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A Coupled Random Fixed Point Result With Application in Polish Spaces

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Abstract. In this paper, we present a new concept of random contraction and prove a coupled random fixed point theorem under this condition which generalizes stochastic Banach contraction principle. Finally, we apply our contraction to obtain a solution of random nonlinear integral equations and we present a numerical example.

1. Introduction

Throughout this paper, we will refer to $\mathbb{R}$ by the set of all real numbers, $\mathbb{R}^+$ by the set of all positive real numbers, $\mathbb{N}$ by the set of all natural numbers, (X, d) by a complete metric or polish space with a metric d and (Ω, Σ) by a measurable space where Σ is a $\sigma-$algebra of Borel subsets of Ω.

There are many extensions of the Banach contraction principle [6], which states that, let T be a self mapping defined on a complete metric space (X, d) and satisfy

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X \text{ and } k \in (0, 1), \tag{1.1}$$

then T has a unique fixed point and for all $x_\circ \in X$ a sequence $\{T^n x_\circ\}_{n\in\mathbb{N}}$ is convergent to this fixed point. Some authors worked in right side of inequality (1.1) by replacing k with a mapping (see [10, 20]) and others impressed by the underlying space is more general (see [4, 27]). There are also many different types of fixed point theorems not mentioned above extending the Banach's result.

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Definition 1.1. A metric space (X, d) is said to be a polish space, if it is satisfied the following conditions:

(i) X is complete,
(ii) X is separable.

In 2006, the concept of coupled fixed point introduced by Bhaskar et al. [9], for some works on a coupled fixed point in different spaces (see [1, 5]).

Definition 1.2 ([9])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$T(x, y) = x, \qquad T(y, x) = y. \tag{1.2}$$

In 2012, Wardowski [28] presented a new type of contraction called F-contraction where $F : \mathbb{R}^+ \to \mathbb{R}$ and proved a new fixed point theorem concerning $F-$contraction. He gave also some examples to obtain the variety of a type known in the literature of contractions. A lot of authors worked in this direction and proved some new fixed point results in various spaces (see [2, 3, 7, 12, 16]).

Definition 1.3 ([28])**.** Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F_1) F is strictly increasing i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;
(F_2) for every sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} F(a_n) = -\infty$;
(F_3) there exists $\lambda \in (0, 1)$ such that $\lim_{a\to 0^+} a^\lambda F(a) = 0$.

A mapping $T : X \to X$ is called F-contraction if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad \forall x, y \in X. \tag{1.3}$$

Theorem 1.4 ([28])**.** *Let T be a self mapping on a complete metric space (X, d) satisfying the condition* (1.3)*. Then T has a unique fixed point x^*. Moreover, for any $x_\circ \in X$, the sequence $\{T^n x_\circ\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Fixed point theory has a solution of many problems that appear in approximation theory, game and potential theory, theory of integral and differential equations and others. The study of random fixed point theorems was initiated by the Prague school of probabilistic in 1950's. The introduction of randomness leads to several new questions of measurability of solutions probabilistic and statistical aspects of random solutions. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Hanš [13] and Špaček [26].

In 1972, Bharucha-Reid [8] attracted the attention of several mathematicians and gave wings to this theory. Itoh [14] extended the results of Špaček and Hanš in multi-valued contractive mappings and obtained random fixed point theorems with an application to random differential equations in Banach spaces.

Recently, some authors [19, 21, 24, 25], applied a random fixed point theorem to prove the existence of a solution in a sparable Banach space of a random nonlinear integral equations.

Random and coupled fixed point theorems are stochastic generalizations of classical or deterministic fixed point theorems. Lakshmikantham et al. [17], gave the notion of coupled random fixed points and proved some theorems in partially ordered metric spaces under this notion. Very recently, some authors generalized this results and obtained some theorems in different spaces (see [22, 23]).

In this paper, we give a stochastic version for F-contraction (1.3) and use it to obtain a coupled random fixed point theorem in a polish space. Also, a solution of random nonlinear integral equations are discussed. Finally, a numerical example, to verify our result, is given.

2. Some Basics of Valued Random Variables

The following preliminaries appear in [15].

Let (Ω, Σ, μ) be a complete probability measure space with measure μ and Σ be a $\sigma-$algebra of subsets of Ω.

