C*-semi-inner product spaces

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Abstract. In this paper, we introduce a generalization of Hilbert C*-modules which are pre-Finsler modules, namely, C*-semi-inner product spaces. Some properties and results of such spaces are investigated, specially the orthogonality in these spaces will be considered. We then study bounded linear operators on C*-semi-inner product spaces.

1. Introduction

The semi-inner product (s.i.p., in brief) spaces were introduced by Lumer in [13]. He considered vector spaces on which, instead of a bilinear form, there is defined a form \([x,y]\) which is linear in one component only, strictly positive, and satisfies Cauchy-Schwarz’s inequality. Six years after Lumer’s work, Giles in [8] explored fundamental properties and consequences of semi-inner product spaces. Also, a generalization of semi-inner product spaces was considered by replacing Cauchy-Schwarz’s inequality by Holder’s inequality in [16]. The concept of +semi-inner product algebras of type (p) was introduced and some properties of such algebras were studied by Siham Galal El-Sayyad and S. M. Khaleelulla in [6]. Also, they obtained some interesting results about generalized adjoints of bounded linear operators on semi-inner product spaces of type (p). In the sequel, a version of adjoint theorem for maps on semi-inner product spaces of type (p), which is obtained by Endre Pap and Radoje Pavlovic in [17]. The concept of s.i.p. has been proved useful both theoretically and practically. The applications of s.i.p. in the theory of functional analysis was demonstrated, for example, in [1, 3, 6, 12, 14, 20, 23, 24].
On the other hand, the concept of Hilbert $C^*$-modules which are generalization of the notion of Hilbert spaces, first made by I. Kaplansky in 1953 ([11]). The research on Hilbert $C^*$-modules began in the 70s (W.L. Paschke, [18]; M.A. Rieffel, [21]). Since then, this generalization of Hilbert spaces was considered by many mathematicians. For more details about Hilbert $C^*$-modules we refer also to [13]. Also, Finsler modules over $C^*$-algebras as a generalization of Hilbert $C^*$-modules, first investigated in [19]. For more on Finsler modules, one may see [1, 2].

In this paper we are going to introduce a new generalization of Hilbert $C^*$-modules which are between Hilbert $C^*$-modules and Finsler modules. Furthermore, $C^*$-semi-inner product spaces are natural generalization of a semi-inner product spaces arising under replacement of the field of scalars $\mathbb{C}$ by $C^*$-algebras.

2. $C^*$-SEMI-INNER PRODUCT SPACE

In this section we investigate basic properties of $C^*$-semi-inner product spaces.

**Definition 2.1.** Let $\mathcal{A}$ be a $C^*$-algebra and $X$ be a right $\mathcal{A}$-module. A mapping $[\cdot, \cdot]: X \times X \to \mathcal{A}$ is called a $C^*$-semi-inner product or $C^*$-s.i.p., in brief, if the following properties are satisfied:

(i) $[x, x] \geq 0$, for all $x \in X$ and $[x, x] = 0$ implies $x = 0$;
(ii) $[\alpha x_1 + \beta x_2, y] = \alpha [x_1, y] + \beta [x_2, y]$, for all $x_1, x_2, y \in X$ and $\alpha, \beta \in \mathbb{C}$;
(iii) $[x, \alpha y] = [x, y] \alpha$ and $[\alpha x, y] = \alpha^* [x, y]$, for all $x, y \in X$ and $\alpha \in \mathcal{A}$;
(iv) $\| [y, x] \|^2 \leq \| [y, y] \| [x, x]$.

The triple $(X, \mathcal{A}, [\cdot, \cdot])$ is called a $C^*$-semi-inner product space or we say $X$ is a semi-inner product $\mathcal{A}$-module.

