Abstract. In this paper, we study $L^p$-conjecture on locally compact hypergroups and by some technical proofs we give some sufficient and necessary conditions for a weighted Lebesgue space $L^p(K, w)$ to be a convolution Banach algebra, where $1 < p < \infty$, $K$ is a locally compact hypergroup and $w$ is a weight function on $K$. Among the other things, we also show that if $K$ is a locally compact hypergroup and $p$ is greater than 2, $K$ is compact if and only if $m(K)$ is finite and $f * g$ exists for all $f, g \in L^p(K)$, where $m$ is a left Haar measure for $K$, and in particular, if $K$ is discrete, $K$ is finite if and only if the convolution of any two elements of $L^p(K)$ exists.

1. Introduction

$L^p$-conjecture on a locally compact group $G$ states that $L^p(G)$ is a Banach algebra if and only if $G$ is compact. This conjecture was formulated by Rajagopalan in his Ph.D. thesis in 1963, and then it has been studied in several papers (for instance see [11, 12]). In this work we study this problem on weighted Lebesgue spaces related to locally compact hypergroups and generalize the main results of recent papers [1, 2]. For a locally compact hypergroup $K$ and $p > 2$, we show that $K$ is compact if and only if $m(K)$ is finite and $f * g$ exists for all $f, g \in L^p(K)$, where $m$ is a Haar measure of $K$. As a corollary of our main theorem, if $L^p(K)$ is a convolution algebra and $m(K) < \infty$, then $K$ is compact.

First we recall the definition of hypergroups and some related concepts.

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Hypergroups were introduced in a series of papers by R.I. Jewett [8], C.F. Dunkl [5], and R. Spector [13] in 70’s. Roughly speaking, a hypergroup is a locally compact space which has enough structure so that a convolution on the space of finite regular Borel measures can be defined. Examples include locally compact groups, double-coset hypergroups, orbit hypergroups, polynomial hypergroups, etc. (see [3] for more examples and details).

Let \( K \) be a locally compact Hausdorff space. We denote by \( \mathcal{M}(K) \) the space of all Radon measures on \( K \) and by \( \mathcal{M}^+(K) \) the set of all nonnegative measures in \( \mathcal{M}(K) \). For any Borel measurable function \( f : K \to \mathbb{C} \) and \( \mu \in \mathcal{M}(K) \), \( \text{supp}(f) \) and \( \text{supp}(\mu) \) denote support of \( f \) and \( \mu \), respectively. Point mass measure at \( x \in K \) is denoted by \( \delta_x \).

**Definition 1.1.** Let \( K \) be a locally compact Hausdorff space and assume there exists a positive-continuous mapping \( (\mu, \nu) \mapsto \mu \ast \nu \) from \( \mathcal{M}(K) \times \mathcal{M}(K) \) into \( \mathcal{M}(K) \) such that:

(i) \( (\mathcal{M}(K), +, \ast) \) is an algebra;
(ii) for all \( x, y \in K \), \( \delta_x \ast \delta_y \) is a probability measure with compact support;
(iii) there exists an element \( e \in K \) such that \( \delta_e \ast \delta_e = \delta_e = \delta_x \ast \delta_e \) for all \( x \in K \);
(iv) there exists a topological involution \( x \mapsto x^- \) from \( K \) onto \( K \) such that for all \( x, y \in K \), we have \( (\delta_x \ast \delta_y)^- = \delta_y^- \ast \delta_x^- \), where for \( \mu \in \mathcal{M}(K) \), \( \mu^- \) is defined by \( \int_K f(t) d\mu^-(t) = \int_K f(t^-) d\mu(t) \).
Also, \( e \in \text{supp}(\delta_x \ast \delta_y) \) if and only if \( y = x^- \);
(v) the mapping \( (x, y) \mapsto \text{supp}(\delta_x \ast \delta_y) \) from \( K \times K \) into the space \( \mathcal{C}(K) \) of compact subsets of \( K \) is continuous, where \( \mathcal{C}(K) \) is equipped with the topology whose subbasis is given by all \( \mathcal{C}_{U,V} = \{ A \in \mathcal{C}(K) : A \cap U \neq \emptyset \text{ and } A \subseteq V \} \), in which \( U \) and \( V \) are open subsets of \( K \);

Then \( K = (K, \ast, -, e) \) is called a locally compact hypergroup.

