a_{0.4} < a_s < a_{0.6}$, then $a_s$ is $r$-fro and $r$-frc. For $a_{0.6} \in Q_{\tau}(a_t, r)$, there exists $a_t \in E_{\tau}(a_t, r)$ with $eC_{\tau}(a_t, r) \leq a_{0.6}$.

(b) Let $a_{0.6} \vee \chi_{\{b, c\}} \in Q_{\tau}(x_t, r)$. If $(x = a)_t$, by (a), there exist $a_s \in E_{\tau}(a_t, r)$ such that $eC_{\tau}(a_s, r) = a_s \leq a_{0.6} \vee \chi_{\{b, c\}}$. If $(x = b)_t$ or $(x = c)_t$, there exists $x_s$ with $s + t > 1$ and $a_s \in E_{\tau}(x_t, r)$ such that $eC_{\tau}(x_t, r) = x_s \leq a_{0.6} \vee \chi_{\{b, c\}}$.

Hence, $(X, \tau)$ is $r$-fuzzy $e^*$-regular.

Example 3.17. Let $X$ be a set containing at least three points. We define the fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.6}, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < r \leq 1/2$, if $\lambda \leq 0.4$, $\lambda$ is $r$-fro and if $\mu \geq 0.6$, $\mu$ is $r$-frc. Let $\lambda \in E_{\tau}(a_t, r)$. Since $\lambda \leq 0.4$, there exists $y \in X$ such that $\lambda(y) > 0.4$, $\lambda(a) + t > 1$. Put $\mu \in I^X$ as

$$\mu(x) = \begin{cases} \lambda(x) & \text{if } x \in \{a, y\}, \\ \min\{0.5, \lambda(x)\} & \text{otherwise.} \end{cases}$$
So, \( \mu \) is \( r \)-feo and \( \mu \in \mathcal{E}_r(a_t, r) \) such that \( eC_r(\mu, r) = \mu \leq \lambda \). Hence \((X, \tau)\) is \( r \)-fuzzy \( ee^* \)-regular. But it is neither \( r \)-fuzzy \( e \)-regular nor \( r \)-fuzzy regular because \( 0 \notin Q_r(a_t, r) \) and for all \( \lambda \in Q_r(a_t, r), C_r(\lambda, r) \not\leq 0 \).

4. Strongly \( e \)-irresolute mappings

**Definition 4.1.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s, a function \( f : (X, \tau) \to (Y, \eta) \) is called:

(i) strongly \( \theta \)-\( e \)-continuous (resp. strongly \( e \)-irresolute) iff for each \( \mu \in Q_\eta(f(x)_t, r) \) (resp. \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \)), there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(eC_r(\lambda, r)) \leq \mu \),

(ii) \( \theta \)-\( e \)-irresolute (resp. quasi \( e \)-irresolute) iff for each \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \), there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(eC_r(\lambda, r)) \leq eC_\eta(\mu, r) \) (resp. \( f(\lambda) \leq eC_\eta(\mu, r) \)).

**Theorem 4.2.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) a function. Then the following statements are equivalent:

1. \( f \) is \( e \)-irresolute.
2. For each \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \), there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(\lambda) \leq \mu \).
3. \( f(eC_r(\lambda, r)) \leq eC_\eta(f(\lambda, r)) \) for each \( \lambda \in I^X \).
4. \( eC_r(f^{-1}(\mu), r) \leq f^{-1}(eC_\eta(\mu, r)) \) for each \( \mu \in I^Y \).

**Proof.**

(1) \( \Rightarrow \) (2) For \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \), by (1), there exists \( f^{-1}(\mu) \in \mathcal{E}_\tau(x_t, r) \) such that \( f(f^{-1}(\mu)) \leq \mu \).