The property (iv) is called the Cauchy-Schwarz inequality. If $\mathcal{A}$ is a unital $C^*$-algebra, then one may see that $[\lambda x, y] = \overline{\lambda} [x, y]$, for all $x, y \in X$ and $\lambda \in \mathbb{C}$. Indeed, by property (iii) we have

$$[\lambda x, y] = [x(\lambda 1), y] = (\lambda 1)^* [x, y] = \overline{\lambda} [x, y].$$

One can easily see that every Hilbert $C^*$-module is a $C^*$-semi-inner product space, but the converse is not true in general. The following is an example of a $C^*$-semi-inner product space which is not a Hilbert $C^*$-module. First we recall that a semi-inner-product (s.i.p.) in the sense of Lumer and Giles on a complex vector space $X$ is a complex valued function $[x, y]$ on $X \times X$ with the following properties:
1. \( [\lambda y + z, x] = \lambda [y, x] + [z, x] \) and \( [x, \lambda y] = \overline{\lambda} [x, y] \), for all complex \( \lambda \);

2. \( [x, x] \geq 0 \), for all \( x \in X \) and \( [x, x] = 0 \) implies \( x = 0 \);

3. \( |[x, y]|^2 \leq [x, x] [y, y] \).

A vector space with a s.i.p. is called a semi-inner-product space (s.i.p. space) in the sense of Lumer-Giles (see [13]). In this case one may prove that \( \|x\| := [x, x]^{1/2} \) defines a norm on \( X \). Also, it is well-known that for every Banach space \( X \), there exists a semi-inner product whose norm is equal to its original norm.

It is trivial that every Banach space is a semi-inner product \( \mathbb{C} \)-module.

**Example 2.2.** Let \( \Omega \) be a set and let for any \( t \in \Omega \), \( X_t \) be a semi-inner product space with the semi inner product \( [\cdot, \cdot]_t \). Define

\[
[x, y]_{X_t} := [x, y]_{X_t}, \quad x, y \in X_t,
\]

trivially \( [x, \alpha y + z]_{X_t} = \alpha [x, y]_{X_t} + [x, z]_{X_t} \) and \( [\alpha x, y]_{X_t} = \overline{\alpha} [x, y]_{X_t} \). Let \( B = \bigcup_t X_t \) be a bundle of these semi-inner product spaces over \( \Omega \). Suppose \( A = Bd(\Omega) \), the set of all bounded complex-valued functions on \( \Omega \), and \( X \) is the set of all maps \( f : \Omega \to B \) such that \( f(t) \in X_t \), for any \( t \in \Omega \), with \( \sup_{t \in \Omega} \|f(t)\| < \infty \). One can easily see that \( X \) is naturally a \( Bd(\Omega) \)-module. Furthermore it has a \( Bd(\Omega) \)-valued semi-inner product defined by

\[
[f, g](t) = [f(t), g(t)]_{X_t},
\]

for \( t \in \Omega \), hence it is a \( C^* \)-semi-inner product space. One can easily verify that the properties of \( C^* \)-semi-inner product are valid.

Suppose \( (A_i, \|\cdot\|_i) \)'s, \( 1 \leq i \leq n \), are \( C^* \)-algebras, then \( \bigoplus_{i=1}^n A_i \) with its point-wise operations is a \( C^* \)-algebra. Moreover,

\[
\|(a_1, \ldots, a_n)\| = \max_{1 \leq i \leq n} \|a_i\|,
\]

is a \( C^* \)-norm on \( \bigoplus_{i=1}^n A_i \). Note that \( (a_1, \ldots, a_n) \in (\bigoplus_{i=1}^n A_i)_+ \) if and only if \( a_i \in (A_i)_+ \). Now we may construct the following example.

**Example 2.3.** Let \( (X_i, [\cdot, \cdot])_i \) be a semi-inner product \( A_i \)-module, \( 1 \leq i \leq n \). If for \( (a_1, \ldots, a_n) \in A \) and \( (x_1, \ldots, x_n) \in \bigoplus_{i=1}^n X_i \), we define \( (x_1, \ldots, x_n)(a_1, \ldots, a_n) = (x_1 a_1, \ldots, x_n a_n) \) and the \( C^* \)-s.i.p. is defined as follows

\[
[(x_1, \ldots, x_n), (y_1, \ldots, y_n)] = ([x_1, y_1], \ldots, [x_n, y_n]),
\]

then the direct sum \( \bigoplus_{i=1}^n X_i \) is a semi-inner product \( A \)-module, where \( A = \bigoplus_{i=1}^n A_i \).
Let \((X, \mathcal{A}, [\cdot, \cdot])\) be a \(C^\ast\)-semi-inner product space. For any \(x \in X\), put \(\|\| x\|\| := \| [x, x] \|^{\frac{1}{2}}\). The following proposition shows that \((X, \|\|.,\|\|)\) is a normed \(\mathcal{A}\)-module.

