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Products of EP operators on Hilbert C*-modules

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Abstract. In this paper, the special attention is given to the product of two modular operators, and when at least one of them is EP, some interesting results is made, so the equivalent conditions are presented that imply the product of operators is EP. Also, some conditions are provided, for which the reverse order law is hold. Furthermore, it is proved that $P(RPQ)^\dagger$ is idempotent, if RPQ has closed range, for orthogonal projections P, Q and R.

1. Introduction and Preliminaries

The generalized inverses is very important in practical applications in diverse fields like optimization, statistics, economics, networks and so on. In this paper, we specialize the investigations to the Moore-Penrose inverse of closed range operators on Hilbert $\mathcal{A}$-modules. The closedness of range of operators is an attractive and important problem which appears in operator theory. We will investigate that the product of two operators with closed ranges has closed range, too. The notion of EP operator was extended by Campbell and Meyer for Hilbert Spaces in [1, 2] and so on by Sharifi [16–18] and Mohammadzadeh Karizaki [11–13] for Hilbert $\mathcal{A}$-modules. In fact the operator $T \in L(\mathcal{X})$ is called EP, if T and T^* have the same ranges. EP operators have been studied by many authors ever since. The reader is referred to [7, 8, 10, 11, 13] and the references cited therein for more details.

This paper is organized as follows. In the remainder of this section, some preliminaries is given, which are used in the following sections. Section 2 describes the conditions provided that the reverse order law is

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hold and state that the equivalent conditions for the product of operators is EP and obtain condition the relation reverse order law between operators that product one is EP. Section 3 prove that P, Q, R are orthogonal projections, if RPQ has closed range, then $P(RPQ)^\dagger$ is idempotent.

Hilbert C^*-modules are objects like Hilbert spaces, except that the inner product take its values in a C^*-algebra, instead of being complex-valued. Throughout the paper, $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,

for each $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$, the so-called adjoint of T such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [10, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ for the kernel and the range of operators, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where $\mathcal{Y}^\perp := \{y \in \mathcal{X} : \ \langle x, y\rangle = 0 \ \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$.

Throughout this paper, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance in [10] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1 ([10] Theorem 3.2). *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* ran(T*).

- ran(T) *is orthogonally complemented in* $\mathcal{Y}$*, with complement* $\ker(T^*)$.
- *The map* $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *has closed range.*

Definition 1.2. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T (if it exists) is an element in $L(\mathcal{Y}, \mathcal{X})$ which satisfies

(a) $TT^\dagger T = T$,
(b) $T^\dagger TT^\dagger = T^\dagger$,
(c) $(TT^\dagger)^* = TT^\dagger$,
(d) $(T^\dagger T)^* = T^\dagger T$.

Motivated by these conditions $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections, in the sense that those are selfadjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. The following theorem is known.

Theorem 1.3 ([19] Theorem 2.2)**.** *Suppose that* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$*. Then the Moore-Penrose inverse* $T^\dagger$ *of* T *exists if and only if* T *has closed range.*

By Definition 1.2, we have

$$\mathrm{ran(T)} = \mathrm{ran(T\,T^\dagger)}, \qquad \mathrm{ran(T^\dagger)} = \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)},$$
$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*),$$

and by Theorem 1.1, we have

$$\mathcal{X} = \ker(T) \oplus \mathrm{ran(T^\dagger)} = \ker(\mathrm{T^\dagger T}) \oplus \mathrm{ran(T^\dagger T)},$$
$$\mathcal{Y} = \ker(T^\dagger) \oplus \mathrm{ran(T)} = \ker(\mathrm{T\,T^\dagger}) \oplus \mathrm{ran(T\,T^\dagger)}.$$

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{1.1}$$

where, $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_\mathcal{M}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_\mathcal{N} T P_\mathcal{M}$, $T_2 = P_\mathcal{N} T(1 - P_\mathcal{M})$, $T_3 = (1 - P_\mathcal{N})TP_\mathcal{M}$ and $T_4 = (1 - P_\mathcal{N})T(1 - P_\mathcal{M})$.

