Sahand Communications in Mathematical Analysis (SCMA) Vol. 11, No. 1 (2018), 25-41
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.60715.215

Meir-Keeler Type Contraction Mappings in c_0-triangular Fuzzy Metric Spaces

Masoomeh Hezarjaribi

Abstract. Proving fixed point theorem in a fuzzy metric space is not possible for Meir-Keeler contractive mapping. For this, we introduce the notion of c_0-triangular fuzzy metric space. This new space allows us to prove some fixed point theorems for Meir-Keeler contractive mapping. As some pattern we introduce the class of $\alpha\Delta$-Meir-Keeler contractive and we establish some results of fixed point for such a mapping in the setting of c_0-triangular fuzzy metric space. An example is furnished to demonstrate the validity of these obtained results.

1. Introduction

In order to generalize the well known Banach contraction principle, many authors introduced various types of contraction inequalities. One of them was introduced by Meir and Keeler [8] in 1969. Meir-Keeler's fixed point theorem has been extended and generalized in many directions (e.g., see [5, 6, 9] and references therein).

In 1965, Zadeh [13] introduced the interesting and important concept of fuzzy set and it is known that there are various types of fuzzy metric spaces in fuzzy topology. One of these fuzzy metric spaces was introduced by Kramosil and Michálek [7] which can be a suitable generalization of the statistical (probabilistic) metric spaces. Evidently, this notion provides an important basis for the establish of fixed point theory in fuzzy metric spaces.

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. c_0-triangular fuzzy metric space, $\alpha\Delta$-Meir-Keeler contractive, Fixed point.
Received: 12 March 2017, Accepted: 12 June 2018.

Next, Grabiec [4] introduced the notion of G-complete fuzzy metric space and then established the Banach contraction principle in setting of G-complete fuzzy metric space. Afterwards, George and Veeramani [3] modified the notion of the G-Cauchy sequence introduced by Grabiec.

Also, they modified the definition of fuzzy metric space of Kramosil and Michálek and so introduced a Hausdorff and first countable topology. Thereafter, the concept of a complete fuzzy metric space defined by George and Veeramani has appeared as another characterization of completeness.

Now we know such a metric space as a complete fuzzy metric space. There are many papers related to fixed point theory in the setting of the above two kinds of complete fuzzy metric spaces (e.g., [1, 2, 12] and the references therein).

Proving fixed point theorem in a fuzzy metric space is not possible for Meir-Keeler contractive mapping. For this, we introduce the notion of c_0-triangular fuzzy metric space. This new space allows us to prove some fixed point theorems for Meir-Keeler contractive mapping.

As some pattern we introduce the class of $\alpha\Delta$-Meir-Keeler contractive and we establish some results of fixed point for such mapping in the setting of c_0-triangular fuzzy metric space. An example is furnished to demonstrate the validity of the obtained results.

We are going to recall some necessary concepts, required definitions and primary results to coherence with the literature.

Definition 1.1 ([3], Schweizer and Sklar)**.** A binary operation $\star : [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:

(T1) $\star$ is commutative and associative;
(T2) $\star$ is continuous;
(T3) $a \star 1 = a$ for all $a \in [0,1]$;
(T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Definition 1.2 ([3], George and Veeramani)**.** A fuzzy metric space is an ordered triple $(X, M, \star)$ whence $X \neq \emptyset$, $\star$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,+\infty)$ satisfying the following assertions, for all $x, y, z \in X$ and $t, s > 0$:

(F1) $M(x, y, t) > 0$;
(F2) $M(x, y, t) = 1$ if and only if $x = y$;
(F3) $M(x, y, t) = M(y, x, t)$;
(F4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$;
(F5) $M(x, y, \cdot) : (0,+\infty) \to (0, 1]$ is continuous.

Definition 1.3 ([3], George and Veeramani)**.** Let $(X, M, \star)$ be a fuzzy metric space. Then

(i) a sequence $\{x_n\}$ converges to $x \in X$, if and only if

$$\lim_{n\to+\infty} M(x_n, x, t) = 1,$$

for all $t > 0$,

(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\epsilon \in (0, 1)$ and $t > 0$, there exists n_0 whence

$$M(x_n, x_m, t) > 1 - \epsilon,$$

for all $m, n \geq n_0$,

(iii) a fuzzy metric space is called complete if every Cauchy sequence converges to some $x \in X$.

In recent years, Samet et al. [11] defined the notion of α-admissible mappings as follows.

Definition 1.4 ([11])**.** Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if and only if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Finally, we recall that Karapinar et al. [6] introduced the notion of triangular α-admissible mapping as follows.

Definition 1.5 ([6])**.** Let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that a self-mapping $T : X \to X$ is triangular α-admissible

(i) if $x, y \in X$ and $\alpha(x, y) \geq 1$ then

$$\alpha(Tx, Ty) \geq 1,$$

(ii) if $x, y, z \in X$, then

$$\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \quad \Rightarrow \quad \alpha(x, y) \geq 1.$$

Lemma 1.6 ([6])**.** *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \textit{ for all } m, n \in \mathbb{N} \textit{ with } m < n.$$

2. c_0-Triangular Fuzzy Metric Spaces

In this section, we introduce the concept of c_0-triangular fuzzy metric spaces which allows us to prove some fixed point theorems for Meir-Keeler contractive mapping. Note that the notion of triangular fuzzy metric was introduced in [2].

Definition 2.1. Let $(X, M, *)$ be a fuzzy metric space and

$$M^{-1}(x,y,t) = \frac{1}{M(x,y,t)},$$

for all $x, y \in X$ and $t > 0$. Let $c_0 > 0$. We say the fuzzy metric $(X, M, *)$ is a c_0-triangular fuzzy metric space when

$$M^{-1}(x,y,c_0) - 1 \leq M^{-1}(x,z,c_0) - 1 + M^{-1}(z,y,c_0) - 1,$$

for all $x, y, \in X$.

