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L^p-Conjecture on Hypergroups

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Abstract. In this paper, we study L^p-conjecture on locally compact hypergroups and by some technical proofs we give some sufficient and necessary conditions for a weighted Lebesgue space $L^p(K,w)$ to be a convolution Banach algebra, where $1<p<\infty$, K is a locally compact hypergroup and w is a weight function on K. Among the other things, we also show that if K is a locally compact hypergroup and p is greater than 2, K is compact if and only if $m(K)$ is finite and $f*g$ exists for all $f,g\in L^p(K)$, where m is a left Haar measure for K, and in particular, if K is discrete, K is finite if and only if the convolution of any two elements of $L^p(K)$ exists.

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1. Introduction

L^p-conjecture on a locally compact group G states that $L^p(G)$ is a Banach algebra if and only if G is compact. This conjecture was formulated by Rajagopalan in his Ph.D. thesis in 1963, and then it has been studied in several papers (for instance see [11, 12]). In this work we study this problem on weighted Lebesgue spaces related to locally compact hypergroups and generalize the main results of recent papers [1, 2]. For a locally compact hypergroup K and $p>2$, we show that K is compact if and only if $m(K)$ is finite and $f*g$ exists for all $f,g\in L^p(K)$, where m is a Haar measure of K. As a corollary of our main theorem, if $L^p(K)$ is a convolution algebra and $m(K)<\infty$, then K is compact.

First we recall the definition of hypergroups and some related concepts.

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Hypergroups were introduced in a series of papers by R.I. Jewett [8], C.F. Dunkl [5], and R. Spector [13] in 70's. Roughly speaking, a hypergroup is a locally compact space which has enough structure so that a convolution on the space of finite regular Borel measures can be defined. Examples include locally compact groups, double-coset hypergroups, orbit hypergroups, polynomial hypergroups, etc. (see [3] for more examples and details).

Let K be a locally compact Hausdorff space. We denote by $\mathcal{M}(K)$ the space of all Radon measures on K and by $\mathcal{M}^+(K)$ the set of all nonnegative measures in $\mathcal{M}(K)$. For any Borel measurable function $f : K \to \mathbb{C}$ and $\mu \in \mathcal{M}(K)$, supp(f) and supp(μ) denote support of f and μ, respectively. Point mass measure at $x \in K$ is denoted by δ_x.

Definition 1.1. Let K be a locally compact Hausdorff space and assume there exists a positive-continuous mapping $(\mu, \nu) \mapsto \mu * \nu$ from $\mathcal{M}(K) \times \mathcal{M}(K)$ into $\mathcal{M}(K)$ such that:

- (i) $(\mathcal{M}(K), +, *)$ is an algebra;
- (ii) for all $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;
- (iii) there exists an element $e \in K$ such that $\delta_e * \delta_x = \delta_e = \delta_x * \delta_e$ for all $x \in K$;
- (iv) there exists a topological involution $x \mapsto x^-$ from K onto K such that for all $x, y \in K$, we have $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$, where for $\mu \in M(K)$, μ^- is defined by $\int_K f(t) d\mu^-(t) = \int_K f(t^-) d\mu(t)$. Also, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = x^-$;
- (v) the mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space $\mathbf{C}(K)$ of compact subsets of K is continuous, where $\mathbf{C}(K)$ is equipped with the topology whose subbasis is given by all $\mathbf{C}_{U,V} = \{A \in \mathbf{C}(K) \ : \ A \cap U \neq \varnothing \text{ and } A \subseteq V\}$, in which U and V are open subsets of K;

Then $K = (K, *, ^-, e)$ is called a locally compact hypergroup.

