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C^*-semi-inner product spaces

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Abstract. In this paper, we introduce a generalization of Hilbert C^*-modules which are pre-Finsler modules, namely, C^*-semi-inner product spaces. Some properties and results of such spaces are investigated, specially the orthogonality in these spaces will be considered. We then study bounded linear operators on C^*-semi-inner product spaces.

1. Introduction

The semi-inner product (s.i.p., in brief) spaces were introduced by Lumer in [13]. He considered vector spaces on which, instead of a bilinear form, there is defined a form $[x, y]$ which is linear in one component only, strictly positive, and satisfies Cauchy-Schwarz's inequality. Six years after Lumer's work, Giles in [8] explored fundamental properties and consequences of semi-inner product spaces. Also, a generalization of semi-inner product spaces was considered by replacing Cauchy-Schwarz's inequality by Holder's inequality in [16]. The concept of $*$-semi-inner product algebras of type (p) was introduced and some properties of such algebras were studied by Siham Galal El-Sayyad and S. M. Khaleelulla in [6]. Also, they obtained some interesting results about generalized adjoints of bounded linear operators on semi-inner product spaces of type (p). In the sequel, a version of adjoint theorem for maps on semi-inner product spaces of type (p), which is obtained by Endre Pap and Radoje Pavlovic in [17]. The concept of s.i.p. has been proved useful both theoretically and practically. The applications of s.i.p. in the theory of functional analysis was demonstrated, for example, in [4, 5, 7, 12, 14, 20, 23, 24].

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On the other hand, the concept of Hilbert C^*-modules which are generalization of the notion of Hilbert spaces, first made by I. Kaplansky in 1953 ([11]). The research on Hilbert C^*-modules began in the 70s (W.L. Paschke, [18]; M.A. Rieffel, [21]). Since then, this generalization of Hilbert spaces was considered by many mathematicians. For more details about Hilbert C^*-modules we refer also to [15]. Also, Finsler modules over C^*-algebras as a generalization of Hilbert C^*-modules, first investigated in [19]. For more on Finsler modules, one may see [1, 2].
In this paper we are going to introduce a new generalization of Hilbert C^*-modules which are between Hilbert C^*-modules and Finsler modules. Furthermore, C^*-semi-inner product spaces are natural generalization of a semi-inner product spaces arising under replacement of the field of scalars $\mathbb{C}$ by C^*-algebras.

2. C^*-semi-inner product space

In this section we investigate basic properties of C^*-semi-inner product spaces.

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and X be a right $\mathcal{A}$-module. A mapping $[.,.] : X \times X \to \mathcal{A}$ is called a C^*-semi-inner product or C^*-s.i.p., in brief, if the following properties are satisfied:

(i) $[x,x] \geq 0$, for all $x \in X$ and $[x,x] = 0$ implies $x = 0$;
(ii) $[x, \alpha y_1 + \beta y_2] = \alpha [x, y_1] + \beta [x, y_2]$, for all $x, y_1, y_2 \in X$ and $\alpha, \beta \in \mathbb{C}$;
(iii) $[x, ya] = [x,y]\, a$ and $[xa, y] = a^* [x,y]$, for all $x, y \in X$ and $a \in \mathcal{A}$;
(iv) $|\, [y,x]\, |^2 \leq \|\, [y,y]\, \|\, [x,x]$.

The triple $(X, \mathcal{A}, [.,.])$ is called a C^*-semi-inner product space or we say X is a semi-inner product $\mathcal{A}$-module.

The property (iv) is called the Cauchy-Schwarz inequality. If $\mathcal{A}$ is a unital C^*-algebra, then one may see that $[\lambda x, y] = \overline{\lambda}\, [x,y]$, for all $x, y \in X$ and $\lambda \in \mathbb{C}$. Indeed, by property (iii) we have

$$[\lambda x, y] = [x(\lambda 1), y] = (\lambda 1)^* [x,y] = \overline{\lambda}\, [x,y]\,.$$

One can easily see that every Hilbert C^*-module is a C^*-semi-inner product space, but the converse is not true in general. The following is an example of a C^*-semi-inner product space which is not a Hilbert C^*-module. First we recall that a semi-inner-product (s.i.p.) in the sense of Lumer and Giles on a complex vector space X is a complex valued function $[x,y]$ on $X \times X$ with the following properties:

1. $[\lambda y+z,x]=\lambda\,[y,x]+[z,x]$ and $[x,\lambda y]=\overline{\lambda}\,[x,y]$, for all complex λ;
2. $[x,x]\geq 0$, for all $x\in X$ and $[x,x]=0$ implies $x=0$;
3. $|\,[x,y]\,|^2\leq [x,x]\,[y,y]$.