Recall that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then $TT^\dagger = P_{\mathrm{ran(T)}}$ and $T^\dagger T = P_{\mathrm{ran(T^*)}}$.

Lemma 1.4 ([11] Theorem 2.2)**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}.$$

In this paper we shall use the following results.

Theorem 1.5 ([14] Corollary 2.4)**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$.*

Theorem 1.6 ([13] Theorem 2.1)**.** *Let $T \in L(\mathcal{X})$ has closed range and $S \in L(\mathcal{X})$ is an arbitrary operator which is commutative with respect to T. Then S commutes with $T^\dagger$.*

2. Product of EP Modular Operators

In the following theorem the conditions provided that the reverse order law holds.

Theorem 2.1. *Suppose that $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $T, S \in L(\mathcal{X})$ are EP operators with closed ranges. If $\mathrm{ran(T)} = \mathrm{ran(S)}$, then the reverse order law is holds.*

Proof. Since $\mathrm{ran(T^*)} = \mathrm{ran(T)} = \mathrm{ran(S)} = \mathrm{ran(S^*)}$ we get

$$\mathrm{ran(T^*TS)} = \mathrm{T^*T(ran(T^*))} = \mathrm{T^*(ran(T))} = \mathrm{ran(T^*)} = \mathrm{ran(S)},$$

and

$$\mathrm{ran(SS^*T^*)} = \mathrm{SS^*(ran(S))} = \mathrm{ran(S)} = \mathrm{ran(T^*)}.$$

Hence, by [18, Corollary 2.3.] the result is obtained. □

The following theorem states that the equal condition for the product of operators is EP.

Theorem 2.2. *Let $T, S \in L(\mathcal{X})$ be EP operators with closed ranges. Then TS is EP if and only if $(1 - SS^\dagger)(TS) = 0$.*

Proof. For necessity, [17, Proposition 2.2] implies that there exists an isomorphism $V \in L(\mathcal{X})$ for which $(TS)^* = VTS$ for some . From the fact $(1 - SS^\dagger)S = S(1 - SS^\dagger) = 0$, we have $(1 - SS^\dagger)(TS) = ((TS)^*(1 - SS^\dagger))^* = (VTS(1 - SS^\dagger))^* = 0$.

For sufficiency, from $(1-SS^\dagger)(TS) = 0$ and $\ker(1-SS^\dagger) = \mathrm{ran(SS^\dagger)} = \mathrm{ran(S)}$, we have: $\mathrm{ran(TS)} \subseteq \mathrm{ran(S)}$. Also, $\mathrm{ran(TS)} \subseteq \mathrm{ran(T)}$, then ran(TS) $\subseteq$ ran(S) $\cap$ ran(T). To show the opposite inclusion, since $y \in$ (ran(T) $\cap$ ran(S)) $\cap$ ran(TS)$^{\perp}$, then $y = 0$ which conclude that, ran(T) $\cap$ ran(S) = ran(TS). Similarly, ran(T*) $\cap$ ran(S*) = ran(TS)*, hence, ran(TS) = ran(T) $\cap$ ran(S) = ran(T*) $\cap$ ran(S*) = ran(TS)*. □

Theorem 2.3. *Let $T \in L(\mathcal{Y}, \mathcal{Z}), S \in L(\mathcal{X}, \mathcal{Y})$ have closed ranges. Then TS has Moore-Penrose inverse if and only if $T^\dagger TSS^\dagger$ has Moore-Penrose inverse.*

Proof. First, let V be the Moore-Penrose inverse of TS, then

$$\begin{aligned} T^\dagger TSS^\dagger(SVT)T^\dagger TSS^\dagger &= (T^\dagger T)(SS^\dagger S)V(TT^\dagger T)(SS^\dagger) \\ &= T^\dagger TSVTSS^\dagger \\ &= T^\dagger TSS^\dagger. \end{aligned}$$

Similarly, $SVT(T^\dagger TSS^\dagger)SVT = SVT$. Therefore, SVT is Moore-Penrose inverse of $T^\dagger TSS^\dagger$. Conversely, let U be the Moore-Penrose inverse of $T^\dagger TSS^\dagger$. Put $P = SS^\dagger$ and $Q = T^\dagger T$, then $QPUQP = QP$.