**Proposition 2.4.** Let \(X\) be a right \(\mathcal{A}\)-module and \([\cdot, \cdot]\) be a \(C^\ast\)-s.i.p. on \(X\). Then the mapping \(x \to \| [x, x] \|^{\frac{1}{2}}\) is a norm on \(X\). Moreover, for each \(x \in X\) and \(a \in \mathcal{A}\) we have \(\| [xa] \| \leq \|\| x\|\| \|a\|\).

**Proof.** Clearly \(\|\| x\|\| = \| [x, x] \|^{\frac{1}{2}} \geq 0\) and \(\|\| x\|\| = 0\) implies that \(x = 0\). Also for each \(x \in X\), \(\lambda \in \mathbb{C}\), by the Cauchy-Schwarz inequality,

\[
\|\| \lambda x\|\| \leq |\lambda| \|\| x\|\|.
\]

Hence, \(\|\| \lambda x\|\| \leq |\lambda| \|\| x\|\|\). On the other hand, we have

\[
\|\| x\|\| = \left\| \frac{1}{\lambda} \lambda x \right\| \leq \frac{1}{|\lambda|} \|\| \lambda x\|\|,
\]

therefore \(\|\| \lambda x\|\| = |\lambda| \|\| x\|\|\).

Finally for each \(x, y \in X\),

\[
\|\| x + y\|\| \leq \|\| x\|\| + \|\| y\|\|.
\]

Also we have

\[
\|\| xa\|\| \leq \|\| x\|\| \|a\|\|,
\]

hence \(\|\| xa\|\| \leq \|\| x\|\| \|a\|\|\).

\(\square\)
As another result for this norm one can see that for each \( x \in X \), 
\[ \| [x,x] \| || = ||x||^3. \]
Indeed 
\[ \| [x,x] \| || = \| [x,x] \|^3 = \| [x,x] \|. \]
The last equality follows from the fact that in any \( C^* \)-algebra, we have 
\[ \| a \|^3 = \| a \|^3, \]
for any self-adjoint element \( a \in A \).

**Proposition 2.5.** Let \( A \) and \( B \) be two \( C^* \)-algebras and \( \psi : A \to B \) be an \(*\)-isomorphism. If \( (X,[ , ]_A) \) is a \( C^* \)-semi-inner product \( A \)-module, then \( X \) can be represented as a right \( B \)-module with the module action 
\( x\psi(a) = xa \) and is a \( C^* \)-semi-inner product \( B \)-module with the \( C^* \)-semi-inner product defined by
\[ [ , ]_B = \psi ([ , ]_A). \]

**Proof.** It is clear that \( X \) is a right \( B \)-module with the mentioned module product. It is easy to verify that the properties (i) to (iii) of definition of \( C^* \)-semi-inner product holds for \([ , ]_B\). Now, we prove the property (iv) for \([ , ]_B\). Since \( \psi : A \to B \) is an \(*\)-isomorphism, so it is isometric and \( \psi(A^+) \subseteq B^+ \). Thus we have 
\[ \| [x,y]_B \|^2 = |\psi([x,y]_A)|^2 \]
\[ = \psi([x,y]_A)^*\psi([x,y]_A) \]
\[ = \psi([x,y]_A)^*[x,y]_A \]
\[ = \psi([x,y]_A)^2 \]
\[ \leq \|[x,x]_A\| \psi([y,y]_A) \]
\[ = \|\psi([x,x]_A)\| \psi([y,y]_A) \]
\[ = \| [x,x]_B \| [y,y]_B. \]

\[ \square \]

We will establish a converse statement to the above proposition. Consider that a semi-inner product \( A \)-module \( X \) is said to be full if the linear span of \( \{[x,x] : x \in X\} \), denoted by \([X,X] \), is dense in \( A \).