A non-zero nonnegative regular measure m on K is called a (left) Haar measure if for all $x \in K$, $\delta_x * m = m$. It has not been proved that *every* locally compact hypergroup has a Haar measure. However, commutative hypergroups, compact hypergroups, discrete hypergroups, and double-coset hypergroups have a Haar measure (see [8, 14]). Throughout this paper, we assume that K is a hypergroup with a Haar measure m. For each $1 \leq p < \infty$ we denote $L^p(K) = L^p(K, m)$. Let f and g be Borel functions on K and $\mu \in M(K)$. For any $x, y \in K$ we denote

$$f_x(y) = f(x^- * y) := \int_K f \, d\left(\delta_{x^-} * \delta_y\right),$$

and

$$(f * g)(x) := \int_K f(x * y) g\left(y^-\right) dm(y).$$

If $x, y \in K$ and $A, B \subseteq K$ we denote

$$A^- := \{x^- : x \in A\}, \qquad \{x\} * \{y\} := \operatorname{supp}\left(\delta_x * \delta_y\right),$$

and

$$A * B := \bigcup_{x \in A, y \in B} \{x\} * \{y\}.$$

Weight functions on hypergroups have been studied in several papers. F. Ghahramani and A.R. Medghalchi in [6, 7] studied compact multipliers on weighted hypergroup algebras and W.R. Bloom and P. Ressel in [4] obtained a Bochner representation for w-bounded positive-definite functions on a commutative hypergroup, where w is a special weight function. See also [9, 10, 15].

Definition 1.2. Let K be a locally compact hypergroup. A continuous function $w : K \to (0, \infty)$ is called a weight function if for all $x, y \in K$, $w(x * y) \leq w(x)w(y)$.

Let K be a hypergroup with a Haar measure m, $1 \leq p < \infty$ and w be a weight function on K. The set of all Borel measurable complex-valued functions f on K such that $fw \in L^p(K)$ is denoted by $L^p(K, w)$, and as usual, two functions $f, g \in L^p(K, w)$ are considered identified if $f = g$ a.e..

$L^p(K, w)$ with the norm $\|f\|_{p,w} := \|fw\|_p \quad (f \in L^p(K, w))$ is a Banach space.

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space $L^p(K, w)^*$ is $L^q(K, \frac{1}{w})$, the space of all Borel complex-valued functions g on K such that $\frac{g}{w} \in L^q(K)$, under the following duality

$$\langle f, g \rangle := \int_K f(x) g(x) dm(x),$$

where $f \in L^p(K, w)$ and $g \in L^q(K, \frac{1}{w})$.

Let $f, g : K \to \mathbb{C}$ be Borel measurable functions. We put

$$\Omega_w(f, g) := \int_K f(x) g(y) w(x) dm(x).$$

In sequel K is a locally compact hypergroup with (left) Haar measure m, and w is a weight function on K.

2. Main Results

In the following theorem, by $\left(\frac{f}{w}\right)_{(.)}$ we mean the mapping $y \mapsto \left(\frac{f}{w}\right)_y$ $(y \in K)$, where for each $x \in K$,

$$\left(\frac{f}{w}\right)_y (x) = \left(\frac{f}{w}\right)(y^- * x) := \int_K \frac{f}{w}\, d\left(\delta_{y^-} * \delta_x\right).$$

Also, by $\Omega_w\left(g, \left(\frac{f}{w}\right)_{(.)}\right)$ we mean the mapping

$$y \mapsto \Omega_w\left(g, \left(\frac{f}{w}\right)_y\right), \quad (y \in K).$$

Theorem 2.1. *If $1 < p < \infty$, then the following are equivalent:*

(i) *$L^p(K, w)$ is a Banach algebra;*
(ii) *for each $f \in L^p(K)$ and $g \in L^q(K)$, we have*

$$\Omega_w\left(g, \left(\frac{f}{w}\right)_{(.)}\right) \in L^q\left(K, \frac{1}{w}\right),$$

and

$$\left\|\Omega_w\left(g, \left(\frac{f}{w}\right)_{(.)}\right)\right\|_{q, \frac{1}{w}} \le \|f\|_p \|g\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof.(i)⇒(ii): Let $L^p(K, w)$ be a Banach algebra, $f \in L^p(K)$ and $g \in L^q(K)$. For each $h \in L^p(K)$ we have

