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A Class of New Results in FLM Algebras

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Abstract. In this paper, we first derive some results by using the Gelfand spectrum and spectrum in FLM algebras. Then, the characterizations of multiplicative linear mappings are also discussed in these algebras.

1. Introduction

Non-normed topological algebras were initially introduced around the year 1950 for the manipulation of certain classes of these algebras that appeared naturally in mathematics and physics. Some results concerning such topological algebras had been published earlier in 1947 by R. Arens [6]. It was in 1952 that Arens and Michael [6] independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras. Here, we would like to mention about the predictions made by the famous Soviet mathematician M.A. Naimark, an expert in the area of Banach algebras, in 1950 regarding the importance of non-normed algebras and the development of their related theory. During his study concerning cosmology, G. Lassner [6] realized that the theory of normed topological algebras was insufficient for his study purposes.
Ansari [1] introduced the notion of fundamental topological spaces and algebras and proved the Cohen's factorization theorem for these algebras. Fundamental locally multiplicative (FLM) topological algebras with a property similar to the normed algebras were introduced later by Ansari [2]. Some celebrated theorems of Banach algebras have been generalized for FLM algebras in the past studies [2, 3].

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The aim of this paper is twofold. Firstly, we derive some results by using the Gelfand spectrum and spectrum in FLM algebras. Secondly, we discuss the problem of how to recognize a multiplicative linear mapping in these algebras. This paper is divided into the following sections. In Section 2, we have gathered a collection of definitions and symbols, and in Section 3, we derive some results by using the Gelfand spectrum and spectrum in FLM algebras. In Section 4, the characterizations of multiplicative linear mappings are discussed in these algebras.

2. Definitions and Symbols

In this section, we present a collection of definitions and symbols, which are also listed in the references.

Definition 2.1. A topological linear space A is said to be fundamental one if there exists $b > 1$ such that for every sequence $\{x_n\}$ of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that $\{x_n\}$ is Cauchy.

Definition 2.2. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

Definition 2.3. A fundamental topological algebra is said to be locally multiplicative, if there exists a neighbourhood U_0 of zero such that for every neighbourhood V of zero, the sufficiently large powers of U_0 lie in V. Such an algebra is known an FLM algebra.

Definition 2.4. Let (A, d_A) be a metrizable topological algebra. We say that A is a sub-multiplicative metrizable topological algebra if

$$d_A(0, xy) \leq d_A(0, x)d_A(0, y),$$

for all $x, y \in A$. For abbreviation, we denote $d_A(0, x)$ by $D_A(x)$ for any $x \in A$.

Definition 2.5. Let A be an algebra with unit element. The set of all invertible elements of A is denoted by $Inv(A)$.

Definition 2.6. Let A be an algebra with the unit element e. The spectrum of $x \in A$ is denoted by $\mathrm{sp}(x)$ and defined as:

$$\mathrm{sp}(x) = \{\lambda \in C : \lambda e - x \text{ is not invertible}\}.$$

Definition 2.7. Let A be a complex algebra with unit element.

i) Define the Gelfand spectrum of A as $\phi_A := Hom(A, C)$ with the topology of pointwise convergence on A. Note that $0 \notin \phi_A$ because we require homomorphisms to respect the unit elements.

ii) Each element $a \in A$ gives rise to a continuous function $\hat{a} : \phi_A \to C$ by $\hat{a}(\varphi) := \varphi(a)$. The function $\hat{a}$ is called the Gelfand transform of a. The map $a \to \hat{a} : A \to C(\phi_A)$, which is a homomorphism of unital algebras, is called the Gelfand homomorphism of the algebra A.

3. Some New Results by Using the Gelfand Spectrum and Spectrum in FLM Algebra

Spectral theory and the Gelfand spectrum have already been studied in Banach algebras. Now we would like to give similar results for FLM algebras.

