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L-topological spaces

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Abstract. By substituting the usual notion of open sets in a topological space X with a suitable collection of maps from X to a frame L, we introduce the notion of L-topological spaces. Then, we proceed to study the classical notions and properties of usual topological spaces to the newly defined mathematical notion. Our emphasis would be concentrated on the well understood classical connectedness, quotient and compactness notions, where we prove the Thychonoff's theorem and connectedness property for ultra product of L-compact and L-connected topological spaces, respectively.

1. Introduction

The concept of "Topology" on a set is a long standing notion covering, with no doubt, entirely whole of the mathematics and many other fields such as science, engineering, pharmacy and etc.. In mathematics, as a natural and immediate extension of metric spaces, topological spaces provide a wide framework to work in geometry, analysis, algebra and etc. One uses a collection of subsets of a set, admitting some properties, to provide a topology on this set. Then various related notions, e.g. connectedness, compactness, closure of sets and etc. appear as consequences.

A frame is a set admitting two operations analogous to maximum and minimum operations, with two elements as its supremum and infimum accompanied by few circumstances. In order to obtain an even larger framework to work on, in topological arguments, the operations in frames seem to be good candidates to be replaced by the notions of union and intersection of sets. Roughly speaking, we substitute the

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maps from a set X to a frame L, with subsets of X and we derive most of the classical notions related to a topological space in detail. We call such a datum, a L-topological space. Our formalism will generalize the classical topological spaces for some suitable choice of L. As well, for a fixed frame L, we introduce the notion of continuous maps within L-topological spaces; showing that the collection of L-topological spaces on a fixed frame, constitutes the structure of a category. Meanwhile, moving L in the category of frames, the collection of topologies on frames admits the structure of multi categories.

The notions of L-connectedness and L-compactness for maps from a L-topological space to a frame will be introduced by the joint and meet operations in the frame. Our choice of frames and the operations therein would be justified by the fact that a subset in the usual topology would be open, connected or compact exactly when its characteristic map is L-open, L-connected or L-compact, respectively. Furthermore, all the mentioned notions are relative in our more extensive context, in the sense that, they might change by varying the base frame.

While the Thychonov's theorem for L-topological spaces might be obtained by a similar argument as in the classical case, we prove it for ultra product of L-topological spaces by means of L-quotient topological spaces.

The notion of closed maps in L-topological spaces are subtle and needs more care in practice. When L admits enough complements, for a map $s : X \to L$, a direct application of the well known "Axiom of choice", guarantees the existence of a map $t : X \to L$, playing the role of complement for s. We use this phenomena to introduce the notion of L-closed maps.

Few elementary facts and notions about lattices are given in Section 1.1. One can consult math. logic theory books, e.g. [3], for more details on theory of Lattices. As the main theme of the paper, we introduce the notion of L-topological spaces in Section 2. Various dependent notions will accompany the notion of L-topological spaces in this section. The L-connected and L-compact topological spaces, as well as L-connected and L-compact maps between L-topological spaces; and various depended arguments would be introduced and discussed in Sections 3 and 4, respectively. Sections 5 and 6 introduce and discuss on the quotient L-topological spaces and on the notion of closedness in a L-topological space. The theory will be accompanied by various explicit examples.

1.1. **Preliminaries and Notations.**

Definition 1.1. (i) A lattice $(L; \vee, \wedge)$, would be called bounded, if there exists elements 0 and 1 in L, such that for each $a \in L$

one has $a \vee 0 = a$ and $a \wedge 1 = a$. This obviously implies that the members 0 and 1 are unique, as well as, for each $a \in L$ one has $0 \leq a \leq 1$.

(ii) A bounded lattice $(L; \vee, \wedge, 0, 1)$, abbreviated by L, is called complete, if an arbitrary joint and arbitrary meet of its elements exist.

(iii) A frame is a complete bounded lattice L in which the arbitrary distribution law is hold for its elements, i.e. the equality

$$x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y}(x \wedge y),$$

is valid for $x \in L$ and for an arbitrary subset Y of L. It can be verified easily that one has $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y}(x \vee y)$, in a frame L.

Definition 1.2. (i) If L is a frame, a subset $A \subseteq L$ is called a filter if it is closed under the meet action of L and for $x \in A$ the relation $y \geq x$ implies $y \in A$.

(ii) For any set I, an ultra-filter Γ of the Boolean algebra $P(I)$ is a filter in $P(I)$ which is maximal with respect to the property: "0 does not belong to Γ". An ultra-filter Γ is called an ultra-filter over I.

(iii) Let $\{X_i\}_{i \in I}$ be a collection of sets and Γ be an ultra-filter over I. One defines a relation R_Γ on $\prod_{i \in I} X_i$, as $(x_1, x_2) \in R_\Gamma$ if and only if $\{i \in I | f(i) = g(i)\} \in \Gamma$. For a family $\{X_i\}_{i \in I}$ of sets and an ultra-filter Γ over I, the ultra-product $\prod_{i \in I} X_i / \Gamma$ is defined to be the quotient set $\prod_{i \in I} X_i / R_\Gamma$. We call $\prod_{i \in I} X_i / \Gamma$, the ultra-product of $\{X_i\}$ with respect to Γ.

Definition 1.3. An element b in a bounded lattice L is called a complement for $a \in L$ if one has $a \vee b = 1$ and $a \wedge b = 0$.

