Some fixed point theorems for $C$-class functions in $b$-metric spaces

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Abstract. In this paper, via $C$-class functions, as a new class of functions, a fixed theorem in complete $b$-metric spaces is presented. Moreover, we study some results, which are direct consequences of the main results. In addition, as an application, the existence of a solution of an integral equation is given.

1. Introduction

In this exciting context, Bakhtin [5] and Czerwik [8, 9] developed the notion of $b$-metric spaces in connection with some problems concerning the convergence of measurable functions with respect to a measure. Moreover they proved some fixed point theorems for single-valued and multi-valued mappings in $b$-metric spaces. In addition, many authors studied the fixed point theory in this space such as [1-4, 6, 12]. Here, we study some fixed point theorems for a $C$-class functions. In order to do this, we recall some concepts as follows:

Definition 1.1 ([8]). Let $X$ be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is a $b$-metric if for all $x, y, z \in X$ the following conditions are satisfied:

$(b_1)$ $d(x, y) = 0$ iff $x = y$,
$(b_2)$ $d(x, y) = d(y, x),$
$(b_3)$ $d(x, z) \leq s[d(x, y) + d(y, z)].$

The pair $(X, d)$ is called a $b$-metric space.
Definition 1.2 (I). Let \((X, d)\) be a \(b\)-metric space. Then a sequence \(\{x_n\}\) in \(X\) is called:
(a) \(b\)-convergent if and only if there exists \(x \in X\) such that \(d(x_n, x) \to 0\) as \(n \to \infty\). In this case, we write \(\lim_{n \to \infty} x_n = x\).
(b) \(b\)-Cauchy if and only if \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\).

The \(b\)-metric space \((X, d)\) is complete if every \(b\)-Cauchy sequence \(b\)-converges in \(X\).

Definition 1.3 (I). A mapping \(F : [0, \infty)^2 \to \mathbb{R}\) is called a \(C\)-class function if it is continuous and satisfies the following axioms:

I) \(F(s, t) \leq s\).
II) \(F(s, t) = s\) implies that either \(s = 0\) or \(t = 0\); for all \(s, t \in [0, \infty)\).

Note for some \(F, F(0, 0) = 0\). Denote the set of \(C\)-class functions by \(C\).

Example 1.4 (I). The following functions \(F : [0, \infty)^2 \to \mathbb{R}\) are elements of \(C\), for all \(s, t \in [\infty)\):

1) \(F(s, t) = s - t\), \(F(s, t) = s \Rightarrow t = 0\).
2) \(F(s, t) = ms, 0 < m < 1\), \(F(s, t) = s \Rightarrow s = 0\).
3) \(F(s, t) = (s + l)^{(1/(1 + t))} - l, l > 1, r \in (0, \infty)\), \(F(s, t) = s \Rightarrow t = 0\).
4) \(F(s, t) = s \beta(s), \beta : [0, \infty) \to (0, 1)\) and is continuous, \(F(s, t) = s \Rightarrow s = 0\).
5) \(F(s, t) = s - \varphi(s), F(s, t) = s \Rightarrow s = 0\), here \(\varphi : [0, \infty) \to [0, \infty)\)
is a continuous function such that \(\varphi(t) = 0 \Leftrightarrow t = 0\).
6) \(F(s, t) = sh(s, t), F(s, t) = s \Rightarrow s = 0\), where \(h : [0, \infty) \times [0, \infty) \to [0, \infty)\) is a continuous function such that \(h(t, s) < 1\) for all \(t, s > 0\).

Definition 1.5 (I). A function \(\psi : [0, \infty) \to [0, \infty)\) is called an altering distance function if the following properties are satisfied:

1) \(\psi\) is non-decreasing and continuous.
2) \(\psi(t) = 0\) if and only if \(t = 0\).

Definition 1.6 (I). An ultra altering distance function is a continuous, nondecreasing mapping \(\varphi : [0, \infty) \to [0, \infty)\) such that \(\varphi(t) > 0, t > 0\) and \(\varphi(0) \geq 0\).