$$\begin{aligned}
\left|\left\langle \frac{1}{w}\Omega_w\left(g, \left(\frac{f}{w}\right)_{(.)}\right), h\right\rangle\right| &= \left|\int_K \int_K \frac{1}{w(y)} g(x) \left(\frac{f}{w}\right)_y (x) w(x) h(y) dm(x) dm(y)\right| \\
&= \left|\int_K w(x) g(x) \left(\int_K \left(\frac{h}{w}\right)(y) \left(\frac{f}{w}\right)(y^- * x) dm(y)\right) dm(x)\right| \\
&= \left|\int_K \left(\frac{h}{w} * \frac{f}{w}\right)(x) w(x) g(x) dm(x)\right| \\
&\le \left\|\left(\frac{h}{w} * \frac{f}{w}\right) wg\right\|_1 \\
&\le \left\|\left(\frac{h}{w} * \frac{f}{w}\right) w\right\|_p \|g\|_q \\
&= \left\|\left(\frac{h}{w} * \frac{f}{w}\right)\right\|_{p,w} \|g\|_q \\
&\le \left\|\frac{h}{w}\right\|_{p,w} \left\|\frac{f}{w}\right\|_{p,w} \|g\|_q \\
&= \|h\|_p \|f\|_p \|g\|_q .
\end{aligned}$$

Then

$$\left\|\Omega_w\left(g,\left(\frac{f}{w}\right)_{(.)}\right)\right\|_{q,\frac{1}{w}}=\left\|\frac{1}{w}\Omega_w\left(g,\left(\frac{f}{w}\right)_{(.)}\right)\right\|_q\leq\|f\|_p\,\|g\|_q\,.$$

(ii)⇒(i): Let $f,h\in L^p(K)$. Then by hypothesis, for each $g\in L^q(K)$ we have

$$\begin{aligned}\left|\left\langle\left(\frac{f}{w}*\frac{h}{w}\right)w,g\right\rangle\right| &= \left|\int_K\left(\frac{f}{w}*\frac{h}{w}\right)(x)w(x)g(x)dm(x)\right|\\ &= \left|\int_K\int_K\left(\frac{f}{w}\right)(y)\left(\frac{h}{w}\right)_{y^-}(x)w(x)g(x)dm(y)dm(x)\right|\\ &= \left|\int_K f(y)\frac{1}{w(y)}\left(\int_K\left(\frac{h}{w}\right)_{y^-}(x)w(x)g(x)dm(x)\right)dm(y)\right|\\ &= \left|\left\langle\frac{1}{w}\Omega_w\left(g,\left(\frac{h}{w}\right)_{(.)}\right),f\right\rangle\right|\\ &\leq \left\|\frac{1}{w}\Omega_w\left(g,\left(\frac{h}{w}\right)_{(.)}\right)\right\|_q\|f\|_p\\ &= \left\|\Omega_w\left(g,\left(\frac{h}{w}\right)_{(.)}\right)\right\|_{q,\frac{1}{w}}\|f\|_p\\ &\leq \|h\|_p\,\|f\|_p\,\|g\|_q\,.\end{aligned}$$

Then

$$\left\|\frac{f}{w}*\frac{h}{w}\right\|_{p,w}\leq\|f\|_p\,\|h\|_p\,,$$

and the proof is completed.

□

Theorem 2.2. *If*

$$\sup_{\|g\|_q\leq 1}\int_K\int_K\left|\frac{(gw)(y*x)}{w(x)w(y)}\right|^q dm(x)dm(y)<\infty,$$

then $L^p(K,w)$ is a Banach algebra, where $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Put

$$M:=\sup_{\|g\|_q\leq 1}\int_K\int_K\left|\frac{(gw)(y*x)}{w(x)w(y)}\right|^q dm(x)dm(y).$$

Then for all $g\in L^q(K)$ we have

$$\int_K\int_K\left|\frac{(gw)(y*x)}{w(x)w(y)}\right|^q dm(x)dm(y)\leq M\,\|g\|_q^q\,.$$