2. L-Topological Spaces

Definition 2.1. Let $(L; \vee, \wedge, 0, 1)$ be a frame and X be a non-empty set. We denote by f_0 and f_1 the constant maps sending elements of X to 0 and 1, respectively. Particularly, one has $f_0, f_1 \in L^X$.

For $f, g \in L^X$, we define $f \leq g$ if and only if for each $x \in X$ one has $f(x) \leq g(x)$.

Definition 2.2. (I) Let X be a set and $\mathcal{T}_L = \{s_\alpha\}_{\alpha \in I}$ be a collection of L-maps of X, i.e. $\{s_\alpha\}_{\alpha \in I} \subseteq L^X$, such that:

(i) $f_0, f_1 \in \mathcal{T}_L$,

(ii) For a non-empty collection $\{s_\alpha\}_{\alpha \in J}$ in $\mathcal{T}_L$, one has $\vee_{\alpha \in J} s_\alpha \in \mathcal{T}_L$,

(iii) The meet of a finite collection of members of $\mathcal{T}_L$ belongs to $\mathcal{T}_L$, i.e. $\wedge_{i=1}^{i=n} s_i \in \mathcal{T}_L$ provided that $s_i \in \mathcal{T}_L$.

Then, the couple $(X, \mathcal{T}_L)$ will be called a L-topological space and the members of $\mathcal{T}_L$ are the L-open sets of this L-topological space.

(II) We call a set U in X open if $\chi_U \in \mathcal{T}_L$ and closed if $\chi_{U^c} \in \mathcal{T}_L$.

(III) Let X be an L-topological space and Y be a subset of X. The family of maps $\{(s_\alpha)_{|Y} : s_\alpha \in \mathcal{T}_L\}$ impose a L-topological structure on Y. We call this topology, the L-subspace topology on Y.

(IV) An L-open set $s \in \mathcal{T}_L$ is called a L-neighborhood of $x \in X$, if $\chi_{\{x\}} \leq s$. An L-open subset f contains an L-open subset g if $g \leq f$.

(V) Let X be a non-empty set and τ_L^1 and τ_L^2 be L-topologies on X. The L-topology τ_L^1 is called finer than τ_L^2 if each $s \in \tau_L^2$ can be written as $\vee_\alpha h_\alpha$ for some $h_\alpha \in \tau_L^1$. Two L-topologies τ_L^1 and τ_L^2 on X are called equivalent if τ_L^1 is finer than τ_L^2 and τ_L^2 is finer than τ_L^1.

Definition 2.3. Let L_1 and L_2 be frames and $(X, \tau_{L_1}^X)$ and $(Y, \tau_{L_2}^Y)$ are L_1, L_2 ordered topological spaces, respectively. A map $f : (X, \tau_{L_1}^X) \to (Y, \tau_{L_2}^Y)$ is called continuous if $\sigma \circ f \in \tau_{L_1}^X$ for each $\sigma \in \tau_{L_2}^Y$. Obviously the composition of continuous maps within L-topological spaces is continuous. Furthermore, a continuous function f from $(X, \tau_{L_1}^X)$ to $(Y, \tau_{L_2}^Y)$ induces a L_2-topology on X. This L-topology coincides on the original L-topology on X when $L_1 = L_2 := L$ and $f = I_X$.

Remark 2.4. (I) For a fixed frame L, the family of L-topological spaces together with the continuous maps consists a category. Taking $L = \{0, 1\}$ and for an open subset Y of a topological space X setting $s_Y := \chi_Y$, the collection

$$\{s_Y |\ \text{Y is an open subset of X}\},$$

constitutes a L-topological space.

(II) The family of L-topological spaces, when L moves in the family of frames, together with the continuous maps consists a multi-category. See [1] for more details on the notion of multi-categories.

Definition 2.5. An L-topology basis is a set $\mathcal{B}^L \subseteq L^X$ such that

(1) $\vee_{s_\alpha \in \mathcal{B}^L} s_\alpha = f_1$,
(2) for each s and t in $\mathcal{B}^L$ we have $s \wedge t = \vee s_\gamma$, where $s_\gamma \in \mathcal{B}^L$.

If $\mathcal{B}^L$ is L-topology basis, then the set $\mathcal{T}^L_{\mathcal{B}^L} = \{\vee s_\gamma | s_\gamma \in \mathcal{B}^L\}$ is called the L-topology generated by $\mathcal{B}^L$. Obviously any L-topological space admits a L-topological basis.

Example 2.6. (a) In the discrete L-topology $\mathcal{T}^L_X = L^X$, i.e. all L-functions are open.

(b) Let X be a non-empty set and define

$$\mathcal{B}^L := \{s : X \to L| \quad \text{Card (Z(s)) is finite}\} \cup \{f_0\},$$

where $Z(s)$ denotes the set $\{\alpha \in X | s(\alpha) \neq 1\}$. Then $\mathcal{B}^L$ is a L-topology basis on X. The L-topology generated by $\mathcal{B}^L$ is an analogue of the finite complement topology.

(c) Let X be an arbitrary set, $(L, \vee, \wedge, 0, 1)$ be a chain and for each $l \in L$ define $f_l(x) = l$. Then

$$\mathcal{T} := \{f_l| \quad l \in L\},$$

gives rise to a L-topology on X.