2. Main results

Theorem 2.1. Let \((X, d)\) be a complete \(b\)-metric space on \(X\) and \(f : X \to X\) be a self mapping. Suppose

\[\psi(s d(fx, fy)) \leq F(\psi(M(x, y)), \varphi(M(x, y))) + LN(x, y),\]

for all \(x, y \in X\), where \(L \geq 0\), \(F : [0, \infty)^2 \to \mathbb{R}\) is \(C\)-class function, \(\psi : [0, \infty) \to [0, \infty)\) is an altering distance function, \(\varphi : [0, \infty) \to [0, \infty)\) is an altering distance function, \(\varphi : [0, \infty) \to [0, \infty)\)
is an ultra altering distance function and
\[ M(x, y) = \max \left\{ d(x, y); \frac{d(x, fx) d(y, fy)}{1 + d(fx, fy)} \right\}, \]
and
\[ N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}. \]

Then \( f \) has a unique fixed point.

**Proof.** Let \( x_0 \in X \). Define a sequence \( \{x_n\} \subset X \) by \( x_n = f^n(x_0) = fx_{n-1} \) for \( n \in \mathbb{N} \cup \{0\} \). In order to show that \( \{x_n\} \) is a Cauchy sequence, first we show \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \). From \( (2.1) \) we have,
\[
\psi(d(x_n, x_{n+1})) \leq \psi(sd(x_n, x_{n+1})) = \psi(sd(f_{n-1}, x_n)) \leq F(\psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))) + LN(x_{n-1}, x_n),
\]
where
\[
M(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, f x_{n-1}) d(x_n, f x_n)}{1 + d(f x_{n-1}, f x_n)} \right\} = \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \right\} = d(x_{n-1}, x_n),
\]
and
\[
N(x_{n-1}, x_n) = \min \{ d(x_{n-1}, f x_n), d(x_n, f x_n), d(x_{n-1}, f x_{n-1}), d(x_n, f x_{n-1}) \} = \min \{ d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n) \} = 0.
\]

Therefore
\[
(2.2) \quad \psi(d(x_n, x_{n+1})) \leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))).
\]
Thus
\[
(2.3) \quad \psi(d(x_n, x_{n+1})) \leq F(\psi(d(x_n, x_{n-1})), \varphi(d(x_n, x_{n-1}))) \leq \psi(d(x_n, x_{n-1})).
\]
Since \( \psi \) is non-decreasing, then \( d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \). This means \( \{d(x_n, x_{n+1})\} \) is a decreasing sequence. Thus it converges and there exists \( r \geq 0 \) such that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = r \). Let \( n \to \infty \), then \( (2.3) \) implies
\[
\psi(r) \leq F(\psi(r), \lim_{n \to \infty} \inf \varphi(d(x_{n-1}, x_n))) \leq F(\psi(r), \varphi(r)) \leq \psi(r).
\]
Thus \( \psi(r) = 0 \). Therefore \( r = 0 \), that is
\[
(2.4) \quad \lim_{n \to \infty} d(x_{n-1}, x_n) = 0.
\]
Now, we prove that the sequence \( \{x_n\} \) is a Cauchy sequence. Suppose that \( \{x_n\} \) is not a Cauchy sequence. Then there exists an \( \varepsilon > 0 \) for which we can find two sequences of positive integers \( \{m(k)\} \) and \( \{n(k)\} \) such that for all positive integers \( k \), \( n(k) > m(k) > k \) and \( d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \).

Let \( n(k) \) be the smallest such positive integer \( n(k) > m(k) > k \) such that

\[
(2.5) \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon.
\]

By \((2.6)\)

\[
(2.6) \quad \frac{\varepsilon}{s} \leq \liminf_{k \to \infty} d(x_{m(k)}, x_{n(k)-1}),
\]

and by \((2.1)\)

\[
(2.7) \quad \psi(sd(x_{m(k)+1}, x_{n(k)})) \leq F(\psi(M(x_{m(k)}, x_{n(k)-1})), \varphi(M(x_{m(k)}, x_{n(k)-1}))) + LN(x_{m(k)}, x_{n(k)-1}),
\]

where

\[
M(x_{m(k)}, x_{n(k)-1}) = \max \left\{ d(x_{m(k)}, x_{n(k)-1}), \frac{d(x_{m(k)}, f(x_{n(k)-1})) d(x_{m(k)}, f(x_{m(k)}))}{1 + d(fx_{n(k)-1}, fx_{m(k)})} \right\}
\]

\[
= \max \left\{ d(x_{m(k)}, x_{n(k)-1}), \frac{d(x_{n(k)-1}, x_{n(k)}) d(x_{m(k)}, x_{m(k)+1})}{1 + d(x_{n(k)}, x_{m(k)+1})} \right\}.
\]