So, if for every $x, y \in K$ we put

$$\phi(x,y) := \frac{(gw)(y*x)}{w(x)w(y)}, \qquad \phi_y(x) := \phi(x,y),$$

then

$$\int_K \|\phi_y\|_q^q \, dm(y) \leq M \|g\|_q^q .$$

This implies that for almost every $y \in K$, $\phi_y \in L^q(K)$. By duality, we have

$$\|\phi_y\|_q^q = \sup_{\|f\|_p \leq 1} \left| \int_K \phi_y(x) f(x) dm(x) \right|^q .$$

So

$$\int_K \sup_{\|f\|_p \leq 1} \left| \int_K \phi(x,y) f(x) dm(x) \right|^q dm(y) \leq M \|g\|_q^q ,$$

and then for each $f \in L^p(K)$ we have

$$\int_K \left| \int_K \frac{(gw)(y*x)}{w(x)w(y)} f(x) dm(x) \right|^q dm(y) \leq M \|f\|_p^q \|g\|_q^q .$$

Then by [8, 5.1.D],

$$\int_K \left| \int_K \frac{g(x)}{w(y)} \left(\frac{f}{w}\right)(y^- * x) w(x) dm(x) \right|^q dm(y) \leq M \|f\|_p^q \|g\|_q^q .$$

So

$$\left\| \Omega_w \left(g, \left(\frac{f}{w}\right)_{(.)} \right) \right\|_{q,\frac{1}{w}} \leq M^{\frac{1}{q}} \|f\|_p \|g\|_q .$$

Similar to previous proposition we can prove that for all $f, h \in L^p(K,w)$,

$$\|f*h\|_{p,w} \leq M^{\frac{1}{q}} \|f\|_{p,w} \|g\|_{q,w} .$$

Then with considering a new Haar measure on K we have

$$\|f*h\|_{p,w} \leq \|f\|_{p,w} \|g\|_{q,w} ,$$

and proof is completed. □

Theorem 2.3. *Let K be a locally compact hypergroup and $p > 2$. K is compact if and only if $m(K)$ is finite and $f*g$ exists for all $f, g \in L^p(K)$.*

Proof. Obviously, if K is compact, then for all $f, g \in L^p(K)$, $f * g$ exists and $m(K) < \infty$. Conversely, let for all $f, g \in L^p(K)$, $f * g$ exists and $m(K) < \infty$. We consider a compact symmetric neighborhood B of e in K i.e. $B = \{x^- : x \in B\}$. Then since $\operatorname{supp}(m) = K$ (see [8, page 23]), we have $0 < m(B) \leq m(B * B) < \infty$. By continuity of modular function Δ ([8, 5.3B]), there exists a constant $M > 0$ such that for all $x \in B$, $0 < \Delta(x) \leq M$.

On the contrary, suppose that K is not compact. Then there exists an element a in $K \setminus B$. So $a^- \in K \setminus B$ and by [8, 5.3B], $\Delta(a)\Delta(a^-) = 1$. Thus for some $a_1 \in K \setminus B$ we have $\Delta(a_1) \leq 1$. Let

$$E_1 := (\{a_1\} * B * B * B * B) \bigcup \left(B * B * B * B * \{a_1^-\}\right).$$

Then E_1 is a compact symmetric subset of K. So $K \setminus E_1$ is a nonempty symmetric subset of K, and there is an element $a_2 \in K \setminus E_1$ such that $\Delta(a_2) \leq 1$. We show that $\left(B * \{a_1^-\}\right) \bigcap \left(B * \{a_2^-\}\right) = \varnothing$. If there exists $t \in \left(B * \{a_1^-\}\right) \bigcap \left(B * \{a_2^-\}\right)$, then $a_2 \in \{t^-\} * B$, since $t \in B * \{a_2^-\}$ (see [8, 4.1B]). So

$$a_2 \in \{a_1\} * B * B \subseteq \{a_1\} * B * B * B * B,$$

a contradiction.