A vector space with a s.i.p. is called a semi-inner-product space (s.i.p. space) in the sense of Lumer-Giles (see [13]). In this case one may prove that $\|x\|:=[x,x]^{\frac{1}{2}}$ defines a norm on X. Also, it is well-known that for every Banach space X, there exists a semi-inner product whose norm is equal to its original norm.

It is trivial that every Banach space is a semi-inner product $\mathbb{C}$-module.

Example 2.2. Let Ω be a set and let for any $t\in\Omega$, X_t be a semi-inner product space with the semi inner product $[.,.]^{X_t}$. Define

$$[x,y]_{X_t}:=[x,y]^{X_t}\,,\ x,y\in X_t,$$

trivially $[x,\alpha y+z]_{X_t}=\alpha\,[x,y]_{X_t}+[x,z]_{X_t}$ and $[\alpha x,y]_{X_t}=\overline{\alpha}\,[x,y]_{X_t}$. Let $B=\cup_t X_t$ be a bundle of these semi-inner product spaces over Ω. Suppose $\mathcal{A}=Bd(\Omega)$, the set of all bounded complex-valued functions on Ω, and X is the set of all maps $f:\Omega\to B$ such that $f(t)\in X_t$, for any $t\in\Omega$, with $\sup_{t\in\Omega}\|f(t)\|<\infty$. One can easily see that X is naturally a $Bd(\Omega)$-module. Furthermore it has a $Bd(\Omega)$-valued semi-inner product defined by

$$[f,g]\,(t)=[f(t),g(t)]_{X_t}\,,$$

for $t\in\Omega$, hence it is a C^*-semi-inner product space. One can easily verify that the properties of C^*-semi-inner product are valid.

Suppose $(\mathcal{A}_i,\|.\|_i)$'s, $1\leq i\leq n$, are C^*-algebras, then $\bigoplus_{i=1}^n\mathcal{A}_i$ with its point-wise operations is a C^*-algebra. Moreover,

$$\|\,(a_1,\dots,a_n)\,\|=\max_{1\leq i\leq n}\|a_i\|,$$

is a C^*-norm on $\bigoplus_{i=1}^n\mathcal{A}_i$. Note that $(a_1,\dots,a_n)\in(\bigoplus_{i=1}^n\mathcal{A}_i)_+$ if and only if $a_i\in(\mathcal{A}_i)_+$. Now we may construct the following example.

Example 2.3. Let $(X_i,[.,.]_i)$ be a semi-inner product $\mathcal{A}_i$-module, $1\leq i\leq n$. If for $(a_1,\dots,a_n)\in\mathcal{A}$ and $(x_1,\dots,x_n)\in\bigoplus_{i=1}^n X_i$, we define $(x_1,\dots,x_n)\,(a_1,\dots,a_n)=(x_1a_1,\dots,x_na_n)$ and the C^*-s.i.p. is defined as follows

$$[(x_1,\dots,x_n)\,,(y_1,\dots,y_n)]=([x_1,y_1]_1\,,\dots,[x_n,y_n]_n)\,,$$

then the direct sum $\bigoplus_{i=1}^n X_i$ is a semi-inner product $\mathcal{A}$-module, where $\mathcal{A}=\bigoplus_{i=1}^n\mathcal{A}_i$.

Let $(X, \mathcal{A}, [.,.])$ be a C^*-semi-inner product space. For any $x \in X$, put $|||x||| := \| [x,x] \|^{\frac{1}{2}}$. The following proposition shows that $(X, |||.|||)$ is a normed $\mathcal{A}$-module.