Theorem 3.1 (The Gelfand homomorphism)**.** *Let A be a commutative complete metrizable FLM algebra with the unit element e and ϕ_A be the Gelfand spectrum of A. Then the following statements hold:*

i) *Every element $a \in A$ satisfies*

$$\operatorname{sp}(x) = \{\varphi(a) : \varphi \in \phi_A\} = \hat{a}(\varphi_A);$$

ii) *The Gelfand spectrum ϕ_A is a compact Hausdorff space.*

Proof. The proof follows easily from [3, 5.5] and [3, 5.6]. □

Theorem 3.2. *Let A be a commutative complete metrizable FLM algebra with the unit element e and ϕ_A be the Gelfand spectrum of A. Then the following statements hold:*

i) $\operatorname{sp}(x+y) \subseteq \operatorname{sp}(x) + \operatorname{sp}(y)$*;*
ii) $\operatorname{sp}(xy) \subseteq \operatorname{sp}(x)\operatorname{sp}(y)$.

Proof. By the preceding theorem, we have

$$\operatorname{sp}(x+y) = \operatorname{Im}(\widehat{(x+y)}) = \operatorname{Im}(\hat{x} + \hat{y}) \subseteq \operatorname{Im}(\hat{x}) + \operatorname{Im}(\hat{y}) = \operatorname{sp}(x) + \operatorname{sp}(y).$$

The analogous calculation holds for the product xy. □

The fact that $\exp(0) = 1$ and $\exp(-a)\exp(a) = 1$ and for all $x, y \in A$ when $xy = yx$, $\exp(x)\exp(y) = \exp(x+y)$, have been considered in [3, 5.1]. These are of course some well-known results of Banach algebras. Now, we would like to prove the following theorem for commutative complete metrizable FLM algebras.

Theorem 3.3. *Let A be a commutative complete metrizable FLM algebra with the unit element e_A and ϕ_A be the Gelfand spectrum of A . Suppose that*

$$\exp(a) = \exp(b), \qquad \hat{a} = \hat{b}, \qquad a, b \in A,$$

then $a = b$.

Proof. By [3, 5.5], and Definition 2.7, we have

$$\hat{a}(\phi_A) = \{\, \varphi(a) : \quad \varphi \in \phi_A \,\} = \mathrm{sp}(a).$$

Set $c = b - a$ *or* $c + a = b$. Then we have

$$\exp(a)\exp(c) \ = \exp(a + c) \ = \ \exp(b),$$

hence, $\exp(c) \ = e_A$ and so

$$c\left(e_A \ + \sum_{k=1}^{\infty} \frac{c^k}{(k+1)!}\right) = 0.$$

By hypothesis, since $c \in rad(A)$, it follows from [2, 4.1] that

$$\left(e_A \ + \sum_{k=1}^{\infty} \frac{c^k}{(k+1)!}\right) \in Inv(A),$$

and so $c = 0$. Thus, $a = b$. □

The following theorem is another application of J. Zemanek's theorem for complete metrizable FLM algebras [9, 3.4].

Theorem 3.4. *Let A be a semisimple complete metrizable FLM algebra with sub-multiplicative meter d_A. If x , $y \in A$ do not commute, then there exists $a \in A$ such that*

$$\mathrm{sp}(a - xy) \neq \ \mathrm{sp}(a - yx).$$

Proof. Let $\mathrm{sp}(a - xy) = \ \mathrm{sp}(a - yx)$ for all $a \in A$. Writing $a + xy$ instead of a, it follows that

$$\mathrm{sp}(a) = \ \mathrm{sp}(a + xy - yx), \quad \forall a \in A,$$

by [9, 3.4], $xy - yx \in rad(B)$. Since A is semisimple, we conclude that $xy - yx = 0$ or $xy = yx$, which is a contradiction. □

4. Multiplicative Linear Mappings

A characterization of multiplicative linear functionals in a Banach algebras was given by Gleason-Kahane-Zelazko [4]. A version of the Gleason-Kahane-Zelazko theorem is also proved for FLM algebras [3, 5.5]. Now we weaken the assumption of this theorem and prove it for sub-multiplicative FLM algebras without linearity. Before giving the main theorem, we need the following lemma.