A non-zero nonnegative regular measure \( m \) on \( K \) is called a (left) Haar measure if for all \( x \in K \), \( \delta_x \ast m = m \). It has not been proved that every locally compact hypergroup has a Haar measure. However, commutative hypergroups, compact hypergroups, discrete hypergroups, and double-coset hypergroups have a Haar measure (see [8, 13]). Throughout this paper, we assume that \( K \) is a hypergroup with a Haar measure \( m \). For each \( 1 \leq p < \infty \) we denote \( L^p(K) = L^p(K, m) \). Let \( f \) and \( g \) be Borel functions on \( K \) and \( \mu \in \mathcal{M}(K) \). For any \( x, y \in K \) we denote

\[
 f_x(y) = f(x^- \ast y) := \int_K f \ d(\delta_x^- \ast \delta_y),
\]
and

\[(f \ast g)(x) := \int_{K} f(x \ast y)g(y) \, dm(y).\]

If \(x, y \in K\) and \(A, B \subseteq K\) we denote

\[A^{-} := \{x^{-} : x \in A\}, \quad \{x\} \ast \{y\} := \text{supp}(\delta_{x} \ast \delta_{y}),\]

and

\[A \ast B := \bigcup_{x \in A, y \in B} \{x\} \ast \{y\}.\]

Weight functions on hypergroups have been studied in several papers. F. Ghahramani and A.R. Medghalchi in [6, 7] studied compact multipliers on weighted hypergroup algebras and W.R. Bloom and P. Ressel in [4] obtained a Bochner representation for \(w\)-bounded positive-definite functions on a commutative hypergroup, where \(w\) is a special weight function. See also [9, 10, 15].

**Definition 1.2.** Let \(K\) be a locally compact hypergroup. A continuous function \(w : K \to (0, \infty)\) is called a weight function if for all \(x, y \in K\),

\[w(x \ast y) \leq w(x)w(y),\]

Let \(K\) be a hypergroup with a Haar measure \(m\), \(1 \leq p < \infty\) and \(w\) be a weight function on \(K\). The set of all Borel measurable complex-valued functions \(f\) on \(K\) such that \(fw \in L^{p}(K)\) is denoted by \(L^{p}(K, w)\), and as usual, two functions \(f, g \in L^{p}(K, w)\) are considered identified if \(f = g\) a.e..

\(L^{p}(K, w)\) with the norm \(\|f\|_{p, w} := \|fw\|_{p} \quad (f \in L^{p}(K, w))\) is a Banach space.

Let \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then the dual space \(L^{p}(K, w)^{*}\) is \(L^{q}(K, \frac{1}{w})\), the space of all Borel complex-valued functions \(g\) on \(K\) such that \(\frac{g}{w} \in L^{q}(K)\), under the following duality

\[ \langle f, g \rangle := \int_{K} f(x)g(x) \, dm(x),\]

where \(f \in L^{p}(K, w)\) and \(g \in L^{q}(K, \frac{1}{w})\).

Let \(f, g : K \to \mathbb{C}\) be Borel measurable functions. We put

\[\Omega_{w}(f, g) := \int_{K} f(x)g(y)w(x) \, dm(x).\]

In sequel \(K\) is a locally compact hypergroup with (left) Haar measure \(m\), and \(w\) is a weight function on \(K\).
2. Main Results

In the following theorem, by \( \left( \frac{f}{w} \right)_y \) we mean the mapping \( y \mapsto \left( \frac{f}{w} \right)_y \), where for each \( x \in K \),
\[
\left( \frac{f}{w} \right)_y (x) = \left( \frac{f}{w} \right) \left( y^{-} * x \right) := \int_K \frac{f}{w} \delta_{y^{-}} * \delta_x.
\]

Also, by \( \Omega_{w} \left( g, \left( \frac{f}{w} \right)_y \right) \) we mean the mapping
\[
y \mapsto \Omega_{w} \left( g, \left( \frac{f}{w} \right)_y \right), \quad (y \in K).
\]

**Theorem 2.1.** If \( 1 < p < \infty \), then the following are equivalent:

(i) \( L^p(K, w) \) is a Banach algebra;