(2) \( \Rightarrow \) (1) For each \( r \)-feo \( \mu \), we only show that

\[
f^{-1}(\mu) = \bigvee \{ \lambda | \lambda \leq f^{-1}(\mu), \lambda \text{ is } r \text{-feo} \}.
\]

Suppose there exist \( x \in X \) and \( t \in I_0 \) such that

\[
f^{-1}(\mu)(x) = \mu(f(x))
> 1 - t
> \bigvee \{ \lambda(x) | \lambda \leq f^{-1}(\mu), \lambda \text{ is } r \text{-feo} \}.
\]

For each \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \), by (2), there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(\lambda) \leq \mu \). Thus \( \lambda \leq f^{-1}(\mu) \) and \( \lambda q x_t \) implies \( 1 - t < \lambda(x) \).

It is a contradiction. Hence \( f^{-1}(\mu) \) is \( r \)-feo.
(1) ⇒ (3)
\[ eC_\eta(f(\lambda), r) = \bigwedge \{ \mu | f(\lambda) \leq \mu, isr - fec \} \]
\[ \geq \bigwedge \{ \mu | f(\lambda) \leq \mu, f^{-1}(\mu)ISR - fec \} \]
\[ \geq f \left( \bigwedge \{ f^{-1}(\lambda) | \lambda \leq f^{-1}(\mu), f^{-1}(\mu)ISR - fec \} \right) \]
\[ \geq f(eC_\tau(\lambda, r)). \]

(3) ⇒ (4) Put \( \lambda = f^{-1}(\mu) \). Then
\[ eC_\tau(f^{-1}(\mu), r) \leq f^{-1}(f(eC_\tau(f^{-1}(\mu), r))) \]
\[ \leq f^{-1}(eC_\eta(\mu, r)). \]

(4) ⇒ (1) For each \( r \)-feo \( \mu \in I^Y \), we have \( eC_\eta(\overline{1} - \mu, r) = \overline{1} - \mu \). By (4),
\[ eC_\tau(\overline{1} - f^{-1}(\mu), r) \leq f^{-1}(eC_\eta(\overline{1} - \mu, r)) \]
\[ = \overline{1} - f^{-1}(\mu). \]

So, \( eC_\tau(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu) \). By Theorem \ref{thm:4.2} (2), \( f^{-1}(\mu) \) is \( r \)-feo.

**Corollary 4.3.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) a function. Then the following statements are equivalent:

1. \( f \) is \( e \)-continuous (resp. supercontinuous).
2. \( f(eC_\tau(\lambda, r)) \leq C_\eta(f(\lambda, r)) \) (resp. \( f(RC_\tau(\lambda, r)) \leq C_\eta(f(\lambda, r)) \)), for each \( \lambda \in I^X \).
3. \( eC_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_\eta(\mu, r)) \) (resp. \( RC_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_\eta(\mu, r)) \)), for each \( \mu \in I^Y \).

**Theorem 4.4.** The following implications hold:

\begin{align*}
\text{strongly } e & \text{- irresolute } \Rightarrow \text{ strongly } \theta & \text{- } e \text{- continuous,} \\
\text{strongly } e & \text{- irresolute } \Rightarrow \text{ } e & \text{- irresolute,} \\
\theta & \text{- } e & \text{- irresolute } \Rightarrow \text{ quasi } - \text{ irresolute.}
\end{align*}

**Proof.** We show that \( e \)-irresolute \( \Rightarrow \theta\)-e-irresolute. Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) a function. For each \( \mu \in E_\eta(f(x), r) \), by e-irresolutity and Theorem \ref{thm:4.2} (4), \( f^{-1}(\mu) \in E_\tau(x, r) \) such that
\[ eC_\tau(f^{-1}(\mu), r) \leq f^{-1}(eC_\eta(\mu, r)). \]

It implies
\[ f(eC_\tau(f^{-1}(\mu), r)) \leq eC_\eta(\mu, r). \]
Example 4.5. Let $X = \{a\}$ be a set. We define the fuzzy topologies $\tau, \eta : I^X \to I$ as follows:

$$
\tau(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = 0.6, \\
0 & \text{otherwise.}
\end{cases} 
\quad \eta(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = 0.6, \\
0 & \text{otherwise.}
\end{cases}
$$

For $r = 1/2$, $\lambda = 0.6$ is $r$-feo in $(Y, \eta)$ and $\lambda$ is $r$-feo in $(X, \tau)$. Hence the identity function $id_X : (X, \tau) \to (X, \eta)$ is fuzzy $e$-irresolute and strongly $\theta$-$e$-continuous because for $0.6 \in Q_\eta(a_t, r)$, there exists $0.6 \in E_\tau(a_t, r)$ such that $eC_\tau(0.6, r) = 0.6 \leq 0.6$.