Let \( k \to \infty \) in the above inequalities and applying \((2.7)\), \((2.8)\) and \((2.9)\), we get

\[
(2.8) \quad \frac{\varepsilon}{s} \leq \liminf_{k \to \infty} M(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k \to \infty} M(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon.
\]

Also

\[
\lim_{k \to \infty} N(x_{m(k)}, x_{n(k)-1})
= \lim_{k \to \infty} \min \left\{ d(x_{n(k)-1}, f(x_{n(k)-1})), d(x_{m(k)}, f(x_{m(k)})) \right\}
\]

\[
+ \min \left\{ d(x_{n(k)-1}, f(x_{n(k)})), d(x_{m(k)}, f(x_{n(k)-1})) \right\}
\]

\[
= \lim_{k \to \infty} \min \left\{ d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}) \right\}
\]

\[
= 0.
\]

Note that

\[
(2.9) \quad d(x_{m(k)}, x_{n(k)}) - sd(x_{m(k)}, x_{m(k)+1}) \leq sd(x_{m(k)+1}, x_{n(k)}).
\]
By (2.7), (2.8) and (2.9) we obtain

\[ \psi (\varepsilon) \leq \psi \left( \lim_{k \to \infty} \sup d \left( x_{m(k)}, x_{n(k)} \right) \right) \]
\[ = \psi \left( \lim_{k \to \infty} \sup s d \left( x_{m(k)+1}, x_{n(k)} \right) \right) \]
\[ \leq F \left( \psi \left( \lim_{k \to \infty} M \left( x_{m(k)}, x_{n(k)-1} \right) \right) \right) \]
\[ , \varphi \left( \lim_{k \to \infty} \inf M \left( x_{m(k)}, x_{n(k)-1} \right) \right) \]
\[ \leq F \left( \psi (\varepsilon), \varphi \left( \lim_{k \to \infty} \inf M \left( x_{m(k)}, x_{n(k)-1} \right) \right) \right) \]
\[ \leq \psi (\varepsilon) , \]

so

\[ \psi (\varepsilon) = 0, \quad \varphi \left( \lim_{k \to \infty} \inf M \left( x_{m(k)}, x_{n(k)-1} \right) \right) = 0, \]

that is \( \varepsilon = 0 \) or \( \lim_{k \to \infty} \inf M \left( x_{m(k)}, x_{n(k)-1} \right) = 0 \) which is a contradiction. Thus \( \{x_n\} \) is a \( b \)-Cauchy sequence in \( X \). Since \((X, d)\) is a complete \( b \)-metric space, there exists \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \).

Now, we show \( u \) is a fixed point of \( f \). We know that

\[ \psi (d (u, fu)) \leq \psi (s d (u, fu)) , \]

where \( s \geq 1 \). Inequality (2.4) implies

\[ \psi (s d (u, fu)) \leq F \left( \psi (M (u, fu)), \varphi (M (u, fu)) \right) + LN \left( u, fu \right) . \]

But it is easy to see that \( M (u, fu) = d (u, fu) \) and \( N (u, fu) = 0 \). Thus

\[ \psi (d (u, fu)) \leq \psi (d (u, fu)) , \]

and so \( \psi (d (u, fu)) = 0 \), or \( \varphi (d (u, fu)) = 0 \) so \( d (u, fu) = 0 \) and \( fu = u \). Moreover, \( u \) is a unique fixed point of \( f \). Let \( v \neq u \) be another fixed point of \( f \). From (2.4) we have

\[ \psi (d (u, v)) \leq \psi (s d (u, v)) \]
\[ = \psi (s d (fu, fv)) \]
\[ \leq F \left( \psi (M (u, v)), \varphi (M (u, v)) \right) + LN \left( u, v \right) , \]

where \( M(u, v) = d(u, v) \) and \( N(u, v) = 0 \). Therefore

\[ \psi (d (u, v)) \leq F \left( \psi (M (u, v)), \varphi (M (u, v)) \right) + LN \left( u, v \right) \]
\[ = F \left( \psi (d (u, v)), \varphi (d (u, v)) \right) \]
\[ \leq \psi (d (u, v)) . \]
So \( \psi (d(u,v)) = 0 \) or \( \varphi (d(u,v)) = 0 \), thus \( d(u,v) = 0 \) that is \( u = v \). This shows \( f \) has a unique fixed point. \( \square \)

Here as an application of Theorem 2.1, the existence of a solution of an integral equation is proved.