Definition 2.1 ([15])**.** A mapping $x : \Omega \to X$ is called:

1. $X-$valued random variable, if the inverse image under the mapping x of every Borel set B of X belongs to Σ, that is, $x^{-1}(B) \in \Sigma$ for all $B \in \Sigma$.
2. Finitely valued random variable, if it is constant on each of a finite number of disjoint sets $A_i \in \Sigma$ and is equal to 0 on
$$\Omega - \left(\bigcup_{i=1}^{n} A_i \right).$$
3. Simple random variable if it is finitely valued and
$$\mu\{\omega : \|x(\omega)\| > 0\} < \infty.$$
4. Strong random variable, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \sum$ with $\mu(A_0) = 0$ such that
$$\lim_{n\to\infty} x_n(\omega) = x(\omega), \quad \omega \in \Omega - A_0.$$

5. Weak random variable, if the function $x^*(x(\omega))$ is a real valued random variable for each $x^* \in X^*$, the space X^* denoting the dual space of X.

Definition 2.2 ([15])**.** Let Y be another Banach space.

1. A mapping $T : \Omega \times X \to Y$ is said to be a random mapping if $T(\omega, x) = Y(\omega)$ is a $Y-$valued random variable for every $x \in X$.
2. A mapping $T : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $T(\omega, x)$ is a continuous function of x has measure one.
3. An equation of the type $T(\omega, x(\omega)) = x(\omega)$, where $T : \Omega \times X \to Y$ is a random mapping, is called a random fixed point equation.
4. Any mapping $x : \Omega \to X$ which satisfies the random fixed point equation $T(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.
5. Any $X-$valued random variable $x(\omega)$ which satisfies

$$\mu\{\omega : T(\omega, x(\omega)) = x(\omega)\} = 1,$$

is said to be a random solution of the fixed point equation or a random fixed point of T.

Definition 2.3. Let X be a polish space, an element $(x(\omega), y(\omega))$ for all $x, y \in X$ and $\omega \in \Omega$ is called a coupled random fixed point of a self random mapping T if

$$T(\omega, (x(\omega), y(\omega))) = x(\omega) \text{ and } T(\omega, (y(\omega), x(\omega))) = y(\omega).$$

Remark 2.4 ([15])**.** A random solution is a wide sense solution of the fixed point equation but the converse is not necessarily true.

3. Some Random Contractive Conditions

In this section, we give some extensions of F-contraction (1.3) in stochastic version and obtain some known contractive conditions.

Definition 3.1. Consider (X, d) be a polish space and F be the family of functions given by $\varphi : (0, \infty) \to (-\infty, \infty)$ satisfying:

(i) φ is strictly increasing;
(ii) for a sequence $(t_n) \subset (0, \infty)$,

$$\lim_{n\to\infty} t_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \varphi(t_n) = -\infty;$$

(iii) there exists $\lambda \in (0, 1)$ such that $\lim_{t\to 0^+} t^\lambda \varphi(t) = 0$.

A random mapping $T : \Omega \times X \to X$ is called φ-contraction if there exists $\tau > 0$ such that

$$d\,(T(\omega, x), T(\omega, y)) > 0 \tag{3.1}$$

$$\Rightarrow \quad \tau + \varphi\left(d(T\left(\omega, x\right), T(\omega, y))\right) \leq \varphi\left(d\left(x(\omega), y(\omega)\right)\right),$$

for all $x, y \in X$ and $\omega \in \Omega$.

Remark 3.2. If we consider the φ-contractive condition (3.1), then we obtain other types of contractions as follows:

► Taking $\varphi(t) = \ln(t)$, $t > 0$. It is clear that φ satisfies (i)-(iii) i.e., $\varphi \in \digamma$ (for $\lambda \in (0,1)$). Each self random mapping $T : \Omega \times X \to X$ satisfying (3.1) is a φ-contraction such that

$$d\left(T(\omega, x), T(\omega, y)\right) \leq e^{-\tau}\left(d\left(x(\omega), y(\omega)\right)\right), \tag{3.2}$$

for all $x, y \in X$ and $\omega \in \Omega, T(\omega, x) \neq T(\omega, y)$.