But the identity function $id_X : (X, \tau) \to (X, \eta)$ is not strongly $e$-irresolute because for $0.75 \in E_\eta(a_{0.3}, r)$, and for all $a_s \in E_\tau(a_{0.3}, r)$ we have $eC_\tau(a_s, r) = 1 \not\leq 0.75$. Moreover, $id_X$ is quasi $e$-irresolute but not $\theta$-$e$-irresolute.

Theorem 4.6. Let $(X, \tau)$ and $(Y, \eta)$ be fts’s and $f : X \to Y$ a function. If $f$ is $e$-irresolute, then $f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$ for each $\mu = eeT_\eta(\mu, r)$.

Proof. Let $\mu = eeT_\eta(\mu, r)$. For each $x_tqT_1 - f^{-1}(\mu)$, we have $f(x_t)qT_1 - f^{-1}(\mu)$.

By Lemma 4.5 (2), there exists $\rho \in e_\eta(f(x_t), r)$ such that $eC_\eta(\rho, r) \leq T_1 - f^{-1}(\mu)$. Since $f$ is $e$-irresolute, by Theorem 4.4 (4), there exists $f^{-1}(\rho) \in E_\tau(x_t, r)$ such that

$$
eC_\tau(f^{-1}(\rho), r) \leq f^{-1}(eeT_\eta(\rho, r)) 
\leq T_1 - f^{-1}(\mu).
$$

By Lemma 4.5 (2), $f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$. \hfill \square

Theorem 4.7. Let $(X, \tau)$ and $(Y, \eta)$ be fts’s and $f : X \to Y$ a function. Then the following statements are equivalent:

1. $f$ is $\theta$-$e$-irresolute.
2. $f(eeT_\tau(\lambda, r)) \leq eeT_\eta(\lambda, r)$ for each $\lambda \in I^X$.
3. $eeT_\tau(f^{-1}(\mu), r) \leq f^{-1}(eeT_\eta(\mu, r))$ for each $\mu \in I^Y$.
4. $eeT_\tau(f^{-1}(\mu), r) \leq f^{-1}(eeT_\eta(\mu, r))$ for each $r$-feo $\mu \in I^Y$.

Proof. (1) $\Rightarrow$ (2) Suppose there exist $\lambda \in I^Y$ and $r \in I_0$ such that

$$
f(eeT_\tau(\lambda, r)) \not\leq eeT_\eta(f(\lambda), r).
$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$
f(eeT_\tau(\lambda, r))(f(x)) \geq eeT_\tau(\lambda, r)(x)
> t 
> eeT_\eta(f(\lambda), r)(f(x)).
$$
Let \( f(x)_t \notin eeT_\eta(f(\lambda, r)) \). Then there exists \( \rho \in E_\eta(f(x)_t, r) \) such that \( eC_\eta(\rho, r) \leq \bar{T} - f(\lambda) \). Since \( f \) is \( \theta \)-irresolute, for \( \rho \in E_\eta(f(x)_t, r) \), there exists \( \mu \in E_\tau(x_t, r) \) such that

\[
\begin{align*}
    f(eC_\tau(\mu, r)) \leq eC_\eta(\rho, r) \\
    \leq \bar{T} - f(\lambda).
\end{align*}
\]

It implies

\[
\begin{align*}
    eC_\tau(\mu, r) \leq f^{-1}(f(eC_\tau(\mu, r))) \\
    \leq f^{-1}(eC_\eta(\rho, r)) \\
    \leq \bar{T} - f^{-1}(f(\lambda)) \\
    \leq \bar{T} - \lambda.
\end{align*}
\]

Hence \( x_t \) is not an \( r \)-\( \theta \)-e cluster point of \( \lambda \). It is a contradiction. Thus (2) holds.