Consider the following integral equation

\[
(2.10) \quad x(t) = p(t) + \int_0^T s(t,r) f(r,x(r)) \, dr, \quad t \in [0,T],
\]

where \( T > 0, \quad f : [0,T] \times \mathbb{R} \to \mathbb{R}, \quad p : [0,T] \to \mathbb{R} \) and \( s : [0,T] \times \mathbb{R} \to [0,\infty) \) are continuous. Assume for all \( x,y \in X \)

\[
0 \leq f(r,y) - f(r,x) \leq q \sqrt{F \left( |y-x|^q, \varphi \left( |y-x|^q \right) \right) - 2q^{-1}},
\]

where \( F : [0,\infty)^2 \to \mathbb{R} \) is a \( C \)-class function and \( \varphi : [0,\infty) \to [0,\infty) \) is an ultra altering distance function. Suppose \( \max_{t \in I} \left( \int_0^T |s(t,r)| \, dr \right)^q \leq 1 \). Then \((2.10)\) has a solution in \( C ([0,T], \mathbb{R}) \), where \( C ([0,T], \mathbb{R}) \) is the set of continuous real functions defined on \([0,T]\). Note that this space with the \( b \)-metric given by \( d(x,y) = \max_{t \in I} \left| x(t) - y(t) \right|^q \) for all \( x,y \in C ([0,T], \mathbb{R}) \) is a complete \( b \)-metric space with \( s = 2^{q-1} \) and \( q \geq 1 \).

Set \( X := C ([0,T], \mathbb{R}) \). Define \( G : X \to X \) by

\[
G(x(t)) = p(t) + \int_0^T s(t,r) f(r,x(r)) \, dr.
\]

For \( x,y \in X \) we have

\[
2^{q-1} \left| Gx(t) - Gy(t) \right|^q = 2^{q-1} \left| \int_0^T s(t,r) [f(r,x(r)) - f(r,y(r))] \, dr \right|^q
\]

\[
\leq 2^{q-1} \left( \int_0^T |s(t,r)| \left| f(r,x(r)) - f(r,y(r)) \right| \, dr \right)^q
\]

\[
\leq \max_{r \in I} F \left( |y-x|^q, \varphi \left( |y-x|^q \right) \right) \left( \int_0^T |s(t,r)| \, dr \right)^q
\]

\[
\leq F \left( d(x,y), \varphi \left( d(x,y) \right) \right)
\]

\[
\leq F \left( M(x,y), \varphi \left( M(x,y) \right) \right),
\]

where

\[
M(x,y) = \max \left\{ d(x,y), \frac{d(x,fx) \cdot d(y,fy)}{1 + d(fx,fy)} \right\}.
\]

For all \( x,y \in X \)

\[
sd(G(x),G(y)) \leq F \left( M(x,y), \varphi \left( M(x,y) \right) \right).
\]

By Theorem 2.1, \( G \) has a unique fixed point. This means \((2.10)\) has a solution.
Some corollaries are presented as follows:

**Corollary 2.2.** Let $(X,d)$ be a complete $b$-metric space on $X$ and $f : X \to X$ be a self mapping. Suppose

$$\psi (sd (fx, fy)) \leq \psi (M(x,y)) - \psi (M(x,y)) + LN(x,y),$$

for all $x, y \in X$, where $L \geq 0$, $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function $\phi : [0, \infty) \to [0, \infty)$ is an ultra altering distance function and

$$M(x,y) = \max \left\{ d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)} \right\},$$

and

$$N(x,y) = \min \left\{ d(x,fx), d(x,fy), d(y,fx), d(y,fy) \right\}.$$

Then $f$ has a unique fixed point.

**Proof.** Set $F(s,t) = s - t$ in Theorem 2.1. □

**Corollary 2.3.** Let $(X,d)$ be a complete $b$-metric space $f : X \to X$ be a self mapping. Suppose

$$\psi (sd (fx, fy)) \leq \psi (M(x,y)) + LN(x,y),$$

for all $x, y \in X$, where $L \geq 0$, $\beta : [0, \infty) \to [0, 1)$ is continuous and

$$M(x,y) = \max \left\{ d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)} \right\},$$

and

$$N(x,y) = \min \left\{ d(x,fx), d(x,fy), d(y,fx), d(y,fy) \right\}.$$

Then $f$ has a unique fixed point.