► If we take $\varphi(t) = \ln(t) + t$, $t > 0$. Then $\varphi \in \digamma$ and the condition (3.1) is of the form

$$\frac{d\left(T(\omega, x), T(\omega, y)\right)}{d\left(x(\omega), y(\omega)\right)} e^{d(T(\omega,x),T(\omega,y)) - d(x(\omega),y(\omega))} \leq e^{-\tau}, \tag{3.3}$$

for all $x, y \in X$ and $\omega \in \Omega, T(\omega, x) \neq T(\omega, y)$.

► Consider $\varphi(t) = \frac{-1}{\sqrt{t}}$, $t > 0$ then, $\varphi \in \digamma$, (iii) for $\lambda \in \left(\frac{1}{2}, 1\right)$ and φ-contraction T satisfies

$$d\left(T(\omega, x), T(\omega, y)\right) \leq \frac{1}{\left(1 + \tau\sqrt{d(x(\omega), y(\omega))}\right)^2} d\left(x(\omega), y(\omega)\right), \tag{3.4}$$

$\forall x, y \in X$ and $\omega \in \Omega, T(\omega, x) \neq T(\omega, y)$.

► Putting $\varphi(t) = \ln(t^2 + t)$, $t > 0$ therefore, $\varphi \in \digamma$ and for φ-contraction T, the following condition holds:

$$\frac{d\left(T(\omega, x), T(\omega, y)\right)\left[d(T(\omega, x), T(\omega, y)) + 1\right]}{d\left(x(\omega), y(\omega)\right)\left[d(x(\omega), y(\omega)) + 1\right]} \leq e^{-\tau}, \tag{3.5}$$

$\forall x, y \in X$ and $\omega \in \Omega, T(\omega, x) \neq T(\omega, y)$.

Note that, all random contractive conditions (3.2)-(3.5) are satisfied for $x, y \in X$ and $\omega \in \Omega$, such that $T(\omega, x) = T(\omega, y)$.

Remark 3.3. Consider $\varphi(t) = \frac{-1}{t^{\frac{1}{p}}}$, $p > 1$, $t > 0$, then $\varphi \in \digamma$.

Proof. Since $\varphi'(t) = \frac{1}{p.t^{1+\frac{1}{p}}} > 0$ then, φ satisfies (i) and it is clear that the condition (ii) holds. Finally, since $p > 1$, $\frac{1}{p} < 1$, we take $\frac{1}{p} < \lambda < 1$ and then,

$$\lim_{t \to 0^+} t^\lambda \varphi(t) = \lim_{t \to 0^+}\left(-t^{\lambda - \frac{1}{p}}\right) = 0.$$

So φ satisfies (iii). This gives us $\varphi \in \digamma$. □

Question. Is it possible to define random contractive condition under $\varphi(t) = \frac{-1}{t^{\frac{1}{p}}}$?

Now, we state a stochastic version of Theorem 1.4.

Theorem 3.4. *Let (X, d) be a polish space and T be a self random mapping on X satisfying the φ-contractive condition (3.1), then T has a unique random fixed point $x^*(\omega)$. Moreover, for any $x_\circ(\omega) \in \Omega \times X$, the sequence $\{T^n(\omega, x_\circ\}\}_{n\in\mathbb{N}}$ is convergent to $x^*(\omega)$.*

Proof. We can get the proof easily from [28, Theorem 1.4] by taking Ω be a singleton in our theorem and X is a complete metric space. □

4. A Coupled Random Fixed Point Result

The following Lemma is useful in our results.

Lemma 4.1. *Let (X, d) be a polish space, $X \times X$ be a cartesian product and $\widetilde{d}$ is defined by*

$$\widetilde{d}\left((x, y), (u, v)\right) = \max\left\{d(x, u), d(y, v)\right\}.$$

Then, the pair $(X \times X, \widetilde{d})$ is a polish space

Proof. It's obvious that the distance $\widetilde{d}$ satisfies all conditions of a metric space and $X \times X$ endowed with $\widetilde{d}$ is complete, hence $(X \times X, \widetilde{d})$ is a complete metric space. Since X is separable, then $X \times X$ is separable too [18]. Therefore the required is obtained. □

Now, we present the main results of the paper.