Lemma 2.7. *Let $\{(X_i, \tau^i_L)\}_{i\in I}$ be an arbitrary collection of L-topological spaces and $\{\mathcal{B}^L_{X_i}\}_{i\in I}$ be L-topology basis for X_i's, respectively. Then, the set*

$$\mathcal{B}^L_{\prod X_i} := \prod \mathcal{B}^L_{X_i} = \{\{s_i\}_{i\in I} | s_i \in \mathcal{B}^L_{X_i}\},$$

is a L-topology basis for $\prod_{i\in I} X_i$, where $\{s_i\}_{i\in I}$ is the map from $\prod_{i\in I} X_i$ to L defined by $(\{s_i\}_{i\in I})(\{x_i\}_{i\in I}) = \wedge_{i\in I} s_i(x_i)$.

Proof. Since $\mathcal{B}^L_{X_i}$ is L-topology basis for X_i, one has

$$\begin{aligned} \vee_{i\in I}\{s_{i,j}\}(\{x_i\}) &= \vee_{s_i\in I}[\wedge_{j\in I} s_j(x_i)] \\ &= \wedge_{j\in I}(\vee_{s_i\in I} s_j(x_i)) \\ &= \{f_1^{X_i}\}_{i\in I} \\ &= f_1^{\prod X_i}. \end{aligned}$$

This verifies property (1).

For each $\{s_i\}$ and $\{t_i\}$ in $\mathcal{B}^L_{\prod X_i}$ one has

$$\begin{aligned} (\{s_i\} \wedge \{t_i\})(\{x_i\}) &= [\{s_i\}(\{x_i\})] \wedge [\{t_i\}](\{x_i\}) \\ (\wedge s_i(x_i)) \wedge (\wedge t_i(x_i)) &= \wedge((s_i \wedge t_i)(x_i)) \\ \wedge(\vee s_{i,j}(x_i)) &= \vee\{s_{i,j}\}\{x_i\}, \end{aligned}$$

establishing property (2). $\square$

3. L-Connected Topological Spaces

Definition 3.1. (I) Assume (X, τ_L^X) be a L-topological space and $(s, \acute{s})$ a couple of L-open subsets of X such that $s, \acute{s} \in \tau_L^X \setminus \{f_0^X, f_1^X\}$. We call $(s, \acute{s})$ an L-separation for (X, τ_L^X) if $s \vee \acute{s} = f_1^X$ and $s \wedge \acute{s} = f_0^X$. An L-topological space (X, τ_L^X) is called L-connected if it fails to admit any L-separation. $Y \subseteq X$ is called a connected subset of X, if it is connected with respect to the induced L-subspace topology.

(II) A couple of L-open sets (s_1, s_2) with $s_1, s_2 \in \tau_L^X \setminus \{f_0^X, f_1^X\}$, is called an L-separation for a map $f : X \to L$ if $f \wedge (s_1 \wedge s_2) = f_0^X$ and $f \leq s_1 \vee s_2$. A map $f : X \to L$ is called L-connected if it fails to admit any L-separation.

Remark 3.2. It is clear that a set Y in a L-topological space X is L-connected if and only if χ_Y is L-connected.

Example 3.3. (a) Let $X = \mathbb{R}$ and $L = \{0, \frac{1}{2}, 1\}$. Define s_1 by $s_1(t) = 0$ for $t \leq \frac{1}{2}$ and $s_1(t) = \frac{1}{2}$ for $t > \frac{1}{2}$. Similarly define s_2 by $s_2(t) = 0$ for $t \leq 0$, $s_2(t) = \frac{1}{2}$ for $0 < t \leq \frac{1}{2}$ and $s_2(t) = 1$ for $\frac{1}{2} < t$. The L-topology generated by the L-topological basis $\mathcal{B}_1 = \{f_0, f_1, s_1, s_2, s_1 \wedge s_2\}$ is easily seen to be L-connected.

(b) The L-topology generated by $\mathcal{B}_2 = \{f_0, f_1, s_1, s_2, s_1 \wedge s_2\}$ fails to be an L-connected topological space, where we have defined s_1 by $s_1(t) = 0$ for $t \leq 0$, $s_1(t) = 1$ for $t \geq 0$ and s_2 by $s_2(t) = 1$ for $t \leq 0$, $s_2(t) = 0$ for $t \geq 0$.

Theorem 3.4. *Assume that $f : X \to Y$ is L-continuous and surjective with X an L-connected topological space. Then Y is L-connected.*

Proof. This is easy to prove and we leave it to the reader. □

A function of frames $e : L_1 \to L_2$ is said to be monotone if for each $x, y \in L_1$ the inequality $x \leq y$ implies $e(x) \leq e(y)$.

Lemma 3.5. *Let $e : L_1 \to L_2$ be a monotone map of frames. Then*

$$e(x \vee y) = e(x) \vee e(y), \qquad e(x \wedge y) = e(x) \wedge e(y).$$

Proof. This is easy to prove and we leave it to the reader. □

Theorem 3.6. *Let L_1 and L_2 be frames and $e : L_1 \to L_2$ a monotone function within frames with $e(0_{L_1}) = 0_{L_2}$. Then a map $f : X \to L_1$ is L_1-connected if and only if $e \circ f : X \to L_2$ is L_2-connected.*

Proof. If $\{s_1, s_2\}$ is a L_1-separation for f with respect to $\mathcal{T}_X^{L_1}$, then we would have

$$(e \circ s_1) \vee (e \circ s_2) = e \circ (s_1 \vee s_2) \geq e \circ f,$$
$$(e \circ s_1) \wedge (e \circ s_2) \wedge (e \circ f) = e \circ (s_1 \wedge s_2 \wedge f) = f_0^X,$$

by Lemma 3.5. So $\{e \circ s_1, e \circ s_2\}$ is a L_2-separation for $e \circ f$ with respect to $\mathcal{T}_X^{L_2}$. This verifies the "if" part.