\( (2) \Rightarrow (3) \) Put \( \lambda = f^{-1}(\mu) \). It is easy.

\( (3) \Rightarrow (4) \) Since \( eeT_\eta(\mu, r) = eC_\eta(\mu, r) \) for each \( r \)-feo \( \mu \in I^Y \) from Theorem 3.5 (9), it is trivial.

\( (4) \Rightarrow (1) \) Let \( \mu \in e_\eta(f(x)_t, r) \). Then \( eC_\eta(\mu, r) \) is not an \( r \)-\( \theta \)-e cluster point of \( \bar{T} - eC_\eta(\mu, r) \). Hence \( f(x)_t \) is not an \( r \)-\( \theta \)-e cluster point of \( \bar{T} - eC_\eta(\mu, r) \). By Theorem, \( f(x)_t \) is not an \( r \)-e-cluster point of \( \bar{T} - eC_\eta(\mu, r) \). Thus,

\[
t > eC_\eta(\bar{T} - eC_\eta(\mu, r), r)(f(x)) \\
    = f^{-1}(eC_\eta(\bar{T} - eC_\eta(\mu, r), r))(x).
\]

Since \( \bar{T} - eC_\eta(\mu, r) \) is \( r \)-feo, by (4),

\[
f^{-1}(eC_\eta(\bar{T} - eC_\eta(\mu, r), r)) \geq eeT_\tau(f^{-1}(\bar{T} - eC_\eta(\mu, r)), r).
\]

It implies

\[
t > eeT_\tau(f^{-1}(\bar{T} - eC_\eta(\mu, r)), r)(x).
\]

Hence \( x_t \) is not an \( r \)-\( \theta \)-e-cluster point of \( f^{-1}(\bar{T} - eC_\eta(\mu, r)) \).

There exists \( \rho \in E_\tau(x_t, r) \) such that

\[
eC_\tau(\rho, r) \leq \bar{T} - f^{-1}(\bar{T} - eC_\eta(\mu, r)) \\
    = f^{-1}(eC_\tau(\mu, r)).
\]

Thus,

\[
f(eC_\tau(\rho, r)) \leq eC_\eta(\mu, r).
\]

\( \square \)

**Theorem 4.8.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) a function. Then the following statements are equivalent:

1. \( f \) is strongly \( \theta \)-e-continuous.
2. \( \bar{T} - f^{-1}(\mu) = eeT_\tau(\bar{T} - f^{-1}(\mu), r) \) for each \( \eta(\mu) \geq r \).
3. \( f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r) \) for each \( \eta(\bar{T} - \mu) \geq r \).
(4) \( f(eeT_\tau(\lambda, r)) \leq C_\eta(f(\lambda), r) \) for each \( \lambda \in I^X \).

(5) \( eeT_\eta(f^{-1}(\mu), r) \leq f^{-1}(C_\tau(\mu, r)) \) for each \( \mu \in I^Y \).

**Proof.**

(1) \( \Rightarrow \) (2) Suppose there exists \( \mu \in I^Y \) with \( \eta(\mu) \geq r \) such that

\[
\mathcal{T} - f^{-1}(\mu) \neq eeT_\tau(\mathcal{T} - f^{-1}(\mu), r).
\]

Then there exist \( x \in X \) and \( t \in I_0 \) such that

\[
(4.1) \quad (\mathcal{T} - f^{-1}(\mu))(x) < t < eeT_\tau(\mathcal{T} - f^{-1}(\mu), r)(x).
\]

Since \( x_tq f^{-1}(\mu) \) implies \( f(x)_tq \mu \); we have \( \mu \in Q_\eta(f(x)_t, r) \).