Theorem 4.2. *Let $F : \Omega \times (X \times X) \to X$ be a continuous random mapping of a polish space (X, d) such that there exist $\tau > 0$ and $\varphi \in F$, satisfying*

$$d\left(F\left(\omega, (x, y)\right), F\left(\omega, (u, v)\right)\right) > 0,$$

then,

$$\tau + \varphi\left(d\left(F\left(\omega, (x, y)\right), F\left(\omega, (u, v)\right)\right)\right) \leq \varphi\left(\max\left\{d\left(x(\omega), u(\omega)\right), d(y(\omega), v(\omega))\right\}\right), \tag{4.1}$$

for all $x, y \in X$ and $\omega \in \Omega$. So, F has a unique coupled random fixed point.

Proof. Consider a random mapping $\widetilde{F} : \Omega \times (X \times X) \to (X \times X)$ defined by

$$\widetilde{F}\left(\omega, (x, y)\right) = \left(F(\omega, (x, y)), F(\omega, (y, x))\right).$$

Next, we check that $\widetilde{F}$ satisfies the contractive condition (3.1) appearing in Theorem 3.4 in a polish space $X \times X$ (see Lemma 4.1).
For $x, y, u, v \in X$ and $\omega \in \Omega$, suppose that

$$\widetilde{d}\Big(\widetilde{F}\left(\omega, (x, y)\right), \widetilde{F}\left(\omega, (u, v)\right)\Big) = \widetilde{d}\left(\left[F\left(\omega, (x, y)\right), F\left(\omega, (y, x)\right)\right],\right. \left.[F(\omega, (u, v)), F(\omega, (v, u))]\right)$$

$$= \max \{ d(F(\omega,(x,y)), F(\omega,(u,v))), \\ d(F(\omega,(y,x)), F(\omega,(v,u))) \} \\ > 0.$$

We can distinguish two cases:

Case 1. If

$$\max \{ d(F(\omega,(x,y)), F(\omega,(u,v))), d(F(\omega,(y,x)), F(\omega,(v,u))) \} \\ = d(F(\omega,(x,y)), F(\omega,(u,v))),$$

since

$$d(F(\omega,(x,y)), F(\omega,(u,v))) > 0,$$

by using the contractive condition (4.1), we have

$$\begin{aligned} &\tau + \varphi\left(\widetilde{d}\left(\widetilde{F}(\omega,(x,y)), \widetilde{F}(\omega,(u,v))\right)\right) \\ &= \tau + \varphi(d(F(\omega,(x,y)), F(\omega,(u,v)))) \\ &\le \varphi(\max\{d(x(\omega),u(\omega)), d(y(\omega),v(\omega))\}) \\ &= \varphi(\max\{d(x(\omega),y(\omega)), d(u(\omega),v(\omega))\}). \end{aligned}$$

Case 2. Let

$$\max \{ d(F(\omega,(x,y)), F(\omega,(u,v))), d(F(\omega,(y,x)), F(\omega,(v,u))) \} \\ = d(F(\omega,(y,x)), F(\omega,(v,u))).$$

Since

$$d(F(\omega,(y,x)), F(\omega,(v,u))) > 0,$$

by using our assumption, we get

$$\begin{aligned} &\tau + \varphi\left(\widetilde{d}\left(\widetilde{F}(\omega,(x,y)), \widetilde{F}(\omega,(u,v))\right)\right) \\ &= \tau + \varphi(d(F(\omega,(y,x)), F(\omega,(v,u)))) \\ &\le \varphi(\max\{d(y(\omega),v(\omega)), d(x(\omega),u(\omega))\}) \\ &= \varphi(\max\{d(x(\omega),y(\omega)), d(u(\omega),v(\omega))\}). \end{aligned}$$

Therefore, in both cases the contractive condition (4.1) holds.

By Theorem 3.4, $\widetilde{F}$ has a unique coupled random fixed point $(x^*(\omega), y^*(\omega)) \in X \times X$, for all $\omega \in \Omega$.

This means that

$$\begin{aligned} (x^*(\omega), y^*(\omega)) &= \widetilde{F}(\omega,(x^*(\omega),y^*(\omega))) \\ &= (F(\omega,(x^*(\omega),y^*(\omega))), F(\omega,(y^*(\omega),x^*(\omega)))), \end{aligned}$$

consequently,

$$F(\omega,(x^*(\omega),y^*(\omega))) = x^*(\omega),$$

and

$$F(\omega,(y^*(\omega),x^*(\omega))) = y^*(\omega).$$

This means that $(x^*(\omega), y^*(\omega))$ is a coupled random fixed point of the random mapping F.