Now, give $W = PUQ$ so, $PWQ = W$, $QWP = QP$, then $Q(1 - W)P = 0$. Hence, $T(1 - W)S = 0$ and therefore, $TS(S^\dagger WT^\dagger)TS = TPWQS = TWS = TS$.

On the other hand,

$$\begin{aligned} S^\dagger WT^\dagger &= S^\dagger PUQT^\dagger \\ &= S^\dagger SS^\dagger UT^\dagger TT^\dagger \\ &= S^\dagger UT^\dagger, \end{aligned}$$

which implies that $(S^\dagger WT^\dagger)TS(S^\dagger WT^\dagger) = S^\dagger UT^\dagger = S^\dagger WT^\dagger$.

Thus, $S^\dagger WT^\dagger$ is the Moore-Penrose inverse of TS.

□

Theorem 2.4. *Let $T, S \in L(\mathcal{X})$. If S is EP and $(TS)^\dagger = S^\dagger T^\dagger$, then TS is EP.*

Proof. Regading to [17, Proposition 2.2], there exists an isometry V such that

$$VTV^* = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where T_1 is invertible. Set

$$VSV^* = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix},$$

where S_1, S_2, S_3 and S_4 as mentioned before, so by above mentioned properties of matrix decomposition (1.1) we have ,

$$VTSV^* = \begin{bmatrix} T_1S_1 & T_1S_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix},$$

and

$$VSTV^* = \begin{bmatrix} S_1T_1 & 0 \\ S-3T_1 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ S_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & J' \end{bmatrix},$$

where J, J' are parts of decomposition (1.1). Hence, S_1 is invertible. By [11] one can consider an operator F for wich

$$VSV^* = \begin{bmatrix} S_1 & S_1F^* \\ FS_1 & FS_1F^* \end{bmatrix}.$$

Also, by the Penroses representation we have

$$VS^\dagger V^* = (VSV^*)^\dagger = \begin{bmatrix} D & DF^* \\ FQ & FQF^* \end{bmatrix},$$

where $D = (I + F^*F)^{-1}S_1^{-1}(I + F^*F)^{-1}$, and Q is an operator coresponding to the considered decomposition, in form (1.1) .

We know that,

$$(VTV^*)^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, by using the matrix technique we can see

$$\begin{aligned} VTSV^* &= (VTSV^*)(VTSV^*)^\dagger(VTSV^*) \\ &= (VTSV^*)V(TS)^\dagger V^*(VTSV^*) \\ &= (VTSV^*)(VS^\dagger V^*)(VT^\dagger V^*)(VTSV^*). \end{aligned}$$

Therefore, $T_1S_1 = T_1S_1(I + F^*F)DS_1$ and so $DS_1 = (I + F^*F)^{-1}$, which implies that $D^{-1} = S_1$. Thus, $I + F^*F = I$, because $DT_1^{-1} = (I + F^*F)DT_1^{-1}$. The former equation implies that $F^*F = 0$ and finally we have $F = 0$. This completes the proof. □

Corollary 2.5. *Let $T, S \in L(\mathcal{X})$ be EP operators. If $(TS)^\dagger = T^\dagger S^\dagger$, then TS is EP.*

Proof. We know that, ran(S) = ran(S*) = ran(S*T*) = ran((TS)*) = ran((TS)†) = ran(T†S†) ⊂ ran(T†) = ran(T*) = ran(T), hence, ran(S) ⊂ ran(T). Similarly, ran(T) ⊂ ran(S) and therefore ran(S) = ran(T). Finally, ran((TS)*) = ran(S*T*) = ran(S*) = ran(S) = ran(T) = ran(TS). which shows that TS is EP. □

Corollary 2.6. *Suppose that $T, S \in L(\mathcal{X})$ are EP operators. Then $(TS)^\dagger = T^\dagger S^\dagger$ if and only if $TS = ST$.*

Proof. ($\Leftarrow$) Since T, S are EP and commutative, then $(TS)^\dagger = T^\dagger S^\dagger$ and $(TS)^\dagger = S^\dagger T^\dagger$.