Assume that f is L_1-connected and $\{\bar{s}_1, \bar{s}_2\}$ are L_2-separations for $e \circ f$. For $i = 1, 2$ define $s_i(x) = e^{-1}(\bar{s}_i)(x)$ if $x \in (s_i)^{-1}(e(L_1))$ and $s_i(x) = 1_{L_1}$ otherwise. Then $\{s_1, s_2\}$ is a L_1-separation for f, which is impossible. □

Theorem 3.7. *Assume that $\{X_\alpha\}_{\alpha \in I}$ is a family of L-connected topological subspaces of a L-topological space X. If $\cap_{\alpha \in I} X_\alpha \neq \emptyset$, then X is L-connected too.*

Proof. Suppose $|I| = 2$, $X = X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$ and assume on the contrary that $(s, \acute{s})$ is a L-separation for X. Setting $s_i = s_{|X_i}, \acute{s}_i = \acute{s}_{|X_i}$ for $i = 1, 2$, we prove that $(s_i, \acute{s}_i)$ imposes a L-separation on X_i either for $i = 1$ or $i = 2$. If $(s_i, \acute{s}_i)$ fails to be a L-seperation on X_i for $i = 1$ and $i = 2$, then we may assume that $s_1 = f_0^{X_1}$. This implies that $s_2 \neq f_0^{X_2}$ and so, observing that $(s_2, \acute{s}_2)$ is a L-separation on X_2, we obtain $\acute{s}_2 = f_0^{X_2}$. This by our very definition of a L-separation on X is equivalent to $s_2 = s_{|X_2} = f_1^{X_2}$. Evaluating s at x_0 in two different ways:

$$s(x_0) = s_{|X_1}(x_0) = 0,$$
$$s(x_0) = s_{|X_2}(x_0) \neq 0,$$

we derive a contradiction. This contradiction implies that either X_1 or X_2 has to be disconnected, which is impossible. So X has to be L-connected. The proof goes verbatim when I is an arbitrary indexed set. □

Proposition 3.8. *Let (X, τ_X^L) and (Y, τ_Y^L) be L-connected topological spaces. Then $X \times Y$ is L-connected too.*

Proof. Take $(a, b) \in X \times Y$ and observe that using Theorem 3.7, for arbitrary $x \in X$ the set $\Gamma_x = X \times \{b\} \cup \{x\} \times Y$ turns to be L-connected. Using the obvious equality $X \times Y = \cup_{x \in X} \Gamma_x$ together with Theorem 3.7, we obtain the result. □

Theorem 3.9. *Let $(L_j)_{j \in J}$ be a family of frames, $\{X_\alpha, \tau_{X_\alpha}^{L_j}\}_{\alpha \in J}$ a collection of L_j-topological spaces and $\{f_\alpha : X_\alpha \to L_\alpha\}_{\alpha \in J}$ a collection of maps. Then $\prod_{\alpha \in J} f_\alpha$ is $(\prod_{\alpha \in J} L_\alpha)$-connected if and only if f_j is L_j-connected for each $j \in J$.*

Proof. Let $\{\bar{s}_1 := (s^1_\alpha)_{\alpha\in J}, \bar{s}_2 := (s^2_\alpha)_{\alpha\in J}\}$ be a $(\prod_{\alpha\in J} L_\alpha)$-separation for $\prod_{\alpha\in J} f_\alpha$. Then, for at least one $j \in J$ we have

$$s^1_j \vee s^2_j \geq f_j, \qquad s^1_j \wedge s^2_j \wedge f_j = f_0^{X_j}, \qquad s^1_j \neq f^0_{X_j}, \qquad s^2_j \neq f_j.$$

So f_j fails to be L_j-connected, which is impossible.

For $j \in J$, if s_1, s_2 be a L_j-separation for f_j, then $s_1 \times (1_{f_\alpha})_{\alpha\in J\setminus\{j\}}, s_2 \times (1_{f_\alpha})_{\alpha\in J\setminus\{j\}}$ is a $(\prod_{\alpha\in J} L_\alpha)$-separation for $\prod_{\alpha\in J} f_\alpha$. □

Theorem 3.10. *Let L be a distributive frame admitting enough complements and $\mathcal{T}^L_X$ a L-topology on X such that the constant functions are open in this L-topology.*

(a) *For $b \in L$ if a map $f : X \to L$ is L-connected then $f \vee b : X \to L$ is L-connected.*
(b) *If the L-maps $f_1 : X \to L$ and $f_2 : Y \to L$ are L-connected, then the map $f_1 \vee f_2 : X \times Y \to L$ is L-connected too.*

Proof. (a) If (s_1, s_2) is a L-separation for $f \vee b$, then the set

$$\{(s_1 \wedge b^c), (s_2 \wedge b^c) \vee (f \wedge b)\},$$

turns to be a L-separation for f.

(b) Assume that f_1, f_2 are L-connected and (s_1, s_2) is a L-separation for $f_1 \vee f_2$. For a fixed $a \in Y$, the restrictions of s_1 and s_2 to $X \times \{a\}$, gives rise to a L-separation for $f_1 \vee f_2(a)$. So $f_1 \vee f_2(a)$ fails to be L-connected by (a), which is absurd.