For the uniqueness, suppose that $(x_1(\omega), y_1(\omega)) \in X \times X$, for all $\omega \in \Omega$ is another coupled random fixed point of F, i.e.,

$$F(\omega, (x_1(\omega), y_1(\omega))) = x_1(\omega),$$

and

$$F(\omega, (y_1(\omega), x_1(\omega))) = y_1(\omega).$$

Or equivalently,

$$(x_1(\omega), y_1(\omega)) = \widetilde{F}(\omega, (x_1(\omega), y_1(\omega))).$$

By the uniqueness of the random fixed point of $\widetilde{F}$, we obtain

$$(x_1(\omega), y_1(\omega)) = (x^*(\omega), y^*(\omega)),$$

and this gives us the uniqueness of the coupled random fixed point of F. □

5. Applications

Before to present an application of our result, we need the following lemma.

Lemma 5.1. *Suppose $p > 1$, $\tau > 0$ and let $\varphi_\tau^p : [0, \infty) \to [0, \infty)$ be the function defined by*

$$\varphi_\tau^p(t) = \frac{t}{(1 + \tau \sqrt[p]{t})^p}.$$

Then

(a) *$\varphi_\tau^p(t)$ is strictly increasing,*
(b) *$\varphi_\tau^p(0) = 0$ and φ_τ^p is a concave function,*
(c) *for $t, s \in [0, \infty)$, $|\varphi_\tau^p(t) - \varphi_\tau^p(s)| \leq \varphi_\tau^p(|t - s|)$.*

Proof. (a) We have

$$\begin{aligned}(\varphi_\tau^p)'(t) &= \frac{\left(1 + \tau \sqrt[p]{t}\right)^p - t.p(1 + \tau \sqrt[p]{t})^{p-1}.\tau.\frac{1}{p}.t^{\frac{1}{p}-1}}{\left(1 + \tau \sqrt[p]{t}\right)^{2p}} \\ &= \frac{\left(1 + \tau \sqrt[p]{t}\right)^{p-1}\left[(1 + \tau \sqrt[p]{t}) - \tau.t^{\frac{1}{p}}\right]}{\left(1 + \tau \sqrt[p]{t}\right)^{2p}} \\ &= \frac{1}{\left(1 + \tau \sqrt[p]{t}\right)^{p+1}} > 0,\end{aligned}$$

and this proves (a).

(b) It is clear that $\varphi_\tau^p(0) = 0$. On the other hand,

$$\left(\varphi_\tau^p\right)''(t) = \frac{-(p+1)\left(1+\tau\sqrt[p]{t}\right)^p.\tau.\frac{1}{p}.t^{\frac{1}{p}-1}}{\left(1+\tau\sqrt[p]{t}\right)^{2p+2}} = \frac{-\left(\frac{p+1}{p}\right)\tau}{t^{1-\frac{1}{p}}\left(1+\tau\sqrt[p]{t}\right)^{p+2}} < 0,$$

and this proves that φ_τ^p is a concave function.

(c) Since $\varphi_\tau^p(0) = 0$ and φ_τ^p is a concave, then φ_τ^p is subadditive, i.e.,

$$\varphi_\tau^p(t+s) \leq \varphi_\tau^p(t) + \varphi_\tau^p(s).$$

Taking $t, s \in [0, \infty)$ and without loss of generality, we can suppose that $t > s$. By (a), we have

$$|\varphi_\tau^p(t) - \varphi_\tau^p(s)| = \varphi_\tau^p(t) - \varphi_\tau^p(s). \tag{5.1}$$

But,

$$\varphi_\tau^p(t) = \varphi_\tau^p(t-s+s) \leq \varphi_\tau^p(t-s) + \varphi_\tau^p(s),$$

(where we have used the subadditivity of φ_τ^p) and consequently,

$$\varphi_\tau^p(t) - \varphi_\tau^p(s) \leq \varphi_\tau^p(t-s) = \varphi_\tau^p\left(|t-s|\right). \tag{5.2}$$

By (5.1) and (5.2), we have

$$|\varphi_\tau^p(t) - \varphi_\tau^p(s)| \leq \varphi_\tau^p(|t-s|),$$

and this completes the proof.