(ii) for each \( f \in L^p(K) \) and \( g \in L^q(K) \), we have
\[
\Omega_{w} \left( g, \left( \frac{f}{w} \right)_y \right) \in L^q \left( K, \frac{1}{w} \right),
\]
and
\[
\left\| \Omega_{w} \left( g, \left( \frac{f}{w} \right)_y \right) \right\|_{q, \frac{1}{w}} \leq \|f\|_p \|g\|_q,
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** (i)\( \Rightarrow \) (ii): Let \( L^p(K, w) \) be a Banach algebra, \( f \in L^p(K) \) and \( g \in L^q(K) \). For each \( h \in L^p(K) \) we have
\[
\left\| \frac{1}{w} \Omega_{w} \left( g, \left( \frac{f}{w} \right)_y \right), h \right\| = \left| \int_K \int_K \frac{1}{w(y)} g(x) \left( \frac{f}{w} \right)_y (x) w(x) h(y) dm(x) dm(y) \right|
\]
\[
= \left| \int_K w(x) g(x) \left( \int_K \frac{h}{w} (y) \left( \frac{f}{w} \right)_y (y^{-} * x) dm(y) \right) dm(x) \right|
\]
\[
= \left| \int_K \left( \frac{h}{w} * \frac{f}{w} \right) (x) w(x) g(x) dm(x) \right|
\]
\[
\leq \left\| \left( \frac{h}{w} * \frac{f}{w} \right) w \right\|_1 \|g\|_q
\]
\[
\leq \left\| \left( \frac{h}{w} * \frac{f}{w} \right) w \right\|_p \|g\|_q
\]
\[
= \left\| \left( \frac{h}{w} * \frac{f}{w} \right) \right\|_{p, w} \|g\|_q
\]
\[
\leq \left\| \frac{h}{w} \right\|_{p, w} \left\| \frac{f}{w} \right\|_{p, w} \|g\|_q
\]
\[
= \|h\|_p \|f\|_p \|g\|_q.
\]
Then
\[ \left\| \Omega_w \left( g, \left( \frac{f}{w} \right) \right) \right\|_{q, \frac{1}{w}} = \left\| \frac{1}{w} \Omega_w \left( g, \left( \frac{f}{w} \right) \right) \right\|_{q} \leq \| f \|_p \| g \|_q. \]

(ii)\(\Rightarrow\)(i): Let \( f, h \in L^p(K) \). Then by hypothesis, for each \( g \in L^q(K) \) we have
\[
\left\langle \left( \frac{f}{w} * \frac{h}{w} \right) w, g \right\rangle = \left| \int_K \left( \frac{f}{w} * \frac{h}{w} \right)(x)w(x)g(x)dm(x) \right|
\]
\[
= \left| \int_K \int_K \left( \frac{f}{w} \right)(y) \left( \frac{h}{w} \right)_{y^{-}}(x)w(x)g(x)dm(y)dm(x) \right|
\]
\[
= \left| \int_K f(y) \frac{1}{w(y)} \left( \int_K \frac{h}{w}(x)w(x)g(x)dm(x) \right) dm(y) \right|
\]
\[
= \left| \frac{1}{w} \Omega_w \left( g, \left( \frac{h}{w} \right) \right), f \right|_{q, \frac{1}{w}} \| f \|_p \| g \|_q,
\]
\[
\leq \| h \|_p \| f \|_p \| g \|_q.
\]

Then
\[ \left\| \frac{f}{w} * \frac{h}{w} \right\|_{p, w} \leq \| f \|_p \| h \|_p, \]
and the proof is completed.