Proposition 2.4. *Let X be a right $\mathcal{A}$-module and $[.,.]$ be a C^*-s.i.p. on X. Then the mapping $x \to \| [x,x] \|^{\frac{1}{2}}$ is a norm on X. Moreover, for each $x \in X$ and $a \in \mathcal{A}$ we have $|||xa||| \leq |||x|||\ \|a\|$.*

Proof. Clearly $|||x||| = \| [x,x] \|^{\frac{1}{2}} \geq 0$ and $|||x||| = 0$ implies that $x = 0$. Also for each $x \in X$, $\lambda \in \mathbb{C}$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |||\lambda x|||^2 &= \| [\lambda x, \lambda x] \| \\ &= |\lambda| \| [\lambda x, x] \| \\ &\leq |\lambda|\ \| \, | [\lambda x, x] | \, \| \\ &\leq |\lambda|\ |||\lambda x|||\ |||x|||. \end{aligned}$$

Hence, $|||\lambda x||| \leq |\lambda|\ |||x|||$. On the other hand, we have

$$|||x||| = \left|\left|\left| \frac{1}{\lambda}.\lambda x \right|\right|\right| \leq \frac{1}{|\lambda|} |||\lambda x|||,$$

therefore $|||\lambda x||| = |\lambda|\ |||x|||$.
Finally for each $x, y \in X$,

$$\begin{aligned} |||x+y|||^2 &= \| [x+y, x+y] \| \\ &\leq \| [x+y, x] \| + \| [x+y, y] \| \\ &\leq \| \, | [x+y, x] | \, \| + \| \, | [x+y, y] | \, \| \\ &\leq |||x+y|||\ |||x||| + |||x+y|||\ |||y||| \\ &\leq |||x+y|||(|||x||| + |||y|||). \end{aligned}$$

Therefore $|||x+y||| \leq |||x||| + |||y|||$.

Also we have

$$\begin{aligned} |||xa|||^2 &= \| [xa, xa] \| \\ &= \| [xa, x] \, a \| \\ &\leq \| [xa, x] \|\ \|a\| \\ &\leq |||xa|||\ |||x|||\ \|a\|, \end{aligned}$$

hence $|||xa||| \leq |||x|||\ \|a\|$.

$\square$

As another result for this norm one can see that for each $x \in X$, $|||x[x,x]||| = |||x|||^3$. Indeed

$$\begin{aligned} |||x\,[x,x]\,|||^2 &= \|\,[x\,[x,x]\,, x\,[x,x]]\,\| \\ &= \|\,[x,x]^3\,\| \\ &= \|\,[x,x]\,\|^3. \end{aligned}$$

The last equality follows from the fact that in any C^*-algebra, we have $\|a^3\| = \|a\|^3$, for any self-adjoint element $a \in \mathcal{A}$.

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras and $\psi : \mathcal{A} \to \mathcal{B}$ be an $*$-isomorphism. If $(X, [.,.]_{\mathcal{A}})$ is a C^*-semi-inner product $\mathcal{A}$-module, then X can be represented as a right $\mathcal{B}$-module with the module action $x\psi(a) = xa$ and is a C^*-semi-inner product $\mathcal{B}$-module with the C^*-semi-inner product defined by*

$$[.,.]_{\mathcal{B}} = \psi\left([.,.]_{\mathcal{A}}\right).$$

Proof. It is clear that X is a right $\mathcal{B}$-module with the mentioned module product. It is easy to verify that the properties (i) to (iii) of definition of C^*-semi-inner product holds for $[.,.]_{\mathcal{B}}$. Now, we prove the property (iv) for $[.,.]_{\mathcal{B}}$. Since $\psi : \mathcal{A} \to \mathcal{B}$ is an $*$-isomorphism, so it is isometric and $\psi(\mathcal{A}_+) \subseteq \mathcal{B}_+$. Thus we have

$$\begin{aligned} |\,[x,y]_{\mathcal{B}}\,|^2 &= |\psi([x,y]_{\mathcal{A}})|^2 \\ &= \psi([x,y]_{\mathcal{A}})^*\psi([x,y]_{\mathcal{A}}) \\ &= \psi([x,y]^*_{\mathcal{A}}\,[x,y]_{\mathcal{A}}) \\ &= \psi(|\,[x,y]_{\mathcal{A}}\,|^2) \\ &\le \|[x,x]_{\mathcal{A}}\|\ \psi([y,y]_{\mathcal{A}}) \\ &= \|\psi([x,x]_{\mathcal{A}})\|\ \psi([y,y]_{\mathcal{A}}) \\ &= \|\,[x,x]_{\mathcal{B}}\,\|\ \ [y,y]_{\mathcal{B}}\,. \end{aligned}$$

□

We will establish a converse statement to the above proposition. Consider that a semi-inner product $\mathcal{A}$-module X is said to be full if the linear span of $\{[x,x] : x \in X\}$, denoted by $[X,X]$, is dense in $\mathcal{A}$.