**Theorem 2.6.** Let \( X \) be both a full complete semi-inner product \( A \)-module and a full complete semi-inner product \( B \)-module such that 
\[ \| [x,x]_A \| = \| [x,x]_B \|, \]
for each \( x \in X \), and let \( \psi : A \to B \) be a map such that \( xa = x\psi(a) \) and 
\( \psi([x,x]_A) = [x,x]_B \) where \( x \in X, a \in A \). Then \( \psi \) is an \(*\)-isomorphism of \( C^* \)-algebras.
Proof. The proof is similar to the proof of Theorem 2.1 [12]. □

We recall that if \( \mathcal{A} \) is a C*-algebra, and \( \mathcal{A}_+ \) is the set of positive elements of \( \mathcal{A} \), then a pre-Finsler \( \mathcal{A} \)-module is a right \( \mathcal{A} \)-module \( E \) which is equipped with a map \( \rho : E \to \mathcal{A}_+ \) such that

1. the map \( \| \cdot \|_E : x \mapsto \| \rho(x) \| \) is a norm on \( E \); and
2. \( \rho(ax)^2 = a^*\rho(x)^2a \), for all \( a \in \mathcal{A} \) and \( x \in E \).

If \( (E, \| \cdot \|_E) \) is complete then \( E \) is called a Finsler \( \mathcal{A} \)-module. This definition is a modification of one introduced by N.C. Phillips and N.Weaver [19]. Indeed it is routine by using an interesting theorem of C. Akemann [19, Theorem 4] to show that the norm completion of a pre-Finsler \( \mathcal{A} \)-module is a Finsler \( \mathcal{A} \)-module. Now, it is trivial to see that every C*-semi-inner product space \( (X, \mathcal{A}, [\cdot, \cdot]) \) is a pre-Finsler module with the function \( \rho : X \to \mathcal{A}_+ \) defined by \( \rho(x) = [x, x]^{1/2} \). Thus every complete C*-semi-inner product space enjoys all the properties of a Finsler module.

**Proposition 2.7** ([19]). Let \( \mathcal{A} = C_0(X) \) and let \( E \) be a Finsler \( \mathcal{A} \)-module. Then \( \rho \) satisfies

\[
\rho(x + y) \leq \rho(x) + \rho(y),
\]

for all \( x, y \in E \).

Replacing the real numbers, as the codomain of a norm, by an ordered Banach space we obtain a generalization of normed space. Such a generalized space, called a cone normed space, was introduced by Rzepecki [22].

**Corollary 2.8.** Let \( (X, [\cdot, \cdot]) \) be a semi-inner \( C(X) \)-module, then \( \| \cdot \|_c : X \to C(X) \) defined by \( \|x\|_c = [x, x]^{1/2} \) is a cone norm on \( X \).

3. Orthogonality in C*-semi-inner product spaces

In this section we study the relations between Birkhoff-James orthogonality and the orthogonality in a C*-semi-inner product spaces.

In a normed space \( X \) (over \( K \in \{ \mathbb{R}, \mathbb{C} \} \)), the Birkhoff-James orthogonality (cf.[3, 14]) is defined as follows

\[
x \perp_B y \iff \forall \alpha \in K; \quad \|x + \alpha y\| \geq \|x\|.
\]

**Theorem 3.1.** Let \( X \) be a right \( \mathcal{A} \)-module and \( [\cdot, \cdot] \) be a C*-s.i.p. on \( X \). If \( x, y \in X \) and \( [x, y] = 0 \) then \( x \perp_B y \).

**Proof.** Let \( [x, y] = 0 \). If \( x = 0 \) then by the definition of Birkhoff-James orthogonality it is obvious that \( x \perp_B y \). Now if \( x \neq 0 \), then for all
\( \alpha \in \mathbb{K}, \)
\[
\|\|x\|\|^2 - |\alpha| \| [x, y] \| \leq \| [x, x + \alpha y] \| \\
\leq \|\|x\|\| \|\|x + \alpha y\|\|.
\]

Hence,
\[
-|\alpha| \| [x, y] \| \leq \|\|x\|\| (\|\|x + \alpha y\|\| - \|\|x\|\|).
\]

But \( x \neq 0 \) and \( [x, y] = 0 \), so by the above inequality we conclude that \( \|\|x + \alpha y\|\| \geq \|\|x\|\| \), which shows that \( x \perp_B y \).