($\Rightarrow$) By the previous corollary, TS is EP, so there are two isometry $U, V \in \mathcal{X}$ such that

$$(UTU^*)^\dagger = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$(VSV^*)^\dagger = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where T_1 and S_1 are invertible. Then $(TS)^\dagger = T^\dagger S^\dagger$ implies that $T_1S_1 = S_1T_1$ so that $TS = ST$. □

3. Orthogonal Projection Results

Lemma 3.1. *Let $P \in L(\mathcal{X})$ be an orthogonal projection and $T \in L(\mathcal{X})$ be self-adjoint with closed range. If $PT(1-P) = 0$, then $PT^\dagger(1-P) = 0$.*

Proof. If $PT(1-P) = 0$, then $PT = PTP$, so by taking adjoint operator on the last relation, we get $TP = PTP$. It follows, P commutes with T. Therefore by Proposition 1.6, we have $T^\dagger P = PT^\dagger$, by multiplying P on the right side of this equation, we have $T^\dagger P = PT^\dagger P$, consequently $PT^\dagger = PT^\dagger P$ and the desired result follows. □

Lemma 3.2. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and let $Q \in \mathcal{L}(\mathcal{X})$ and $P \in \mathcal{L}(\mathcal{Y})$ be orthogonal projections and TQ and PT have closed ranges. Then*

(i) $(TQ)^\dagger = Q(TQ)^\dagger$,
(ii) $(PT)^\dagger = (PT)^\dagger P$.

Proof. (i) TQ has the Moore-Penrose inverse which is due to the fact that TQ has closed range, therefore $\mathrm{ran}((\mathrm{TQ})^\dagger) = \mathrm{ran}((\mathrm{TQ})^*) = \mathrm{ran}(\mathrm{QT}^*) \subseteq \mathrm{ranQ}$. Hence $Q((TQ)^\dagger) = (TQ)^\dagger$. This completes the proof of (i). By a similar argument, the proof of (ii) follows. □

Corollary 3.3. *Let $P, Q \in \mathcal{L}(\mathcal{X})$ be orthogonal projections and $PQ \in \mathcal{L}(\mathcal{X})$ has closed range. Then*

(i) $(PQ)^\dagger = Q(PQ)^\dagger$,
(ii) $(PQ)^\dagger = (PQ)^\dagger P$.

Proof. We only need to consider PQ instead of T. This completes the proof. □

Theorem 3.4. *Let $P, Q \in \mathcal{L}(\mathcal{X})$ be orthogonal projections and PQ has closed range, Then the following statements hold*

(i) $(QP)^\dagger Q = (QP)^\dagger$ *and* $P(QP)^\dagger = (QP)^\dagger$,
(ii) $(QP)^\dagger PQ = PQ$,

(iii) $(PQ)^{\dagger}QP = QP$.

Proof. (i) By taking adjoint on (i) and (ii) in Corollary 3.3, we get $(QP)^{\dagger}Q = (QP)^{\dagger}$ and $P(QP)^{\dagger} = (QP)^{\dagger}$.

(ii) The fact that PQ has closed range [14, Corollary 2.4] implies that $(QP)^{\dagger} = (QP)^{*}(QP(QP)^{*})^{\dagger}$. Multiplying PQ on the right hand of $(QP)^{\dagger} = PQ(QP(QP)^{*})^{\dagger}$, yields

$$\begin{aligned}(QP)^{\dagger}PQ &= PQ(QP(QP)^{*})^{\dagger}PQ \\ &= PQ[((QP)^{*})^{\dagger}(QP)^{\dagger}QP]PQ \\ &= PQ((QP)^{*})^{\dagger}PQ \\ &= PQ(PQ)^{\dagger}PQ \\ &= PQ.\end{aligned}$$

(iii) By [11, Theorem 2.3.] $(i) \Rightarrow (vii)$ we have $(PQ)^{\dagger}QP = QP$.