Since \( f \) is strongly \( \theta \)-e-continuous, for \( \mu \in Q_\eta(f(x)_t, r) \), there exists \( \lambda \in E_\tau(x_t, r) \) such that \( f(eC_\tau(\lambda, r)) \leq \mu \). It implies \( eC_\tau(\lambda, r) \leq f^{-1}(\mu) \). Thus, \( eC_\tau(\lambda, r)q(\mathcal{T} - f^{-1}(\mu)) \). Hence \( x_t \) is not an \( r-e-q \) cluster point of \( \mathcal{T} - f^{-1}(\mu) \). Hence \( eeT_\tau(\mathcal{T} - f^{-1}(\mu), r)(x) < t \). It is a contradiction. Hence (2) holds.

(2) and (3) are equivalent.

(3) \( \Rightarrow \) (4) Suppose there exist \( \lambda \in I^X \) and \( t \in I_0 \) such that

\[
f(eeT_\tau(\lambda, r)) \not\in C_\eta(f(\lambda), r).
\]

Then there exist \( y \in Y \) and \( t \in I_0 \) such that

\[
f(eeT_\tau(\lambda, r))(y) > t > C_\eta(f(\lambda), r)(y).
\]

By the definition of \( f(eeT_\tau(\lambda, r)) \), there exists \( x \in X \) with \( f(x) = y \) such that

\[
f(eeT_\tau(\lambda, r))(f(x)) \geq eeT_\tau(\lambda, r)(x)
\]

\[
> t
\]

\[
> C_\eta(f(\lambda), r)(f(x)).
\]

By the definition of \( C_\eta(f(\lambda), r) \), there exists \( \mu \in I^Y \) with \( f(\lambda) \leq \mu, \eta(\mathcal{T} - \mu) \geq r \) such that

\[
(4.2) \quad f(eeT_\tau(\lambda, r))(f(x)) \geq eeT_\tau(\lambda, r)(x)
\]

\[
> t
\]

\[
> \mu(f(x)).
\]

On the other hand, by (3), \( f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r) \) for each \( \eta(\mathcal{T} - \mu) \geq r \). Then \( \lambda \leq \mu \) implies

\[
eeT_\tau(\lambda, r)(x) \leq eeT_\tau(f^{-1}(\mu), r)(x)
\]

\[
= \mu(f(x))
\]

\[
< t.
\]

It is a contradiction. Hence (4) holds.

(4) \( \Rightarrow \) (5) Put \( \lambda = f^{-1}(\mu) \). It is easy.
(5) \implies (1) For each \( \mu \in \mathcal{Q}_\eta(f(x)_t, r) \), \( C_\eta(\bar{I} - \mu, r) = \Sigma - \mu \). By (5),
\[ e \in T_\eta(\bar{I} - f^{-1}(\mu), r) = \bar{I} - f^{-1}(\mu). \]
Since \( f(x)_t q \mu \) implies \( x_q f^{-1}(\mu) \), by Lemma \( \Box \Box \) (2), there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( e C_\tau(\lambda, r) \leq f^{-1}(\mu) \). It implies \( f(e C_\tau(\lambda, r)) \leq \mu \). Hence \( f \) is strongly \( \theta \)-e-continuous.

**Theorem 4.9.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \rightarrow Y \) a function. Let \((Y, \eta)\) be a fuzzy regular space. Then the following statements are equivalent:

1. \( f \) is weakly \( e \)-continuous.
2. \( f \) is \( e \)-continuous.
3. \( f \) is strongly \( \theta \)-e-continuous.

**Proof.**

(1) \implies (2) For \( \mu \in \mathcal{Q}_\eta(f(x)_t, r) \), since \((Y, \eta)\) is a fuzzy regular space, there exists \( \omega \in \mathcal{Q}_\eta(x_t, r) \) such that \( \mu \leq C_\eta(\omega, r) \leq \mu \).
Since \( f \) is weakly \( e \)-continuous, there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(\lambda) \leq C_\eta(\omega, r) \leq \mu \).