**Theorem 2.2.** If
\[
\sup_{\| g \|_{q} \leq 1} \int_K \int_K \left| \frac{(gw)(y * x)}{w(x)w(y)} \right|^q dm(x)dm(y) < \infty,
\]
then \( L^p(K, w) \) is a Banach algebra, where \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Put
\[ M := \sup_{\| g \|_{q} \leq 1} \int_K \int_K \left| \frac{(gw)(y * x)}{w(x)w(y)} \right|^q dm(x)dm(y). \]
Then for all \( g \in L^q(K) \) we have
\[ \int_K \int_K \left| \frac{(gw)(y * x)}{w(x)w(y)} \right|^q dm(x)dm(y) \leq M \| g \|_{q}^q. \]
So, if for every $x, y \in K$ we put

$$
\phi(x, y) := \frac{(gw)(y * x)}{w(x)w(y)}, \quad \phi_y(x) := \phi(x, y),
$$

then

$$
\int_K \|\phi_y\|_q^q dm(y) \leq M \|g\|_q^q.
$$

This implies that for almost every $y \in K$, $\phi_y \in L^q(K)$. By duality, we have

$$
\|\phi_y\|_q^q = \sup_{\|f\|_p \leq 1} \left| \int_K \phi_y(x)f(x)dm(x) \right|^q.
$$

So

$$
\int_K \sup_{\|f\|_p \leq 1} \left| \int_K \phi(x, y)f(x)dm(x) \right|^q dm(y) \leq M \|g\|_q^q,
$$

and then for each $f \in L^p(K)$ we have

$$
\int_K \left| \int_K \frac{(gw)(y * x)}{w(x)w(y)}f(x)dm(x) \right|^q dm(y) \leq M \|f\|_p^q \|g\|_q^q.
$$

Then by [8, 5.1.D],

$$
\int_K \left| \int_K \frac{g(x)}{w(y)} \left( \frac{f}{w} \right) (y^* * x)w(x)dm(x) \right|^q dm(y) \leq M \|f\|_p^q \|g\|_q^q.
$$

So

$$
\left\| \Omega_w \left( g, \left( \frac{f}{w} \right) \right) \right\|_{q, \frac{1}{q}} \leq M_1 \frac{1}{q} \|f\|_p \|g\|_q.
$$

Similar to previous proposition we can prove that for all $f, h \in L^p(K, w)$,

$$
\|f * h\|_{p, w} \leq M_1 \frac{1}{q} \|f\|_{p, w} \|g\|_{q, w}.
$$

Then with considering a new Haar measure on $K$ we have

$$
\|f * h\|_{p, w} \leq \|f\|_{p, w} \|g\|_{q, w},
$$

and proof is completed. \qed

**Theorem 2.3.** Let $K$ be a locally compact hypergroup and $p > 2$. $K$ is compact if and only if $m(K)$ is finite and $f * g$ exists for all $f, g \in L^p(K)$. 
Proof. Obviously, if $K$ is compact, then for all $f, g \in L^p(K)$, $f * g$ exists and $m(K) < \infty$. Conversely, let for all $f, g \in L^p(K)$, $f * g$ exists and $m(K) < \infty$. We consider a compact symmetric neighborhood $B$ of $e$ in $K$, i.e. $B = \{x^- : x \in B\}$. Then since $\text{supp}(m) = K$ (see [S, page 23]), we have $0 < m(B) \leq m(B * B) < \infty$. By continuity of modular function $\Delta$ ([S, 5.3B]), there exists a constant $M > 0$ such that for all $x \in B$, $0 < \Delta(x) \leq M$.

On the contrary, suppose that $K$ is not compact. Then there exists an element $a$ in $K \setminus B$. So $a^- \in K \setminus B$ and by [S, 5.3B], $\Delta(a) \Delta(a^-) = 1$. Thus for some $a_1 \in K \setminus B$ we have $\Delta(a_1) \leq 1$. Let

$$E_1 := (\{a_1\} * B * B * B) \cup (B * B * B * B * \{a_1^-\}).$$

Then $E_1$ is a compact symmetric subset of $K$. So $K \setminus E_1$ is a nonempty symmetric subset of $K$, and there is an element $a_2 \in K \setminus E_1$ such that $\Delta(a_2) \leq 1$. We show that $(B * \{a_1^-\}) \cap (B * \{a_2^-\}) = \emptyset$. If there exists $t \in (B * \{a_1^-\}) \cap (B * \{a_2^-\})$, then $a_2 \in \{t^-\} * B$, since $t \in B * \{a_2^-\}$ (see [S, 4.1B]). So

$$a_2 \in \{a_1\} * B * B \subseteq \{a_1\} * B * B * B * B,$$

a contradiction.