□

4. L-Compact Topological Spaces

4.1. L-Compactness.

Definition 4.1. (I) A L-topological space X is called L-compact if for each family of L-open subsets $\{s_\alpha\}$ in τ_L such that $\vee s_\alpha = f_1$, one can find a finite number of $s_\alpha, \alpha = 1, \cdots, n$ such that $\vee_{\alpha=1}^{\alpha=n} s_\alpha = f_1$. A L-topological space (Y, τ^L_Y) is called a L-compact subset of X, if it is compact with respect to the induced L-subspace topology.

(II) A map $f : X \to L$ is called L-compact if the existence of a family of L-open subsets $\{s_\alpha\}$ in τ_L such that $\vee s_\alpha = f$ implies that there exist finite number of $s_\alpha, \alpha = 1, \cdots, n$ such that $\vee_{\alpha=1}^{\alpha=n} s_\alpha = f$.

Remark 4.2. (I) A L-subspace $Y \subseteq X$ is L-compact if and only if χ_Y is L-compact.

(II) A L-topological space X is L-compact if and only if for any collection of open sets with the property that no finite sub-collection covers, there exists a point $x \in X$ such that x is not covered by the collection of open sets.

Example 4.3. Let $X = [0,1]$ and $L = (X, \min, \max, 0, 1)$. For any natural number $n \geq 1$ define a function $f_n : X \to L$ by $f_n(t) = 1 - \frac{1}{n}$, for each $t \in X$. With this assignments, the collection of $\{f_n\}_{n\in\mathbb{N}}$ is a L-topology on X and $\{f_n\}_{n\geq 1}$ is an L-open covering for f_1^X. Indeed for any $a \in X$, one has $(\vee_n f_n)(a) = \limsup\{1 - \frac{1}{n} | n \geq 1\} = 1$. This cover of f_X^1, obviously fails to admit any finite sub-collection. So f_X^1 fails to be compact.

Theorem 4.4. *Assume that $f : X \to Y$ is L-continuous and surjective. Then Y is L-compact provided X be L-compact.*

Proof. If $\{s_\alpha\}_{\alpha\in I}$ be an L-cover for Y, then $\{\acute{s}_\alpha := f o s_\alpha\}_{\alpha\in I}$ would be an L-cover for X. This establishes the lemma. □

Theorem 4.5. *Let L_1 and L_2 be frames and $e : L_1 \to L_2$ a function within frames.*

(I) *Consider the L_2-topology on X induced by e, $\mathcal{T}_X^{L_2}$. Then X is compact with respect to $\mathcal{T}_X^{L_1}$ when it is compact with respect to $\mathcal{T}_X^{L_2}$.*

(II) *If $e : L_1 \to L_2$ is strictly monotone with $e(0_{L_1}) = 0_{L_2}$. Then, a map $f : X \to L_1$ is L_1-compact if and only if $e \circ f : X \to L_2$ is L_2-compact.*

Proof. The proof is similar to the proof of theorem 3.6. □

Theorem 4.6. *Let $(L_\alpha)_{\alpha\in J}$ be a family of frames with $|J| = n$ and $\{X_\alpha, \tau_{X_\alpha}^{L_\alpha}\}_{\alpha\in J}$ be a collection of L_α-topological spaces and $\{f_\alpha : X_\alpha \to L_\alpha\}_{\alpha\in J}$ be a collection of maps. Then $\prod_{\alpha\in J} f_\alpha$ is $(\prod_{\alpha\in J} L_\alpha)$-compact if and only if f_α is L_α-compact for each $\alpha \in J$.*

Proof. It is enough to prove the theorem when $|J| = 2$. Assume that f_1 and f_2 are L_1 and L_2-compact, respectively. If $\{s_i\}_{i\in I}$ be a cover for $f_1 \times f_2$; define t_i and $\bar{t}_i$ by $t_i := \pi_1 \circ s_i \circ i_1$ and $\bar{t}_i := \pi_2 \circ s_i \circ i_2$. Then, one can see easily that $\vee_i t_i = f_1$ and $\vee_i \bar{t}_i = f_2$. By compactness of f_1 and f_2, set $\vee_{i=1}^{i=n} t_i = f_1$, $\vee_{i=1}^{i=m} \bar{t}_i = f_2$. Let $m \leq n$ and notice that for each $i \in I$ one has $t_i \times \bar{t}_i = s_i$. Consider now that the equalities

$$\begin{aligned}
[\vee_{i=1}^{i=n}(t_i \times \bar{t}_i)](x,y) &= \vee_{i=1}^{i=n}(t_i \times \bar{t}_i)(x,y) \\
&= (\vee_{i=1}^{i=n} t_i(x,y), \vee_{i=1}^{i=n} \bar{t}_i(x,y)) \\
&= (f_1(x), f_2(y)) \\
&= (f_1 \times f_2)(x,y),
\end{aligned}$$

imply that there exists a finite sub-cover of $\{s_i\}_{i\in I}$ for $f_1 \times f_2$.