In the sequel we try to find a sufficient condition for \( x, y \) to be orthogonal in the \( C^* \)-semi-inner product. For; we need some preliminaries. We remind that in a \( C^* \)-algebra \( A \) and for any \( a \in A \) there exist self-adjoint elements \( h, k \in A \) such that \( a = h + ik \). We apply \( \text{Re}(a) \) for \( h \).

**Definition 3.2.** A \( C^* \)-s.i.p. \( [\cdot, \cdot] \) on right \( A \)-module \( X \) is said to be continuous if for every \( x, y \in X \) one has the equality
\[
\lim_{t \to 0} \text{Re} [x + ty, y] = \text{Re} [x, y],
\]
where \( t \in \mathbb{R} \).

**Example 3.3.** In Example 2.3, \( \Omega = \{1, 2, \ldots, n\} \) and \( X \) be the semi inner product \( Bd(\Omega) \)-module defined in Example 2.2. If \( X_t \) is a continuous s.i.p. space (see [8]), for all \( t \in \Omega \), then \( X \) is a continuous \( C^* \)-s.i.p space. Indeed it is clear that
\[
\sup_{t \in \Omega} \|\text{Re} [f(t) + \alpha g(t), g(t)]_{X_t} - \text{Re} [f(t), g(t)]_{X_t} \|
\]
tends to 0, when \( \alpha \to 0 \).

**Theorem 3.4.** Let \( X \) be a right \( A \)-module and let \([\cdot, \cdot]\) be a continuous \( C^* \)-s.i.p. on \( X \) such that \([x, y] \in A_{sa} \) for each \( x, y \in X \). If for \( x, y \in X \)
\[
[x + ty, x + ty] \geq [x, x]^{\frac{1}{2}} \|\|x + ty\|\|,
\]
for all \( t \in \mathbb{R} \), then \([x, y] = 0 \).

**Proof.** It is clear that for each \( a \in A_{sa} \), we have \( a \leq |a| \). Now assume that
\[
[x + ty, x + ty] \geq [x, x]^{\frac{1}{2}} \|\|x + ty\|\|,
\]
for all \( x, y \in X \) and \( t \in \mathbb{R} \). By Cauchy-Schwarz inequality (iv) and the fact that \([x, y] \in A_{sa} \) for each \( x, y \in X \), we get
\[
[x + ty, x + ty] \geq [x, x]^{\frac{1}{2}} \|\|x + ty\|\|
\]
\[
\geq \| [x + ty, x] \|
\]
\[
\geq [x + ty, x],
\]
so, we have: \( t[x+ty,y] \geq 0 \) for each \( t \in \mathbb{R} \). Thus for \( t \geq 0 \) we have \( [x+ty,y] \geq 0 \) and for \( t \leq 0 \) we have \( [x+ty,y] \leq 0 \). Now, since \([,\,]\) is a continuous \( C^*\)-s.i.p. and \( \mathcal{A}_+ \) is a closed subset of \( \mathcal{A} \), so we have

\[
0 \geq [x,y] = \lim_{t \to 0^-} [x+ty,y] = \lim_{t \to 0^+} [x+ty,y] = [x,y] \geq 0,
\]

thus \([x,y] = 0\). \( \square \)

4. Bounded Linear operators on \( C^*-\text{semi-inner product spaces} \)

**Theorem 4.1.** Let \( X \) be a semi inner product \( \mathcal{A}\)-module. Then for every \( y \in X \) the mapping \( f_y : X \to \mathcal{A} \) defining by \( f_y(x) = [y,x] \) is a \( \mathcal{A}\)-linear continuous operator endowed with the norm generated by \( C^*-\text{s.i.p.} \). Moreover, \( \|f_y\| = \|y\| \).