(2) \implies (3) For \( \mu \in \mathcal{Q}_\eta(f(x)_t, r) \), since \((Y, \eta)\) is a fuzzy regular space, there exists \( \mu \in \mathcal{Q}_\eta(x_t, r) \) such that \( \mu \leq C_\eta(\mu, r) \leq \mu \). Since \( f \) is \( e \)-continuous, there exists \( \lambda \in \mathcal{E}_\tau(x_t, r) \) such that \( f(\lambda) \leq \mu \). We will show that \( f(e C_\tau(\lambda, r)) \leq C_\eta(\mu, r) \).

Suppose
\[ f(e C_\tau(\lambda, r))(y) > t > C_\eta(\mu, r)(y). \]
Then there exist \( x \in X \) with \( f(x) = y \) and \( \rho \in \bar{I}^Y \), \( \mu \leq \rho \) with \( \eta(\bar{I} - \rho) \geq r \) such that
\[ f(e C_\tau(\lambda, r))(y) \geq e C_\tau(\lambda, r)(x) \]
\[ > t \]
\[ > \rho(f(x)) \]
\[ \geq C_\eta(\mu, r)(y). \]

On the other hand, since \( f \) is \( e \)-continuous, for \( \eta(\bar{I} - \rho) \geq r \), there exists \( \omega \in \mathcal{E}_\tau(x_t, r) \) such that \( f(\omega) \leq \bar{I} - \rho \). Thus
\[ \lambda \leq f^{-1}(\mu) \leq f^{-1}(\rho) \leq \bar{I} - \omega. \]
So, \( e C_\tau(\lambda, r)(x) \leq (\bar{I} - \omega)(x) < t \). It is a contradiction. Thus, \( f(e C_\tau(\lambda, r)) \leq C_\eta(\mu, r) \). Hence \( f \) is strongly \( \theta \)-e-continuous.

(3) \implies (1) It is trivial.

**Theorem 4.10.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s.

1. Every fuzzy continuous function \( f : X \rightarrow Y \) is strongly \( \theta \)-e-continuous iff \((X, \tau)\) is fuzzy \( \epsilon \)-regular.
(2) Every e-continuous function \( f : X \to Y \) is strongly \( \theta \)-e-continuous iff \((X, \tau)\) is fuzzy ee*-regular.

(3) Every supercontinuous function \( f : X \to Y \) is strongly \( \theta \)-e-continuous iff \((X, \tau)\) is fuzzy almost ee*-regular.

**Proof.**

(1) \((\Rightarrow)\) For an identity function \( f : (X, \tau) \to (Y, \sigma) \), by hypothesis, \( f \) is fuzzy continuous and strongly \( \theta \)-e-continuous. For each \( \mu \in Q_\eta(f(x)_t, r) \), there exists \( \lambda \in E_\tau(x_t, r) \) such that
\[
 f(eC_\tau(\lambda, r)) \leq \mu.
\]
Since \( eC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \) then
\[
f(eC_\tau(\lambda, r)) \leq f(C_\tau(\lambda, r)) \leq \mu.
\]
We have
\[
f^{-1}(\mu) = f(Q_\eta(f(x)_t, r)) = Q_\tau(f^{-1}f(x)_t, r) = Q_\tau(x_t, r).
\]
Since \( f \) is fuzzy continuous, then we have
\[
f(eC_\tau(\lambda, r)) \leq \mu \quad \Rightarrow \quad eC_\tau(\lambda, r) \leq f^{-1}(\mu)
\]
\[
\Rightarrow \quad eC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \leq f^{-1}(\mu).
\]
By Corollary \(4.14\), \((X, \tau)\) is fuzzy e-regular.

\((\Leftarrow)\) Let \( f \) be fuzzy continuous. For each \( \nu \in Q_\eta(f(x)_t, r) \), \( f^{-1}(\nu) \in Q_\tau(x_t, r) \). Since \((X, \tau)\) is fuzzy e-regular, there exists \( \mu \in E_\tau(x_t, r) \) such that \( \mu \leq eC_\tau(\mu, r) \leq f^{-1}(\nu) \). Thus, \( f(eC_\tau(\mu, r)) \leq \nu \). Hence \( f \) is strongly \( \theta \)-e-continuous.