Assume for the converse that $f_1 \times f_2$ is $L_1 \times L_2$-compact and let $\{s_i\}$ be a L_1-covering for f_1. Then $\{s_i \times f_2\}_i$ would be a $L_1 \times L_2$-cover for $f_1 \times f_2$. Therefore a finite number of them has to cover $f_1 \times f_2$, implying the existence of a finite sub-cover of $\{s_i\}$ for f_1. □

Theorem 4.7. *Let L be a frame admitting enough complements in the sence that any $a \in L$ has at least a complement.*

(a) *For a map $f : X \to L$ and $b \in L$ the map f is L-compact if $f \vee b : X \to L$ and $f \wedge b : X \to L$ are L-compact.*

(b) *For L-maps $f_1 : X \to L$ and $f_2 : Y \to L$ assume that the maps $f_1 \vee f_2 : X \times Y \to L$ and $f_1 \wedge f_2 : X \times Y \to L$ are L-compact. Then $f_1 : X \times Y \to L$ and $f_2 : X \times Y \to L$ are L-compact.*

Proof. (a) If $\{s_i\}$ is a L-cover for f, then $\{s_i \vee b\}$ and $\{s_i \wedge b\}$ would be L-covers for $f \vee b$ and $f \wedge b$, respectively. Therefore one has $\vee_{i=1}^{i=n}(\acute{s}_i \vee b) = f \vee b$ and $\vee_{i=1}^{i=m}(\bar{s}_i \wedge b) = f \wedge b$ for $\acute{s}_i, \bar{s}_j \in \{s_i\}$. Consider moreover that one has $[(f \vee b) \wedge b^c] \vee [f \wedge b] = f$, and so $[(\vee_{i=1}^{i=n}(\acute{s}_i \vee b) \wedge b^c] \vee [\vee_{i=1}^{i=m}(\bar{s}_i \wedge b] = f$. From this we conclude $(\vee_{i=1}^{i=m}(\bar{s}_i)) \vee (\vee_{i=1}^{i=m}(\acute{s}_i)) = f$. It is enough to verify the equality for $m = n = 1$, i.e. assuming $[(\acute{s}_1 \vee b) \wedge b^c] \vee [(\bar{s}_1 \wedge b] = f$ we prove that $\acute{s}_1 \vee \bar{s}_1 = f$. The chain of relations

$$\begin{aligned} f &= [(s_1 \vee b) \wedge b^c] \vee (s_2 \wedge b) \\ &= s_2 \vee (s_1 \wedge b^c) \vee (s_2 \wedge b^c) \\ &= s_2 \vee (s_1 \wedge b^c) \\ &\leq s_2 \vee s_1, \end{aligned}$$

imply that $s_2 \vee s_1 = f$.

(b) For $y \in Y$ define

$$h_y(a, y) = f_1(a) \vee f_2(y), \qquad k_y(a, y) = f_1(a) \wedge f_2(y),$$

and $h_y(a, b) = k_y(a, y) = f_1(a)$ for $b \neq y$. Then one has

$$f_1 \vee f_2 = \vee_{y\in Y} h_y, \qquad f_1 \wedge f_2 = \vee_{y\in Y} k_y,$$

and there exists a point $y_0 \in Y$ which for this y_0 both of the L-functions hy_0 and k_{y_0} are L-compact. An analogue argument as in (a) implies that f_1 is L-compact.

□

Example 4.8. Assume X and L are as in Example 4.3 and $b = 1$. Then, for any L-compact map f one has $f \vee b = f_X^1$, which fails to be compact. Therefore, the L-comactness of f does not guarantee that of $f \vee b$, even if one wishes to take L a chain.

4.2. **Thychonoff's Theorem for L-Topological Spaces.**

Definition 4.9. (I) A point $x \in X$ is called a L-limit point of the set $E \subseteq X$ if for each neighborhood S_x of x, one has $f_0 < S_x \wedge \chi_{E\setminus\{x\}}$.

(II) A point $x \in X$ is called a perfect L-limit point of the set $E \subseteq X$ if for each neighborhood S_x of x, one has

$$|\{\alpha \in X|(S_x \wedge \chi_E)(\alpha)\}| = |E|,$$

where we denote by $|Z|$ the cardinality of a set Z.

Theorem 4.10. *A L-topological space X is L-compact if and only if each infinite subset E of X has at least a perfect L-limit point.*

Proof. If E is an infinite subset of X that fails to admit perfect L-limit points, then for $x \in X$, denoting by S_x a neighborhood of x with $|A_x| < |E|$, we obtain $\vee_{x\in X} S_x = f_1^X$. This together with the L-compactness of X implies that $\vee_{i=1}^n S_{x_i} = f_1^X$ for some $n \in \mathbb{N}$. Therefore, $\chi_E = \chi_E \wedge f_1^X = \vee_{i=1}^n (S_{x_i} \wedge \chi_E)$. So

$$|E| = |\{\alpha \in X : (\vee_{i=1}^n (S_{x_i} \wedge \chi_E))(\alpha) \neq 0\}| = \sum_{i=1}^n |A_{x_i}| < |E|,$$

which is a contradiction. Summarizing, any infinite subset of a L-compact topological space admits perfect L-limit points. The converse is a direct generalization of proof of [6, (C)], which we omit. □

Remark 4.11. A direct equivalent reformulation for L-compactness, as it exists for usual notion of topology, is as follows: A L-topological space X is L-compact if for each collection $\mathcal{S}$ of L-open maps, if no finite sub collection of $\mathcal{S}$ covers f_1, then $\mathcal{S}$ wont cover f_1.