**Proof.** Let $F(s,t) = s\beta(s)$ in Theorem 2.1, where $\beta : [0, \infty) \to [0, 1)$ and is continuous. □

Setting $\psi(t) = t$ in Corollary 2.2, we have the following corollary.

**Corollary 2.4.** Let $(X,d)$ be a complete $b$-metric space and $f : X \to X$ be a self mapping. Suppose

$$d(fx,fy) \leq \left[ \frac{\beta(M(x,y))}{s} \right] M(x,y) + LN(x,y),$$

for all $x, y \in X$, where $L \geq 0$, $\beta : [0, \infty) \to [0, 1)$ is continuous and

$$M(x,y) = \max \left\{ d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)} \right\},$$

and

$$N(x,y) = \min \left\{ d(x,fx), d(x,fy), d(y,fx), d(y,fy) \right\}.$$

Then $f$ has a unique fixed point.
Let \( F(s, t) = \phi(s) \) in Theorem 2.1, where \( \phi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \phi(0) = 0 \) and \( \phi(t) < t \) for \( t > 0 \) and \( \psi(t) = t \). Then we have the following corollary.

**Corollary 2.5.** Let \((X, d)\) be a complete \( b \)-metric space and \( f : X \to X \) be a self mapping. Suppose
\[
\text{sd}(fx, fy) \leq \phi(M(x, y)) + LN(x, y),
\]
for all \( x, y \in X \), where \( L \geq 0 \), \( \phi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \phi(0) = 0 \) and \( \phi(t) < t \) for \( t > 0 \) and
\[
M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) d(y, fy)}{1 + d(fx, fy)} \right\},
\]
and
\[
N(x, y) = \min \{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.
\]
Then \( f \) has a unique fixed point.

Suppose \( \phi(t) = lt, 0 < l < 1 \) in Corollary 2.5 or with choice \( F(s, t) = ls, 0 < l < 1 \) and \( \psi(t) = t \), \( L = 0 \) in Theorem 2.1, we have the following corollary.

**Corollary 2.6.** Let \((X, d)\) be a complete \( b \)-metric space and \( f : X \to X \) be a self mapping. Suppose
\[
\text{sd}(fx, fy) \leq lM(x, y),
\]
for all \( x, y \in X \), where \( 0 < l < 1 \), and
\[
M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) d(y, fy)}{1 + d(fx, fy)} \right\},
\]
Then \( f \) has a unique fixed point.

**Example 2.7.** Let \( X = [0, \infty) \). Suppose the function \( f : X \to X \) is given by
\[
f(x) = \frac{3x + 8}{5}.
\]
If we define the \( b \)-metric \( d \) on \( X \) by \( d(x, y) = |x - y|^2 \), then \((X, d)\) is a complete \( b \)-metric space with \( s = 2 \). We obtain
\[
\text{sd}(fx, fy) = 2d\left(\frac{3x + 8}{5}, \frac{3y + 8}{5}\right) = \frac{18}{25} |x - y|^2 \\
\leq \frac{18}{25} M(x, y).
\]
Hence by Corollary 2.6, \( f \) has a unique fixed point (which is \( x = 4 \)).
Example 2.8. Let $X = \{1, 2, 4\}$. Assume the function $f : X \to X$ is given by

$$f = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 2 \end{pmatrix}.$$ 

First, define the $b$-metric $d$ on $X$ by $d(1, 2) = 7$, $d(1, 4) = 10$, $d(2, 4) = \frac{1}{5}$, and $d(x, x) = 0$. Then $(X, d)$ is a complete $b$-metric space with $s = \frac{25}{18}$.

It is easy to see that $sd(fx, fy) \leq \frac{1}{18} M(x, y)$, so by Corollary 2.6, $f$ has a unique fixed point (which is $x = 2$).

Corollary 2.9. Let $(X, d)$ be a complete $b$-metric space and $f : X \to X$ be a self mapping. Suppose

$$sd(fx, fy) \leq ad(x, y) + bd(fx, y) d(y, fy) 1 + d(fx, fy),$$

for all $x, y \in X$, where $a + b < 1$. Then $f$ has a unique fixed point.