□

Now, we will apply our result to the existence and uniqueness of solutions of random nonlinear integral equations

$$\begin{cases} x(t;\omega) = a(t;\omega) + \int_0^1 \left(\omega; t, s, \max\limits_{0\leq\tau\leq s}\left\{|x(\tau;\omega)|\right\}, y(s;\omega)\right) ds, \\ y(t;\omega) = a(t;\omega) + \int_0^1 \left(\omega; t, s, \max\limits_{0\leq\tau\leq s}\left\{|y(\tau;\omega)|\right\}, x(s;\omega)\right) ds, \end{cases} \tag{5.3}$$

where,

1. $\omega \in \Omega$ where ω is a supporting set of the probability measure space (Ω, Σ, μ),
2. $x(t;\omega)$ and $y(t;\omega)$ are unknown vector-valued random variables for each $t \in [0, 1]$,
3. $a(t;\omega)$ is the stochastic free term defined for $t \in [0, 1]$.

Random integral equations (5.3) in stochastic version is a similar to volterra integral type equation.

System (5.3) will be considered under the following assumptions:
(H_1) $a(t;\omega) \in C([0,1], \mathbb{R})$, where $C([0,1], \mathbb{R}) = \{x : [0,1] \to \mathbb{R} : x$ is continuous$\}$ and $\mathbb{R}$ is a polish space, (note here, we take $X = \mathbb{R}$).
(H_2) $f : \Omega \times [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a random continuous function satisfying

$$\begin{aligned} &|f(\omega; t, s, u(\omega), v(\omega)) - f(\omega; t, s, u_1(\omega), v_1(\omega))| \\ &\qquad \leq \frac{\max\{|u(\omega) - u_1(\omega)|, |v(\omega) - v_1(\omega)|\}}{\left(1 + \tau \sqrt[p]{\max\{|u(\omega) - u_1(\omega)|, |v(\omega) - v_1(\omega)|\}}\right)^p}, \end{aligned}$$

for any $t, s \in [0,1]$ and $u(\omega), v(\omega), u_1(\omega), v_1(\omega) \in \Omega \times \mathbb{R}$, where $\tau > 0$ and $p > 1$.

The following results are important in the sequel (see [11]).

Suppose that $x(\omega) \in C([0,1], \mathbb{R})$ and let $G(\omega, x) = G_x$ be a measurable function defined by

$$G_x(t) = \max_{0 \leq \tau \leq t} |x(\omega; \tau)|.$$

Then $G_x(t) \in C([0,1], \mathbb{R})$. Moreover, for $x(\omega), y(\omega) \in C([0,1], \mathbb{R})$,

$$\begin{aligned} d(G_x, G_y) &= \sup\{|G_x(t) - G_y(t)| : t \in [0,1]\} \\ &\leq d(x(\omega), y(\omega)) \\ &= \sup\{|x(\omega; t) - y(\omega; t)| : t \in [0,1]\}. \end{aligned}$$

Theorem 5.2. *Let (Ω, Σ, μ) be a probability measure space and $\mathbb{R}$ is a polish space, then the system (5.3) under assumptions (H_1) and (H_2) has a unique random solution $(x^*(\omega), y^*(\omega)) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$.*

Proof. For $x(\omega), y(\omega) \in C([0,1], \mathbb{R})$, $\omega \in \Omega$ and $t \in [0,1]$, we define $F(\omega, (x,y))$ by

$$F(\omega, (x,y))(t) = a(t;\omega) + \int_0^1 \left(\omega; t, s, \max_{0 \leq \tau \leq s} \{|x(\tau;\omega)|\}, y(s;\omega)\right) ds.$$

In virtue of (H_1), (H_2) and the fact that a random operator G mentioned above is continuous on $C([0,1], \mathbb{R})$, it is clear that if $x(\omega), y(\omega) \in C([0,1], \mathbb{R})$ then $F(\omega, (x,y)) \in C([0,1], \mathbb{R})$. Therefore,

$$F : \Omega \times C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \to C = ([0,1], \mathbb{R}).$$

Next, we check that F satisfies the contractive condition (4.1).