Similarly $(\{a_1\} * B * B) \bigcap (\{a_2\} * B * B) = \varnothing$. Inductively, we can find $a_1, a_2, \ldots$ in K such that for all distinct $m, n \in \mathbb{N}$,

$$(\{a_n\} * B * B) \bigcap (\{a_m\} * B * B) = \varnothing,$$

and

$$\left(B * \{a_n^-\}\right) \bigcap \left(B * \{a_m^-\}\right) = \varnothing. \tag{2.1}$$

Consider a number L such that $m\left(\{a_n\} * B * B\right) \leq L$, for all $n = 1, 2, 3, \ldots$. For every $x \in K$, we define

$$f(x) := \Delta(x^-)^{\frac{1}{p}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{B * \{a_n^-\}}(x),$$

and

$$g(x) := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{\{a_n\} * B * B}(x).$$

We will show that $f, g \in L^p(K)$ but $f * g$ does not exist. First since $\Delta m^- = m$ ([8, 5.3B]), we have

$$\begin{aligned}
\int_K |f(x)|^p \, dm(x) &= \int_K \Delta\left(x^-\right) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \chi_{B*\{a_n^-\}}(x) dm(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \int_K \Delta\left(x^-\right) \chi_{B*\{a_n^-\}}(x) dm(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \int_{\{a_n\}*B} d\left(\Delta m^-\right)(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} m(\{a_n\} * B) \\
&\leq L \cdot \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
\int_K |g(x)|^p \, dm(x) &= \int_K \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} \chi_{\{a_n\}*B*B}(x) dm(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^p}} m\left(\{a_n\} * B * B\right) \\
&\leq L \cdot \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty.
\end{aligned}$$

On the other hand, if $x \in B$ and $y \in K$, then by relation 2.1, at most for one $n \in \mathbb{N}$ we have $y \in B * \{a_n^-\}$. Then $f(y) = \Delta(y^-)^{\frac{1}{p}} \frac{1}{\sqrt{n}}$. If $t \in \{y^-\} * \{x\}$, then $t \in \{a_n^-\} * B * B$. So $g(t) = \frac{1}{\sqrt{n}}$ and

$$g(y^- * x) = \int_{\{y^-\}*\{x\}} g(t) d\left(\delta_{y^-} * \delta_x\right)(t) = \frac{1}{\sqrt{n}}.$$

Thus

$$\begin{aligned}
(f*g)(x) &= \int_K f(y)g(y^- * x)dm(y) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_K \Delta\left(y^-\right)^{\frac{1}{p}} \chi_{B*\{a_n^-\}}(y)dm(y) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_{\{a_n\}*B} \Delta(y)^{\frac{1}{p}} dm^-(y) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_{\{a_n\}*B} \Delta(y)^{\frac{1}{p}-1} d(\Delta m^-)(y) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_{\{a_n\}*B} \Delta(y)^{\frac{1}{p}-1} dm(y) \\
&\geq M^{\frac{1}{p}-1} \sum_{n=1}^{\infty} \frac{1}{n} m\left(\{a_n\}*B\right) \\
&\geq m(B) M^{\frac{1}{p}-1} \sum_{n=1}^{\infty} \frac{1}{n} = \infty,
\end{aligned}$$

since by [8, 3.3C] we have $m(B) \leq m\left(\{a_n\}*B\right)$, and for each $y \in \{a_n\}*B$, there is $b \in B$ such that $y \in \{a_n\}*B$, and so by [8, 5.3C], $\Delta(y) = \Delta(a_n)\Delta(b)$, which implies that

$$\Delta(y)^{\frac{1}{p}-1} = \Delta(a_n)^{\frac{1}{p}-1}.\Delta(b)^{\frac{1}{p}-1} \geq M^{\frac{1}{p}-1}.$$

Thus for all $x \in B$, $f*g(x) = \infty$. □

Corollary 2.4. *Let $p > 2$. If $m(K) < \infty$ and $L^p(K)$ is a convolution algebra, then K is compact.*

Corollary 2.5. *Let K be a discrete hypergroup, $m(K) < \infty$ and $p > 2$. K is finite if and only if for all $f, g \in L^p(K)$, $f*g$ exists.*

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