Letting $\phi(t) = \frac{t}{1 + t}$ in Corollary 2.8 or with choice $F(s, t) = \frac{s}{t + s}, \psi(t) = t, L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.10. Let $(X, d)$ be a complete $b$-metric space and $f : X \to X$ be a self mapping. Suppose

$$sd(fx, fy) \leq \frac{M(x, y)}{1 + M(x, y)},$$

for all $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), d(fx, fy) 1 + d(fx, fy) \right\}.$$ 

Then $f$ has a unique fixed point.

Letting $\phi(t) = t - \frac{t}{n + t}, n \geq 1$ in Corollary 2.8, we have the following corollary.

Corollary 2.11. Let $(X, d)$ be a complete $b$-metric space and $f : X \to X$ be a self mapping. Suppose

$$sd(fx, fy) \leq M(x, y) - \frac{M(x, y)}{n + M(x, y)},$$

for all $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), d(fx, fy) 1 + d(fx, fy) \right\}.$$ 

Then $f$ has a unique fixed point.

Assume $\psi(t) = \ln(1 + t)$ in Corollary 2.8 or with choice $F(s, t) = \ln(1 + s), \psi(t) = t, L = 0$ in Theorem 2.1, we have the following corollary.
Corollary 2.12. Let \((X,d)\) be a complete \(b\)-metric space and \(f : X \to X\) be a self mapping. Suppose
\[
sd(f(x), f(y)) \leq \ln(1 + M(x, y)),
\]
for all \(x, y \in X\), where
\[
M(x, y) = \max \left\{ d(x, y), \frac{d(x, f(x))d(y, f(y))}{1 + d(f(x), f(y))} \right\}.
\]
Then \(f\) has a unique fixed point.

We recall the following lemma.

Lemma 2.13 ([11]). Let \(X\) be a nonempty set and \(f : X \to X\) be a function. Then there exists a subset \(E \subseteq X\) such that \(f(E) = f(X)\) and \(f : E \to f(X)\) is one-to-one.

Theorem 2.14. Let \((X,d)\) be a complete \(b\)-metric space, \(f, T : X \to X\) be such that \(f(X) \subseteq T(X)\). Suppose that \((T, f)\) satisfy the following condition:
\[
\psi(s_d(f(x), f(y))) \leq F(\psi(M(d(Tx, Ty))), \phi(M(Tx, Ty)))) + L N(Tx, Ty),
\]
for all \(x, y \in X\), where \(L \geq 0\), \(F : [0, \infty)^2 \to \mathbb{R}\) is a \(C\)-class function, \(\psi : [0, \infty) \to [0, \infty)\) is an altering distance function, \(\phi : [0, \infty) \to [0, \infty)\) is an ultra altering distance function and
\[
M(Tx, Ty) = \max \left\{ d(Tx, Ty), \frac{d(Tx, f(x))d(Ty, f(y))}{1 + d(f(x), f(y))} \right\},
\]
and
\[
N(Tx, Ty) = \min \{ d(Tx, f(x)), d(Tx, f(y)), d(Ty, f(x)), d(Ty, f(y)) \}.
\]
Then \((f, T)\) has a unique coincidence point.

Proof. Let \(f : X \to X\). By Lemma 2.13 there exists \(E \subseteq X\) such that \(T(E) = T(X)\) and \(T \mid E\) is one-to-one. Since \(T(E) \subseteq T(X) \subseteq X\), one can define the mapping \(A : T(E) \to X\) by \(A(Tx) = f(x)\) for all \(x \in E\). Since \(T \mid E\) is one-to-one, then \(A\) is well-defined. Now
\[
\psi(s_d(A(Tx), A(Ty))) = \psi(s_d(f(x), f(y)))
\]
\[
\leq F(\psi(M(d(Tx, Ty))), \phi(M(Tx, Ty)))),
\]
for all \(x, y \in X\), where
\[
M(Tx, Ty) = \max \left\{ d(Tx, Ty), \frac{d(Tx, A(Tx))d(Ty, A(Ty))}{1 + d(A(Tx), A(Ty))} \right\},
\]
and
\[
N(Tx, Ty)
\]
\[
= \min \{ d(Tx, A(Tx)), d(Tx, A(Ty)), d(Ty, A(Tx)), d(Ty, A(Ty)) \}.
\]
Thus by Theorem 2.1, there exists a unique fixed point $u \in T(E)$ of $A$, i.e. $Au = u$. Since $u \in T(E)$, there exists $w \in E$ such that

$$fw = A(Tw) = Au = u = Tw.$$ 

Thus $w$ is a unique coincidence point of $f$ and $T$. □

References


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