Suppose that $d(F(\omega, (x,y)), F(\omega, (u,v))) > 0$, then, for $t \in [0,1]$, we have

$$|F(\omega, (x,y))(t) - F(\omega, (u,v))(t)|$$

$$= \left| \int_0^1 f(\omega; t, s, \max_{0\le\tau\le s}\{|x(\tau;\omega)|\}, y(s;\omega))ds \right.$$
$$\left. - \int_0^1 f(\omega; t, s, \max_{0\le\tau\le s}\{|u(\tau;\omega)|\}, v(s;\omega))ds \right|$$
$$= \left| \int_0^1 f(\omega; t, s, G_x(s), y(s;\omega))ds \right.$$
$$\left. - \int_0^1 f(\omega; t, s, G_u(s), v(s;\omega))ds \right|$$
$$\le \int_0^1 |f(\omega; t, s, G_x(s), y(s;\omega)) - f(\omega; t, s, G_u(s), v(s;\omega))| \, ds$$
$$\le \int_0^1 \frac{\max\{|G_x(s) - G_u(s)|, |y(s;\omega) - v(s;\omega)|\}}{\left(1 + \tau \sqrt[p]{\max\{|G_x(s) - G_u(s)|, |y(s;\omega) - v(s;\omega)|\}}\right)^p} ds$$
$$\le \int_0^1 \frac{\max\{d(G_x(s), G_u(s)), d(y(s;\omega), v(s;\omega))\}}{\left(1 + \tau \sqrt[p]{\max\{d(G_x(s), G_u(s)), d(y(s;\omega), v(s;\omega))\}}\right)^p} ds$$
$$\le \int_0^1 \frac{\max\{d(x(\omega), u(\omega)), d(y(s;\omega), v(s;\omega))\}}{\left(1 + \tau \sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(s;\omega), v(s;\omega))\}}\right)^p} ds$$
$$= \frac{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}{\left(1 + \tau \sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}\right)^p},$$

where we used the nondecreasing character of φ_τ^p (Lemma 5.1) and the fact that $d(G_x, G_y) \le d(x(\omega), y(\omega))$.
Therefore

$$d(F(\omega, (x, y)), F(\omega, (u, v))) \le \frac{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}{\left(1 + \tau \sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}\right)^p}.$$

This yields that

$$\sqrt[p]{d(F(\omega, (x, y)), F(\omega, (u, v)))} \le \frac{\sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}}{1 + \tau \sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}},$$

or, equivalently,

$$\frac{1 + \tau \sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}}{\sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}} \le \frac{1}{\sqrt[p]{d(F(\omega, (x, y)), F(\omega, (u, v)))}},$$

or,

$$\frac{1}{\sqrt[p]{\max\{d(x(\omega), u(\omega)), d(y(\omega), v(\omega))\}}} + \tau \le \frac{1}{\sqrt[p]{d(F(\omega, (x, y)), F(\omega, (u, v)))}},$$

or,

$$\tau - \frac{1}{\sqrt[p]{d(F(\omega,(x,y)),F(\omega,(u,v)))}} \leq -\frac{1}{\sqrt[p]{\max\{d(x(\omega),u(\omega)),d(y(\omega),v(\omega))\}}}.$$

This says us that the contractive condition (4.1) is satisfied with $\varphi(t) = \frac{-1}{\sqrt[p]{t}} \in \digamma$ (Remark 3.3). By Theorem 4.2, there exists a unique coupled random fixed point of a random mapping F, i.e., there exists a unique $(x^*(\omega), y^*(\omega)) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that, for any $t \in [0,1]$

$$\begin{aligned} x^*(\omega;t) &= F(\omega,(x_\circ,y_\circ))(t) \\ &= a(t;\omega) + \int_0^1 f(\omega;t,s,\max_{0\leq\tau\leq s}\{|x_\circ^*(\tau;\omega)|\}, y_\circ^*(s;\omega))ds, \\ y^*(\omega;t) &= F(\omega,(y_\circ,x_\circ))(t) \\ &= a(t;\omega) + \int_0^1 f(\omega;t,s,\max_{0\leq\tau\leq s}\{|y_\circ^*(\tau;\omega)|\}, x_\circ^*(s;\omega))ds, \end{aligned}$$

and this completes the proof. □

In the sequel, we present a numerical example.