Similarly $(\{a_1\} * B * B) \cap (\{a_2\} * B * B) = \emptyset$. Inductively, we can find $a_1, a_2, \ldots$ in $K$ such that for all distinct $m, n \in \mathbb{N}$,

$$(\{a_n\} * B * B) \cap (\{a_m\} * B * B) = \emptyset,$$

and

$$(2.1) \quad (B * \{a_n^-\}) \cap (B * \{a_m^-\}) = \emptyset.$$

Consider a number $L$ such that $m(\{a_n\} * B * B) \leq L$, for all $n = 1, 2, 3, \ldots$. For every $x \in K$, we define

$$f(x) := \Delta(x^-)^{\frac{1}{2p}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{B^* \{a_n^-\}}(x),$$

and

$$g(x) := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{\{a_n\} * B^* B}(x).$$
We will show that \( f, g \in L^p(K) \) but \( f \ast g \) does not exist. First since \( \Delta m^- = m (\mathbb{R}, 5.3B) \), we have

\[
\int_K |f(x)|^p \, dm(x) = \int_K \Delta (x^-) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \chi_{B^+ \{a_n^-\}}(x) \, dm(x)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \int_K \Delta (x^-) \chi_{B^+ \{a_n^-\}}(x) \, dm(x)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \int \chi_{\{a_n\} \ast B} \, d(\Delta m^-)(x)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} m(\{a_n\} \ast B)
\]

\[
\leq L \cdot \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty,
\]

and

\[
\int_K |g(x)|^p \, dm(x) = \int_K \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \chi_{\{a_n\} \ast B \ast B}(x) \, dm(x)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} m(\{a_n\} \ast B \ast B)
\]

\[
\leq L \cdot \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty.
\]

On the other hand, if \( x \in B \) and \( y \in K \), then by relation 2.1, at most for one \( n \in \mathbb{N} \) we have \( y \in B \ast \{a_n^-\} \). Then \( f(y) = \Delta(y^-)^{\frac{1}{2}} \frac{1}{\sqrt{n}} \). If \( t \in \{y^-\} \ast \{x\} \), then \( t \in \{a_n^-\} \ast B \ast B \). So \( g(t) = \frac{1}{\sqrt{n}} \) and

\[
g(y^- \ast x) = \int_{\{y^-\} \ast \{x\}} g(t) \, d(\delta_{y^-} \ast \delta_x)(t) = \frac{1}{\sqrt{n}}.
\]
Thus

\[(f * g)(x) = \int_K f(y)g(y - x)dm(y)\]

\[= \sum_{n=1}^{\infty} \frac{1}{n} \int_{K} \Delta(y)^{-\frac{1}{p}} \chi_{B^+\{a_n^\circ\}}(y)dm(y)\]

\[= \sum_{n=1}^{\infty} \frac{1}{n} \int_{\{a_n\} * B} \Delta(y)^{\frac{1}{p}} dm(y)\]

\[\geq M^{\frac{1}{p} - 1} \sum_{n=1}^{\infty} \frac{1}{n} m(\{a_n\} * B)\]

\[\geq m(B) M^{\frac{1}{p} - 1} \sum_{n=1}^{\infty} \frac{1}{n} = \infty,\]

since by [8, 3.3C] we have \(m(B) \leq m(\{a_n\} * B)\), and for each \(y \in \{a_n\} * B\), there is \(b \in B\) such that \(y \in \{a_n\} * B\), and so by [8, 5.3C], \(\Delta(y) = \Delta(a_n) \Delta(b)\), which implies that

\[\Delta(y)^{\frac{1}{p} - 1} = \Delta(a_n)^{\frac{1}{p} - 1} \Delta(b)^{\frac{1}{p} - 1} \geq M^{\frac{1}{p} - 1}.\]

Thus for all \(x \in B\), \(f * g(x) = \infty\). \(\square\)

**Corollary 2.4.** Let \(p > 2\). If \(m(K) < \infty\) and \(L^p(K)\) is a convolution algebra, then \(K\) is compact.

**Corollary 2.5.** Let \(K\) be a discrete hypergroup, \(m(K) < \infty\) and \(p > 2\). \(K\) is finite if and only if for all \(f, g \in L^p(K)\), \(f * g\) exists.

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**References**


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