Theorem 2.6. *Let X be both a full complete semi-inner product $\mathcal{A}$-module and a full complete semi-inner product $\mathcal{B}$-module such that*

$$\|\,[x,x]_{\mathcal{A}}\,\| = \|\,[x,x]_{\mathcal{B}}\,\|,$$

for each $x \in X$, and let $\psi : \mathcal{A} \to \mathcal{B}$ be a map such that $xa = x\psi(a)$ and $\psi([x,x]_{\mathcal{A}}) = [x,x]_{\mathcal{B}}$ where $x \in X,\ a \in \mathcal{A}$. Then ψ is an $*$-isomorphism of C^*-algebras.

Proof. The proof is similar to the proof of Theorem 2.1 [1]. □

We recall that if $\mathcal{A}$ is a C*-algebra, and $\mathcal{A}_+$ is the set of positive elements of $\mathcal{A}$, then a pre-Finsler $\mathcal{A}$-module is a right $\mathcal{A}$-module E which is equipped with a map $\rho : E \to \mathcal{A}_+$ such that

(1) the map $\|.\|_E : x \mapsto \|\rho(x)\|$ is a norm on E; and
(2) $\rho(xa)^2 = a^*\rho(x)^2 a$, for all $a \in \mathcal{A}$ and $x \in E$.

If $(E, \|.\|_E)$ is complete then E is called a Finsler $\mathcal{A}$-module. This definition is a modification of one introduced by N.C. Phillips and N.Weaver [19]. Indeed it is routine by using an interesting theorem of C. Akemann [[19], Theorem 4] to show that the norm completion of a pre-Finsler $\mathcal{A}$-module is a Finsler $\mathcal{A}$-module. Now, it is trivial to see that every C^*-semi-inner product space $(X, \mathcal{A}, [.,.])$ is a pre-Finsler module with the function $\rho : X \to \mathcal{A}_+$ defined by $\rho(x) = [x, x]^{\frac{1}{2}}$. Thus every complete C^*-semi-inner product space enjoys all the properties of a Finsler module.

Proposition 2.7 ([19])**.** *Let $\mathcal{A} = C_0(X)$ and let E be a Finsler $\mathcal{A}$-module. Then ρ satisfies*

$$\rho(x + y) \leq \rho(x) + \rho(y),$$

for all $x, y \in E$,

Replacing the real numbers, as the codomain of a norm, by an ordered Banach space we obtain a generalization of normed space. Such a generalized space, called a cone normed space, was introduced by Rzepecki [22].

Corollary 2.8. *Let $(X, [.,.])$ be a semi-inner $C(X)$-module, then $\|.\|_c : X \to C(X)$ defined by $\|x\|_c = [x, x]^{\frac{1}{2}}$ is a cone norm on X.*

3. Orthogonality in C^*-semi-inner product spaces

In this section we study the relations between Birkhoff-James orthogonality and the orthogonality in a C^*-semi-inner product spaces.

In a normed space X (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), the Birkhoff-James orthogonality (cf.[3, 9]) is defined as follows

$$x \perp_B y \;\Leftrightarrow\; \forall \alpha \in \mathbb{K};\; \|x + \alpha y\| \geq \|x\|.$$

Theorem 3.1. *Let X be a right $\mathcal{A}$-module and $[.,.]$ be a C^*-s.i.p. on X. If $x, y \in X$ and $[x, y] = 0$ then $x \perp_B y$.*

Proof. Let $[x, y] = 0$. If $x = 0$ then by the definition of Birkhoff-James orthogonality it is obvious that $x \perp_B y$. Now if $x \neq 0$, then for all