Proposition 4.12. *Let $\{X_\alpha\}_{\alpha\in I}$ be an arbitrary collection of L-compact topological spaces. Then their product, $\prod_{\alpha\in I} X_\alpha$, is L-compact too.*

Proof. Assume first that $|I| = 2$ and let $\mathcal{S}$ be a collection of open sets of $X_1 \times X_2$ such that no finite sub-collection of $\mathcal{S}$ covers. There exists $a_1 \in X_1$ such that for each L-open neighborhood S_{a_1} of a_1 the L-open set $S_{a_1} \times f_1^{X_2}$ fails to be covered by any finite sub-collection of $\mathcal{S}$. Indeed otherwise if S_a denotes the L-open neighborhood of $a \in X_1$ such that a finite sub-cover of $\mathcal{S}$ covers $S_a \times f_1^{X_2}$, then the L-compactness of X_1 implies $\vee_{t\in T} S_t = f_1^{X_1\times X_2}$, where T is a finite subset of I and $S_t \in \mathcal{S}$. This contradiction asserts the existence of $a_1 \in X_1$ as desired.

There exists $a_2 \in X_2$ such that any L-open neighborhood $S_{a_1} \times S_{a_2}$ of (a_1, a_2) fails to be covered by any finite sub-collection of $\mathcal{S}$. Therefore (a_1, a_2) won't be covered by any element of $\mathcal{S}$. So $X_1 \times X_2$ turns to be compact by remark 4.11.

In order to establish the assertion for an arbitrary collection of L-topological spaces; assume that I is well ordered and $\mathcal{S}$ is a covering such that none of its finite sub-collections covers $X = \prod_{\alpha \in I} X_\alpha$. We inductively choose $a_\beta \in X_\beta$ such that if $\bar{\mathcal{S}}$ is any basic L-open set containing $((a_t)_{t \leq \beta}) \times (\prod X_{t > \beta})$, then no finite sub-collection of $\mathcal{S}$ covers $\bar{\mathcal{S}}$. Therefore $(a_t)_{t \leq \beta}$ fails to be covered by any member of $\mathcal{S}$ and X turns to be compact again by remark 4.11. □

5. Quotient L-Topological Spaces

Definition 5.1. Assume that $(X, \mathcal{T}^L)$ is an L-topological space and $\sim$ an equivalence relation on X. A collection $\bar{\mathcal{T}}_Q^L = \{\bar{s}\}$ of L-functions on X is called an L-quotient topology on $X/\sim$, if for any $\bar{s} \in \bar{\mathcal{T}}_Q^L$, the map $\bar{s} \circ \pi : X \to L$ is open with respect to the topology $\mathcal{T}^L$, i.e. there exists L-opens $\{s_\alpha\}$ with $\{s_\alpha\} \subseteq \mathcal{T}^L$ such that $\vee_\alpha s_\alpha = \bar{s} \circ \pi$, where $\pi : X \to X/\sim$ is the quotient map.

Remark 5.2.
(I) For any equivalence relation on an L-topological $(X, \mathcal{T}^L)$, the set of L-quotient topological spaces on $X/\sim$ is non-empty by the axiom of choice.
(II) Two L-quotient topologies on $X/\sim$ are equivalent. So for each equivalence relation on X, we will take one such L-topology to work on, and we will denote this L-topological space by $(\tilde{X}^L, \tilde{\mathcal{T}}^L)$.
(III) Obviously the quotient map $\pi : X \to \tilde{X}^L$ defined by $x \mapsto [x]$, is continuous with respect to L-quotient topology on $\tilde{X}^L$ by very definition of the L-quotient topology. As a direct consequence of this fact; the L-quotient space of L-compact L-topological spaces turns to be L-compact.

Theorem 5.3.
(a) *For any L-topological space Y, a map $f : \tilde{X}^L \to Y$ is continuous if and only if $f \circ \pi$ is continuous.*
(b) *The L-quotient topology is the unique L- topoloy on $\tilde{X}^L$ for which the property* (*a*) *holds.*
(c) *The composition of L-quotient maps is again an L-quotient map.*

Proof. (a) Since the continuity property remains valid under composition of functions in the category of L-topological spaces, the continuity of f guarantees that of $f \circ \pi$. Conversely, assume $f \circ \pi : X \to \tilde{X}^L \to Y$ is continuous. This implies that for $\sigma \in \mathcal{T}_Y^L$ the L-function $\sigma \circ (f \circ \pi)$ belongs to $\mathcal{T}_X^L$. So $\sigma \circ f$ belongs to $\tilde{\mathcal{T}}_{\tilde{X}}^L$ by very definition of quotient spaces. This completes continuity of f.

(b) Assume $\tilde{\mathcal{T}}_1^L$ and $\tilde{\mathcal{T}}_2^L$ are L-topologies on $\tilde{X}^L$ which property (a) holds for them. Applying property (a) for these L-topologies, we conclude that the identity map

$$id_{\tilde{X}^L} : (\tilde{X}, \tilde{\mathcal{T}}_1^L) \to (\tilde{X}, \tilde{\mathcal{T}}_2^L),$$

is an L-homeomorphism of L-topological spaces. This implies that the L-topologies $\tilde{\mathcal{T}}_1^L$ and $\tilde{\mathcal{T}}_2^L$ are equivalent on $\tilde{X}^L$.

(c) This is easy and we leave it to the reader.