**Proof.** The fact that \( f_y \) is a \( \mathcal{A}\)-linear operator follows by (ii) and (iii) of Definition 2.1. Now, using Schwartz inequality (iv) we get;

\[
\|f_y(x)\| = \|[y,x]\| \leq \|y\| \|x\|,
\]

which implies that \( f_y \) is bounded and

\[
\|f_y\| \leq \|y\|.
\]

On the other hand, we have

\[
\|f_y\| \geq \left\| f_y \left( \frac{y}{\|y\|} \right) \right\| = \|y\|,
\]

and then \( \|f_y\| = \|y\| \). \( \square \)

**Corollary 4.2.** If \( X \) is a right \( \mathcal{A}\)-module and \([,\,]\) a \( C^*-\text{s.i.p.} \) on \( X \), then for all \( x \in X \) we have

\[
\|x\| = \sup \{\|[x,y]\| : \|y\| \leq 1\}.
\]

**Lemma 4.3 ([10?]).** Let \( \mathcal{A} \) be a unital \( C^*\)-algebra. Let \( r : \mathcal{A} \to \mathcal{A} \) be a linear map such that for some constant \( K \geq 0 \) the inequality \( r(a)^*r(a) \leq Ka^*a \) is fulfilled for all \( a \in \mathcal{A} \). Then \( r(a) = r(1)a \) for all \( a \in \mathcal{A} \).

**Theorem 4.4.** Let \( X \) and \( Y \) be semi inner product \( \mathcal{A}\)-modules, \( T : X \to Y \) be a linear map. Then the following conditions are equivalent:

(i) The operator \( T \) is bounded and \( \mathcal{A}\)-linear, i.e. \( T(xa) = Tx.a \) for all \( x \in X \), \( a \in \mathcal{A} \);

(ii) There exists a constant \( K \geq 0 \) such that for all \( x \in X \) the operator inequality \( \|TxTx\| \leq K \|x\| \) holds.
Proof. To obtain the second statement from the first one, assume that $T(xa) = Tx.a$ and $\|T\| \leq 1$. If $C^*$-algebra $A$ does not contain a unit, then we consider modules $X$ and $Y$ as modules over $C^*$-algebra $A_1$, obtained from $A$ by unitization. For $x \in X$ and $n \in \mathbb{N}$, put

$$a_n = \left( [x,x] + \frac{1}{n} \right)^{-\frac{1}{2}}, \quad x_n = xa_n,$$

Then $[x_n, x_n] = a_n^* [x,x] a_n = [x,x] \left( [x,x] + \frac{1}{n} \right)^{-1} \leq 1$, therefore $\|x_n\| \leq 1$, hence $\|Tx_n\| \leq 1$. Then for all $n \in \mathbb{N}$ the operator inequality $[Tx_n, Tx_n] \leq 1$ is valid. But

$$[Tx, Tx] = a_n^{-1} [Tx_n, Tx_n] a_n^{-1} \leq a_n^{-2} = [x,x] + \frac{1}{n}.$$

Passing in the above inequality to the limit $n \to \infty$, we obtain $[Tx, Tx] \leq [x,x]$.

To derive the first statement from the second one, we assume that for all $x \in X$ the inequality $[Tx, Tx] \leq [x,x]$ is fulfilled and it obviously implies that the operator $T$ is bounded, $\|T\| \leq 1$. Let $x \in X$, $y \in Y$. Let us define a map $r : A_1 \to A_1$ by the equality

$$r(a) = [y, T(xa)].$$

Then

$$r(a)^* r(a) = \|[y, T(xa)]\|^2 \leq \|y\|^2 [T(xa), T(xa)] \leq \|y\|^2 [xa, xa] = \|y\|^2 a^* [x,x] a \leq \|y\|^2 ||x||^2 a^* a.$$

Therefore by the above lemma we have $r(a) = r(1)a$, i.e.

$$[y, T(xa)] = [y, Tx] a = [y, Tx.a],$$

for all $a \in A$ and all $y \in Y$. Hence the proof is complete.

Corollary 4.5. Let $X$ and $Y$ be semi inner product $A$-modules, $T : X \to Y$ be a bounded $A$-linear map. Then

$$\|T\| = \inf \left\{ K^\frac{1}{2} : [Tx, Tx] \leq K [x,x] \right\}.$$

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