$\alpha \in \mathbb{K}$,

$$\begin{aligned}|||x|||^2 - |\alpha| \, \| \, [x, y] \, \| &\leq \| \, [x, x + \alpha y] \, \| \\ &\leq |||x||| \; |||x + \alpha y|||.\end{aligned}$$

Hence,

$$-|\alpha| \, \| \, [x, y] \, \| \leq |||x|||(|||x + \alpha y||| - |||x|||).$$

But $x \neq 0$ and $[x, y] = 0$, so by the above inequality we conclude that $|||x + \alpha y||| \geq |||x|||$, which shows that $x \perp_B y$. □

In the sequel we try to find a sufficient condition for x, y to be orthogonal in the C^*-semi-inner product. For; we need some preliminaries. We remind that in a C^*-algebra $\mathcal{A}$ and for any $a \in \mathcal{A}$ there exist self-adjoint elements $h, k \in \mathcal{A}$ such that $a = h + ik$. We apply $Re(a)$ for h.

Definition 3.2. A C^*-s.i.p. $[.,.]$ on right $\mathcal{A}$-module X is said to be continuous if for every $x, y \in X$ one has the equality

$$\lim_{t \to 0} Re \, [x + ty, y] = Re \, [x, y] \, ,$$

where $t \in \mathbb{R}$.

Example 3.3. In Example 2.2, $\Omega = \{1, 2, \ldots, n\}$ and X be the semi inner product $Bd(\Omega)$-module defined in Example 2.2. If X_t is a continuous s.i.p. space (see [8]), for all $t \in \Omega$, then X is a continuous C^*-s.i.p space. Indeed it is clear that

$$\sup_{t \in \Omega} \| Re \, [f(t) + \alpha g(t), g(t)]_{X_t} - Re \, [f(t), g(t)]_{X_t} \|,$$

tends to 0, when $\alpha \to 0$.

Theorem 3.4. *Let X be a right $\mathcal{A}$-module and let $[.,.]$ be a continuous C^*-s.i.p. on X such that $[x, y] \in \mathcal{A}_{sa}$ for each $x, y \in X$. If for $x, y \in X$*

$$[x + ty, x + ty] \geq [x, x]^{\frac{1}{2}} \; |||x + ty|||,$$

for all $t \in \mathbb{R}$, then $[x, y] = 0$.

Proof. It is clear that for each $a \in \mathcal{A}_{sa}$, we have $a \leq |a|$. Now assume that

$$[x + ty, x + ty] \geq [x, x]^{\frac{1}{2}} \; |||x + ty|||,$$

for all $x, y \in X$ and $t \in \mathbb{R}$. By Cauchy-Schwarz inequality (iv) and the fact that $[x, y] \in \mathcal{A}_{sa}$ for each $x, y \in X$, we get

$$\begin{aligned}[x + ty, x + ty] &\geq [x, x]^{\frac{1}{2}} \; |||x + ty||| \\ &\geq | \, [x + ty, x] \, | \\ &\geq [x + ty, x] \, ,\end{aligned}$$

so, we have: $t\,[x+ty,y] \geq 0$ for each $t \in \mathbb{R}$. Thus for $t \geq 0$ we have $[x+ty,y] \geq 0$ and for $t \leq 0$ we have $[x+ty,y] \leq 0$. Now, since $[.,.]$ is a continuous C^*-s.i.p. and $\mathcal{A}_+$ is a closed subset of $\mathcal{A}$, so we have

$$\begin{aligned} 0 \geq [x,y] &= \lim_{t\to 0^-} [x+ty,y] \\ &= \lim_{t\to 0^+} [x+ty,y] = [x,y] \geq 0, \end{aligned}$$

thus $[x,y]=0$. □

4. Bounded Linear operators on C^*-semi-inner product spaces

Theorem 4.1. *Let X be a semi inner product $\mathcal{A}$-module. Then for every $y \in X$ the mapping $f_y : X \to \mathcal{A}$ defining by $f_y(x) = [y,x]$ is a $\mathcal{A}$-linear continuous operator endowed with the norm generated by C^*-s.i.p. Moreover, $\|f_y\| = |||y|||$.*