Example 5.3. Let (Ω,Σ) be a measurable space where Σ is a $\sigma-$algebra of Borel subsets of Ω. Consider $\Omega = [0,1]$ and the following coupled system of random nonlinear integral equations for $t \in [0,1]$ and $\omega \in \Omega$,

(5.4)

$$\begin{cases} x(t;\omega) = e^{t+\omega} \\ \qquad + \int_0^1 \left(t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{\max\limits_{0\leq\tau\leq s}|x(\tau;\omega)|}{\left(1+8\sqrt[3]{\max\limits_{0\leq\tau\leq s}|x(\tau;\omega)|}\right)^3} + \frac{1}{2}\frac{|y(s;\omega)|}{\left(1+5\sqrt[3]{|y(s;\omega)|}\right)^3}\right)ds; \\ y(t;\omega) = e^{t+\omega} \\ \qquad + \int_0^1 \left(t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{\max\limits_{0\leq\tau\leq s}|y(\tau;\omega)|}{\left(1+8\sqrt[3]{\max\limits_{0\leq\tau\leq s}|y(\tau;\omega)|}\right)^3} + \frac{1}{2}\frac{|x(s;\omega)|}{\left(1+5\sqrt[3]{|x(s;\omega)|}\right)^3}\right)ds. \end{cases}$$

System (5.4) is a particular case of system (5.3), where $a(t;\omega) = e^{t+\omega}$ and

$$\begin{aligned} f\left(\omega;t,s,u(s;\omega),v(s;\omega)\right)n = t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{|u(s;\omega)|}{\left(1+8\sqrt[3]{|u(s;\omega)|}\right)^3} \\ + \frac{1}{2}\frac{|v(s;\omega)|}{\left(1+5\sqrt[3]{|v(s;\omega)|}\right)^3}. \end{aligned}$$

It is clear that (H_1) is satisfied.

For (H_2), we have

$$|f\left(\omega;t,s,u(s;\omega),v(s;\omega)\right) - f\left(\omega;t,s,u_1(s;\omega),v_1(s;\omega)\right)|$$

$$\leq \frac{1}{2}\left|\frac{|u(s;\omega)|}{\left(1+8\sqrt[3]{|u(s;\omega)|}\right)^3}-\frac{|u_1(s;\omega)|}{\left(1+8\sqrt[3]{|u_1(s;\omega)|}\right)^3}\right|$$
$$+\frac{1}{2}\left|\frac{|v(s;\omega)|}{\left(1+5\sqrt[3]{|v(s;\omega)|}\right)^3}-\frac{|v_1(s;\omega)|}{\left(1+5\sqrt[3]{|v_1(s;\omega)|}\right)^3}\right|$$
$$=\frac{1}{2}\left|\varphi_8^3\left(|u(s;\omega)|\right)-\varphi_8^3\left(|u_1(s;\omega)|\right)\right|$$
$$+\frac{1}{2}\left|\varphi_5^3\left(|v(s;\omega)|\right)-\varphi_5^3\left(|v_1(s;\omega)|\right)\right|$$
$$\leq \frac{1}{2}\varphi_8^3\left(||u(s;\omega)|-|u_1(s;\omega)||\right)+\frac{1}{2}\varphi_5^3\left(||v(s;\omega)|-|v_1(s;\omega)||\right)$$
$$\leq \frac{1}{2}\varphi_8^3\left(|u(s;\omega)-u_1(s;\omega)|\right)+\frac{1}{2}\varphi_5^3\left(|v(s;\omega)-v_1(s;\omega)|\right)$$
$$\leq \frac{1}{2}\varphi_8^3\left(\max\left\{|u(\omega)-u_1(\omega)|,|v(\omega)-v_1(\omega)|\right\}\right)$$
$$+\frac{1}{2}\varphi_5^3\left(\max\{|u(\omega)-u_1(\omega)|,|v(\omega)-v_1(\omega)|\}\right)$$
$$\leq 2.\frac{1}{2}\varphi_5^3\left(\max\left\{|u(\omega)-u_1(\omega)|,|v(\omega)-v_1(\omega)|\right\}\right)$$
$$=\frac{\max\{|u(\omega)-u_1(\omega)|,|v(\omega)-v_1(\omega)|\}}{(1+5\sqrt[3]{\max\{|u(\omega)-u_1(\omega)|,|v(\omega)-v_1(\omega)|\}})^3},$$

where we have used Lemma 5.1.
Therefore (H_2) is hold with $\tau = 5$ and $p = 3$.
By Theorem 5.2, system (5.4) has a unique solution

$$(x^*(\omega), y^*(\omega)) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}).$$

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