(2) \((\Rightarrow)\) Since every fuzzy continuous function is fuzzy e-continuous then the proof followed by the necessary part of (1).

\((\Leftarrow)\) By Remark \(4.15\), since every fuzzy e-regular space is fuzzy ee*-regular. Then the proof followed by the sufficiency part of (1).

(3) Proof is similar from the above (1) and (2).

\(\square\)

**Theorem 4.11.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) a function. Let \((Y, \eta)\) be a ee*-regular space. Then the following statements are equivalent:

(1) \( f \) is strongly e-irresolute.

(2) \( f \) is e-irresolute.

(3) \( f \) is \( \theta \)-e-irresolute.

(4) \( f \) is quasi-e-irresolute.

**Proof.**

(1) \((\Rightarrow)\) (2), (2) \((\Rightarrow)\) (3) and (3) \((\Rightarrow)\) (4) are trivial from Theorem \(4.14\).

(3) \((\Rightarrow)\) For each \( \nu \in E_\eta(f(x)_t, r) \), since \((X, \eta)\) is fuzzy ee*-regular, there exists \( \mu \in e_\tau(f(x)_t, r) \) such that \( \mu \leq eC_\eta(\mu_t, r) \leq \nu \). for
\[ \mu \in e_*(f(x)_t, r), \text{ by (3), there exists } \lambda \in \mathcal{E}_r(x_t, r) \text{ such that } f(eC_\eta(\lambda, r)) \leq eC_\eta(\mu, r) \leq \nu. \text{ Hence } f \text{ is strongly } e\text{-irresolute.} \]

(4) \Rightarrow (2) For each \( r \)-feo \( \nu \), we only show that

\[ f^{-1}(\nu) = \nu(f(x)) \]

Then there exist \( x \in X \) and \( t \in I_0 \) such that

\[ f^{-1}(\nu)(x) = \nu(f(x)) \]

\[ > 1 - t \]

\[ > \nu(\lambda|x_t|) \leq f^{-1}(\nu), \lambda \text{ is } r \text{-feo}\} \]

For each \( \nu \in \mathcal{E}_\eta(f(x)_t, r) \), since \( (Y, \eta) \) is fuzzy \( ee^* \)-regular, there exists \( \mu \in \mathcal{E}_\eta(f(x)_t, r) \) such that \( \mu \leq eC_\eta(\mu, r) \leq \nu \). By (4), there exists \( \lambda \in \mathcal{E}_r(x_t, r) \) such that \( f(\lambda) \leq eC_\eta(\mu, r) \leq \nu \). Thus \( \lambda \leq f^{-1}(\nu) \) and \( \lambda \in \mathcal{E}_r(x_t, r) \) implies \( 1 - t < \lambda(x) \). It is a contradiction. Thus \( f^{-1}(\nu) \) is \( r \)-feo.

\[ \square \]

**Theorem 4.12.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \rightarrow Y \) a function. Let \((X, \tau)\) be a fuzzy \( ee^* \)-regular space. Then \( f \) is \( \theta \)-e-irresolute iff \( f \) is quasi \( e \)-irresolute.

**Proof.** Let \( f \) be quasi-\( e \)-irresolute. For each \( \nu \in \mathcal{E}_\eta(f(x)_t, r) \), there exists \( \lambda \in \mathcal{E}_r(x_t, r) \) such that \( f(\lambda) \leq eC_\eta(\nu, r) \). Since \((X, \tau)\) is fuzzy \( e^* \)-regular, there exists \( \mu \in \mathcal{E}_r(x_t, r) \) such that \( \mu \leq eC_\tau(\mu, r) \leq \lambda \). Hence \( f(eC_\tau(\mu, r)) \leq eC_\eta(\nu, r) \). Then \( f \) is \( \theta \)-e-irresolute.

\[ \square \]

**Acknowledgment.** The authors would like to thank the referees for their valuable comments and suggestions.

**References**


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