Proof. The fact that f_y is a $\mathcal{A}$-linear operator follows by (ii) and (iii) of Definition 2.1. Now, using Schwartzs inequality (iv) we get;

$$\|f_y(x)\| = \|\,[y,x]\,\| \leq |||y|||\ |||x|||,$$

which implies that f_y is bounded and

$$\|f_y\| \leq |||y|||.$$

On the other hand, we have

$$||f_y|| \geq \left\| f_y\left(\frac{y}{|||y|||}\right)\right\| = |||y|||,$$

and then $\|f_y\| = |||y|||$. □

Corollary 4.2. *If X is a right $\mathcal{A}$-module and $[.,.]$ a C^*-s.i.p. on X, then for all $x \in X$ we have*

$$|||x||| = \sup\{\|\,[x,y]\,\| : \ |||y||| \leq 1\}.$$

Lemma 4.3 ([10?])**.** *Let $\mathcal{A}$ be a unital C^*-algebra. Let $r : \mathcal{A} \to \mathcal{A}$ be a linear map such that for some constant $K \geq 0$ the inequality $r(a)^* r(a) \leq Ka^*a$ is fulfilled for all $a \in \mathcal{A}$. Then $r(a) = r(1)a$ for all $a \in \mathcal{A}$.*

Theorem 4.4. *Let X and Y be semi inner product $\mathcal{A}$-modules, $T : X \to Y$ be a linear map. Then the following conditions are equivalent:*

- (i) *The operator T is bounded and $\mathcal{A}$-linear, i.e. $T(xa) = Tx.a$ for all $x \in X$, $a \in \mathcal{A}$;*
- (ii) *There exists a constant $K \geq 0$ such that for all $x \in X$ the operator inequality $[Tx,Tx] \leq K\,[x,x]$ holds.*

Proof. To obtain the second statement from the first one, assume that $T(xa) = Tx.a$ and $\|T\| \leq 1$. If C^*-algebra $\mathcal{A}$ does not contain a unit, then we consider modules X and Y as modules over C^*-algebra $\mathcal{A}_1$, obtained from $\mathcal{A}$ by unitization. For $x \in X$ and $n \in \mathbb{N}$, put

$$a_n = \left([x,x] + \frac{1}{n}\right)^{-\frac{1}{2}}, \qquad x_n = xa_n,$$

Then $[x_n, x_n] = a_n^* [x,x] a_n = [x,x] ([x,x] + \frac{1}{n})^{-1} \leq 1$, therefore $\|x_n\| \leq 1$, hence $\|Tx_n\| \leq 1$. Then for all $n \in \mathbb{N}$ the operator inequality $[Tx_n, Tx_n] \leq 1$ is valid. But

$$[Tx, Tx] = a_n^{-1} [Tx_n, Tx_n] a_n^{-1} \leq a_n^{-2} = [x,x] + \frac{1}{n}.$$

Passing in the above inequality to the limit $n \to \infty$, we obtain $[Tx, Tx] \leq [x,x]$.

To derive the first statement from the second one, we assume that for all $x \in X$ the inequality $[Tx, Tx] \leq [x,x]$ is fulfilled and it obviously implies that the operator T is bounded, $\|T\| \leq 1$. Let $x \in X$, $y \in Y$. Let us define a map $r : \mathcal{A}_1 \to \mathcal{A}_1$ by the equality

$$r(a) = [y, T(xa)].$$

Then

$$\begin{aligned} r(a)^* r(a) &= |\,[y, T(xa)]\,|^2 \\ &\leq |||y|||^2 [T(xa), T(xa)] \\ &\leq |||y|||^2 [xa, xa] = |||y|||^2 a^* [x,x] a \\ &\leq |||y|||^2 |||x|||^2 a^* a. \end{aligned}$$

Therefore by the above lemma we have $r(a) = r(1)a$, i.e.

$$[y, T(xa)] = [y, Tx] a = [y, Tx.a],$$

for all $a \in \mathcal{A}$ and all $y \in Y$. Hence the proof is complete.

$\square$

Corollary 4.5. *Let X and Y be semi inner product $\mathcal{A}$-modules, $T : X \to Y$ be a bounded $\mathcal{A}$-linear map. Then*

$$\|T\| = \inf\left\{K^{\frac{1}{2}} : [Tx, Tx] \leq K [x,x]\right\}.$$

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