Lemma 4.1. *Let A be a complete metrizable FLM algebra with the unit element e and φ be an R-linear on A such that $\varphi(x) \in \mathrm{sp}(x)$ for each $x \in A$, then φ is C-linear.*

Proof. The proof of this lemma follows by the same reasoning as in [7, 2.1]. □

Theorem 4.2. *Let A be a complete metrizable FLM algebra with the sub-multiplicative meter d_A and the unit element e. Let $f : A \to C$ satisfy $f(0) = 0$ and $f(x) - (y) \in \mathrm{sp}(x - y)$ for each $x, y \in A$. Then f is multiplicative and linear.*

Proof. First we may assume that A is separable. Suppose that f has a differential at a point $a \in A$. We have

$$\frac{f(a + rx) - f(a)}{r} \in \frac{\mathrm{sp}(a + rx - a)}{r} = \mathrm{sp}(x), \quad r \in R,\ r \neq 0, x \in A.$$

So,

$$(Df)_a(x) = \lim_{r \to 0} \frac{f(a + rx) - f(a)}{r} \in \mathrm{sp}(x).$$

Thus, by Lemma 4.1, the differential is C-linear. On the other hand, by [9], the following relation holds for spectral radius

$$r(x) \leq D_A(x).$$

Hence, we have

$$|f(x) - f(y)| \in |\mathrm{sp}(x - y)| \leq D_A(x - y).$$

Thus, f is a Lipschitz function. By [7, 2.3] and [7, 2.4], we obtain that f is an entire function. As in [7, 1.2], we define the function $f_{a,b} : C \to C$ by $f_{a,b}(z) = f(az + b)$, for $a, b \in A$. Therefore, $f_{a,b}$ is Lipschitz and entire. So, it is affine. By the same reasoning as in [6, 1.2], and the Gleason-Kahane-Zelazko theorem for FLM algebras [3, 5.5], we conclude that f is linear and multiplicative, when A is separable. Now we consider the general case. Let $a, b \in A$. Clearly $[e,a,b]$ (the subalgebra of A generated by e,a,b in A) is the FLM subalgebra of A. The function f in Theorem 4.2, restricted to the subalgebra $[e,a,b]$ of A satisfies conditions of the theorem. As $[e,a,b]$ is separable, from the preceding part of the proof, it follows that $f\big|_{[e,a,b]}$ is multiplicative and linear. Since a and b is arbitrary, we deduce that f is multiplicative and linear in the whole of A. □

The next results show the applications of the Gleason-Kahane-Zelazko theorem.

Theorem 4.3. *Let A be a complete metrizable FLM algebra with the sub-multiplicative meter d_A and the unit element e and let B be a semisimple commutative complete metrizable FLM algebra with the unit element e'. Suppose that T is a map from A into B such that*

$$\mathrm{sp}(T(x) + T(y)) \subseteq \mathrm{sp}(x + y), \quad \forall x, y \in A.$$

Then T is linear and multiplicative.

Proof. Since B is semisimple and $\mathrm{sp}(T(y)+T(-y)) \subseteq \mathrm{sp}(y+(-y)) = \{0\}$ for each $y \in A$, we obtain that $T(-y) = -T(y)$ for each $y \in A$. Let φ be an element in ϕ_B . Then $\varphi o T$ is a function from A into C such that $\varphi o T(0) = 0$, and by the Gleason-Kahane-Zelazko theorem in [3, 5.5], we have

$$\begin{aligned} \varphi o T(x) - \varphi o T(y) &= \varphi\left(T(x) - T(y)\right) \in \mathrm{sp}\left(T(x) - T(y)\right) \\ &= \mathrm{sp}\left(T(x) + T(-y)\right) \subseteq \mathrm{sp}(x-y). \end{aligned}$$

Thus, $\varphi o T$ is linear and multiplicative by Theorem 4.2. Now we have

$$\varphi(T(xy) - T(xy)) = \varphi o T(xy) - \varphi o T(x)\varphi o T(y) = 0.$$

Since B is semisimple and $\varphi \in \phi_B$ is arbitrary, it follows that

$$T(xy) = T(x)T(y).$$

□

Theorem 4.4. *Let A be a complete metrizable FLM algebra with the sub-multiplicative meter d_A and the unit element e and let B be a semisimple commutative complete metrizable FLM algebra with the unit element e', and $p(u,v) = \lambda u + \mu v$, $(\lambda\mu \neq 0)$ be two-variable polynomials. Suppose that T is a map from A into B such that*

$$\mathrm{sp}(p(Tf, Tg)) \subseteq \mathrm{sp}(p(f,g)), \quad \forall f, g \in A.$$

If $\lambda + \mu \neq 0$, then T is linear and multiplicative.