□

Lemma 5.4. *Suppose $\pi : X \to \tilde{X}^L$ is a L-quotient map, Z an L-topological space, and $f : X \to Z$ is a continuous map that is constant on the fibers of π. Then there exists a unique continuous map $\tilde{f} : \tilde{X}^L \to Z$ such that $f = \tilde{f} \circ \pi$.*

Proof. The existence of $\tilde{f}$ satisfying $f = \tilde{f} \circ \pi$, is immediate by the universal property of quotient sets. Meanwhile; the continuity of $\tilde{f}$ is a direct consequence of Theorem 5.3 (a). The uniqueness part is trivial. □

Corollary 5.5. (a) *The Thychonoff's theorem is valid for ultra product of L-compact topological spaces.*

(b) *The ultra product of L-connected topological spaces is an L-connected topological space.*

Proof. (a) Theorem 5.3 implies that for each equivalence relation U on an arbitrary collection of L-topological spaces $\{X_\alpha\}_{\alpha \in J}$ the set $\prod_{\alpha \in J} X_\alpha / U$ admits the structure of an L-topological space. This means in its own right that the ultraproduct of L-topological spaces is again an L-topological space. The same reason together with proposition 4.12 implies that $\prod_{\alpha \in J} X_\alpha / U$ is the L-continuous image of an L-compact L-topological space. This, by theorem 4.4, concludes the assertion.

(b) An analogous argument as in (a) together with Proposition 3.8 implies the assertion.

□

6. Closedness and Closure in L-Topological Spaces

Let L be a frame admitting enough complements, in the sense that for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. As an application of the axiom of choice, it turns to be possible to define complements of a L-map $s : X \to L$, namely a map $t : X \to L$ with the properties:

(1) $s \vee t = f_X^1$,

(2) $s \wedge t = f_X^0$.

Definition 6.1. (i) Let $(L; \vee, \wedge, 0, 1)$ be a frame admitting enough complements and $(X, \mathcal{T}_X^L)$ be a L-topology. We call a map $\gamma : X \to L$ a L-closed map, if it has a L-open complement.

(ii) A subset $Y \subseteq X$ is called an L-closed set in $(X, \mathcal{T}_X^L)$ if χ_Y is L-closed.

Definition 6.2. (i) For any L-subspace Y in X, the L-closure of Y is defined to be the intersection of all closed L-subspaces of X containing Y. Obviously, the closure of a set $Y \subseteq X$ is a L-closed subset of X.

(ii) For any L-map $s : X \to L$, the L-closure of s, denoted by $\bar{s}$, is defined by $\bar{s} := \wedge\gamma$, where γ is a L-closed map with $s \leq \gamma$. The meet of any collection of L-closed maps is a L-closed map. So the closure of a L-map is a closed map with $s \leq \bar{s}$.

Remark 6.3. If $(X, \mathcal{T}^L)$ is a L-topology and $\{f_i\}_{i \in I}$ is a collection of L-maps, then one can verify easily that

$$\overline{\vee f_i} \supseteq \vee \bar{f}_i, \qquad \overline{\wedge f_i} = \wedge \bar{f}_i.$$

Example 6.4. Set $X = [0, 1]$ and assume that L is a frame with enough complements. Define $s, t : X \to L$ with $s(x) = 0, t(x) = 1$ for $x \in (-\infty, \frac{1}{2})$ and $s(x) = 1, t(x) = 0$ for $x \in [\frac{1}{2}, +\infty)$.

The set $\mathcal{T} = \{f_0, f_1, s, t\}$ induces a L-topology on X and one has $s^c = t$, so both of the maps s and t are closed and open. Furthermore; $\bar{t} = \bar{s} = f_1$.

Theorem 6.5. *Let X be an L-topology with L admitting complements. Assume moreover that the map $f_X^1 : X \to L$ is a L-compact map. Then any L-closed map $s : X \to L$ is L-compact.*

Proof. Assume $\{s_\alpha\}_{\alpha \in I}$ be a covering for s. So one has $\vee_{\alpha \in I} s_\alpha = s$, which implies $f^1 = \vee_{\alpha \in I}(s_\alpha \vee s^c)$. Applying the L-compactness of f^1 the equality can be rewritten as $f^1 = \vee_{i=1}^{i=n}(s_i \vee s^c)$ for some natural number n. It is now easy to deduce $s = \vee_{i=1}^{i=n} s_i$. □

Theorem 6.6. *Let $s : X \to L$ be a L-connected map. Then $\bar{s}$ is L-connected too.*

Proof. Let s_1, s_2 be a L-separation for $\bar{s}$. Then $\{s_1 \wedge s, s_2 \wedge s\}$ is a L-separation for s. So, either $s_1 \wedge s = s$ or $s_2 \wedge s = s$. The equality $s_1 \wedge s = s$, implies $s \leq s_1$, which implies in its own right that $\bar{s} \leq \bar{s}_1$. Therefore one has $s_2 = \bar{s} \wedge s_2 = f_0$, which is impossible. □

Remark 6.7. An argument as for the usual topological spaces implies that the union of a collection of L-connected topological spaces with a point in common is L-connected.

Theorem 6.8. *Let L be a complemented frame and $\{X_\alpha, \tau^L_{X_\alpha}\}_{\alpha\in I}$ be a collection of L-connected topological spaces. Then $\prod_{\alpha\in J} X_\alpha$ is L-connected.*

Proof. For finite number of L-connected topological spaces, Proposition 3.8 implies that their product is L-connected without any assumption on complementaryness of L. For arbitrary L-cennected topological spaces, choose $x_j \in X_j$ and for any finite subset $T \subset I$ let Γ_T be the product of the subspaces X_t if $t \in T$ and $\{x_t\}$ if t does not belong to T. Abserve that the intersection $\cap_T C_T$, where T moves in the set of finite subsets of I, is non-empty. So Remark 6.7 implies that $\cup_T C_T$ is L-connected. Applying Theorem 6.6 together with the equality $\overline{\cup_T C_T} = \prod_i X_i$, we conclude the assertion. □

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