□

Theorem 3.5. *Let $P, Q \in \mathcal{L}(\mathcal{X})$ be orthogonal projections and PQ has closed range. Then*

(i) *PQ is EP.*
(ii) *$PQP = QPQ$.*
(iii) *PQP is an orthogonal projection.*

Proof. (i) By taking adjoint on (ii) in Theorem 3.4, we have $(QP)^{\dagger}PQ = PQ$. By taking adjoins of Theorem 3.4 (iii) we have $PQ(QP)^{\dagger} = PQ$. Therefore $(QP)^{\dagger}PQ = PQ(QP)^{\dagger}$.

(ii) From (i), we know that PQ commute with $(QP)^{\dagger}$. So Lemma 1.6, implies that PQ commute with QP, Thus, $PQP = QPQ$.

(iii) Considering (ii) and multiplying it by P both on the left and write hand of this equation, one can right $PQP = PQPQP = (PQP)(PQP) = (PQP)^2$. Since $(PQP)^{*} = PQP$, then PQP is an orthogonal projection.

□

Theorem 3.6. *Let $P, Q \in \mathcal{L}(\mathcal{X})$ be orthogonal projections and PQ has closed range. Then $(PQ)^2 = -(QP)^2$.*

Proof. By Theorem 3.4 one can write $1 - P - PQP = 1 - P - QPQ$. It follows that

$$(PQ - QP)(1 - P - PQP) = (PQ - QP)(1 - P - QPQ).$$

A straightforward computation and using (ii) in Theorem 3.4 deduce $(PQ)^2 = -(QP)^2$. □

Corollary 3.7. *let $P, Q \in \mathcal{L}(\mathcal{X})$ be orthogonal projections and PQ has closed range. Then $(PQ)^2 = -PQP$.*

Proof. The assertion can be proved by combining Theorem 3.6 and 3.4. □

Theorem 3.8. *Suppose $Q \in \mathcal{L}(\mathcal{X})$ and $P \in \mathcal{L}(\mathcal{Y})$ and $R \in \mathcal{L}(\mathcal{Z})$ are orthogonal projections and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$. If $RSPTQ$ has closed range, then $SPT(RSPTQ)^\dagger$ and $(RSPTQ)^\dagger SPT$ are idempotent closed range operators.*

Proof. Since ran(RSPTQ) is closed, the operator $U = (RSPTQ)^\dagger$ exists. It follows $\mathrm{ran(U)} = (\mathrm{ran(RSPTQ)^*}) = \mathrm{ran(QT^*PS^*R)}$. So $\mathrm{ran(U)} \subseteq \mathrm{ranQ}$. Also, we have

$$\mathrm{ran(U^*)} = \mathrm{ran(((RSPTQ)^*)^\dagger)} = \mathrm{ran((RSPTQ)^*)^*} = \mathrm{ran(RSPTQ)} \subseteq \mathrm{ranR}.$$

After that, we obtain

$$QU = U, \qquad RU^* = U^*, \qquad UR = U, \tag{3.1}$$

and

$$USPTU = URSPTQU = U(RSPTQ)U = UU^\dagger U = U. \tag{3.2}$$

By multiplying SPT on the left hand of the equation (3.2), we get $SPTU = SPTTUSPTU = (SPTU)(SPTU) = (SPTU)^2$. Once again by multiplying SPT on the right hand of the equation (3.2), we attain $USPT = USPTUSPT = (USPT)(USPT) = (USPT)^2$. Finally, $SPT(RSPTQ)^\dagger$ and $(RSPTQ)^\dagger SPT$ are idempotent. Corollary 3.3. in [10] implies that $SPT(RSPTQ)^\dagger$ and $(RSPTQ)^\dagger SPT$ have closed range. □

Corollary 3.9. *Suppose that $P, Q, R \in \mathcal{L}(\mathcal{X})$ are orthogonal projections. If RPQ has closed range, then $U = P(RPQ)^\dagger$ is idempotent and $U = QUR$.*

Proof. By taking $T, S = 1_{\mathcal{X}}$ in Theorem 3.8, we conclude that $U = P(RPQ)^\dagger$ is idempotent. By using equation (3.1), we have $QU^2R = U^2$. Since U is idempotent, the desired result follows. □

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