Proof. Let $f, g \in A$. Since $\lambda \neq 0$, we have

$$\mathrm{sp}\left(Tf + \frac{\mu}{\lambda}Tg\right) \subseteq \mathrm{sp}\left(f + \frac{\mu}{\lambda}g\right),$$

so,

$$\mathrm{sp}\left(T(-\frac{\mu}{\lambda}g) + \frac{\mu}{\lambda}Tg\right) \subseteq \mathrm{sp}\left(-\frac{\mu}{\lambda}g + \frac{\mu}{\lambda}g\right) = \{0\}.$$

By taking $f = -\frac{\mu}{\lambda}g$, we obtain

$$T(-\frac{\mu}{\lambda}g) = -\frac{\mu}{\lambda}Tg, \quad \forall g \in A.$$

Since B is semisimple, it follows that

$$\begin{aligned} \mathrm{sp}\left(Tf - Tg\right) &= \mathrm{sp}\left(Tf - T\left(\frac{\mu}{\lambda} - \frac{\lambda}{\mu}g\right)\right) \\ &= \mathrm{sp}\left(Tf + \frac{\mu}{\lambda}T\left(-\frac{\lambda}{\mu}g\right)\right) \\ &\subseteq \mathrm{sp}(f-g) \end{aligned}$$

for all $f, g \in A$. Now we prove that $T(0) = 0$ if $\lambda + \mu \neq 0$. By taking $f = g = 0$, we have

$$\operatorname{sp}(\lambda T(0) + \mu T(0)) \subseteq \operatorname{sp}(\lambda.0 + \mu.0) = \{0\}.$$

Hence, $T(0) = 0$. The result follows from the proof of Theorem 4.3. □

Corollary 4.5. *If $\lambda + \mu = 0$, in Theorem 4.4, then $T - T(0)$ is linear and multiplicative.*

Proof. We consider a map W from A into B by $Wf = Tf - T(0)$. Clearly W is onto and we have

$$\operatorname{sp}(Wf - Wg) \subseteq \operatorname{sp}(f - g), \quad \forall f, g \in A.$$

So, the result follows from the proof of Theorem 4.3. □

Theorem 4.6. *Let A and B be two complete metrizable FLM algebras with unit elements e_A and e_B respectively, and let B be commutative and semisimple. If $T : A \to B$ is a linear map such that $T(e_A) = e_B$ and $T(a) \in Inv(B)$ for $a \in Inv(A)$, then T is multiplicative.*

Proof. Let $\varphi \in \phi_B$ and $x, y \in A$. Then we have

$$\varphi o T(a) = \varphi(T(a)) \neq 0, \qquad \varphi o T(e_A) = 1,$$

this implies that $\ker \varphi o T \subseteq \sin g\, A$. By the Gleason-Kahane-Zelazko theorem for FLM algebras [3, 5.5], we conclude that $\varphi o T$ is multiplicative. The remainder of the proof follows from Theorem 4.3. □

Theorem 4.7. *Let A be a complete metrizable FLM algebra with the unit element e_A and let B be a semisimple complete metrizable FLM algebra with the sub-multiplicative meter d_B and the unit element e_B. If $\varphi : A \to B$ is surjective linear map such that for all $x \in A$*

$$\operatorname{sp}(x) \neq \emptyset, \qquad \operatorname{sp}(x) = \operatorname{sp}(\varphi(x)),$$

then $\varphi(e_A) = e_B$.

Proof. We have

$$\operatorname{sp}(\varphi(e_A)) = \operatorname{sp}(e_A) = 1,$$

and so $\varphi(e_A) = e_B + q$ where $q \in B$ is quasi-nilpotent. For all $x \in A$, we obtain

$$\begin{aligned} 1 + \operatorname{sp}(q + \varphi(x)) &= \operatorname{sp}(e_B + q + \varphi(x)) \\ &= \operatorname{sp}(\varphi(e_A) + \varphi(x)) \\ &= \operatorname{sp}(\varphi(e_A + x)) \\ &= \operatorname{sp}(e_A + x) \\ &= 1 + \operatorname{sp}(x), \end{aligned}$$

hence,

$$\mathrm{sp}(q+\varphi(x)) = \mathrm{sp}(x) = \mathrm{sp}(\varphi(x)).$$

By [9, 3.4], $q \in rad(B)$. Since B is semisimple, we conclude that $\varphi(e_A) = e_B$. □

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