Definition 2.2. Let $\{x_n\}$ be a sequence in a c_0-triangular fuzzy metric space $(X, M, *)$.

- $\{x_n\}$ is said to be c_0-convergent to x in X, written $x_n \xrightarrow{c_0} x$ as $n \to \infty$, if
$$\lim_{n\to\infty} M(x_n, x, c_0) = 1.$$
- $\{x_n\}$ is said to be c_0-Cauchy if
$$\lim_{n,m\to\infty} M(x_n, x_m, c_0) = 1.$$
- X is said to be c_0-complete if every c_0-Cauchy sequence is a c_0-convergent sequence.

Definition 2.3. Assume that $(X, M, *)$ be a c_0-triangular fuzzy metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \xrightarrow{c_0} x$ and $y_n \xrightarrow{c_0} y$. Let $f : X \times X \to D \subset \mathbb{R}$ be a function. We say f is c_0-continuous if $\lim_{n\to\infty} f(x_n, y_n) = f(x, y)$.

Definition 2.4. Assume that $(X, M, *)$ be a c_0-triangular fuzzy metric space. Let $\{x_n\}$ be a sequence in X such that $x_n \xrightarrow{c_0} x$. Let $T : X \to X$ be a self mapping. We say T is c_0-continuous if $M(x_n, x, c_0) \to 1$ as $n \to \infty$ implies $M(Tx_n, Tx, c_0) \to 1$ as $n \to \infty$.

Definition 2.5. We say the c_0-triangular fuzzy metric space $(X, M, *)$ is c_0-regular if, $M(x, y, c_0) = 1$ implies, $M(x, y, t) = 1$ for all $t > 0$ (or $x = y$).

Lemma 2.6. *Assume that $(X, M, *)$ is a c_0-triangular fuzzy metric space. Then the function M is c_0-continuous.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \xrightarrow{c_0} x$ and $y_n \xrightarrow{c_0} y$. Then we can write,

$$\begin{aligned} M^{-1}(x_n, y_n, c_0) - 1 &\leq M^{-1}(x_n, x, c_0) - 1 + M^{-1}(y_n, x, c_0) - 1 \\ &\leq M^{-1}(x_n, x, c_0) - 1 + M^{-1}(y_n, y, c_0) - 1 \\ &\quad + M^{-1}(x, y, c_0) - 1, \end{aligned}$$

and

$$\begin{aligned} M^{-1}(x,y,c_0)-1 &\le M^{-1}(x,x_n,c_0)-1+M^{-1}(x_n,y,c_0)-1 \\ &\le M^{-1}(x,x_n,c_0)-1+M^{-1}(x_n,y_n,c_0)-1 \\ &\quad + M^{-1}(y_n,y,c_0)-1, \end{aligned}$$

and hence by taking limit as $n\to\infty$ in the above inequalities we get,

$$\lim_{n\to\infty} M^{-1}(x_n,y_n,c_0)=M^{-1}(x,y,c_0),$$

or

$$\lim_{n\to\infty} M(x_n,y_n,c_0)=M(x,y,c_0).$$

□

Throughout this paper, $Fix(T)$ denotes the set of all fixed points of a self mapping T in c_0-triangular fuzzy metric spaces.

3. Main Results

Assume that $(X,M,*)$ is a c_0-triangular fuzzy metric space. Denote by Γ_Δ the family of functions $\Delta : X\times X\to[-1,\infty)$ whence Δ is c_0-continuous.

Remark 3.1. Note that, $M^{-1}(x,y,,c_0)+\Delta(x,y)\ge 0$ for all $x,y\in X$ and $t>0$. Also if $x\neq y$ then, $M^{-1}(x,y,,c_0)+\Delta(x,y)>0$.

Now we define the notion of $\alpha\Delta$-Meir-Keeler contractive mapping as follows.

Definition 3.2. Let $(X,M,*)$ be a c_0-triangular fuzzy metric space and T be a self-mapping on X. Also, suppose that $\Delta\in\Gamma_\Delta$ and $\alpha : X\times X\to[0,+\infty)$ is a function. We say that T is $\alpha\Delta$-Meir-Keeler contractive if for each $\epsilon>0$ and $x,\ y\in X$ there exists $\delta>0$ such that

$$\epsilon\le M^{-1}(x,y,c_0)+\Delta(x,y)<\epsilon+\delta \tag{3.1}$$

then

$$\alpha(x,y)\left[M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)\right]<\epsilon.$$

Remark 3.3. Let $(X,M,*)$ be a c_0-triangular fuzzy metric space and T be an $\alpha\Delta$-Meir-Keeler contractive mapping. Then

$$\alpha(x,y)[M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)]<M^{-1}(x,y,c_0)+\Delta(x,y),$$

for all $x,y\in X$ with $x\neq y$.

Proof. From Remark 3.1, we know that $M^{-1}(x,y,c_0)+\Delta(x,y)>0$ for all $x,y\in X$ with $x\neq y$. Assume, $\delta>0$ and $\epsilon=M^{-1}(x,y,c_0)+\Delta(x,y)$. Now we can write

$$\begin{aligned}\epsilon &\leq M^{-1}(x,y,c_0)+\Delta(x,y)\\ &< M^{-1}(x,y,c_0)+\Delta(x,y)+\delta\\ &=\epsilon+\delta,\end{aligned}$$

and then by using (3.1) we derive,

$$\begin{aligned}\alpha(x,y)\left[M^{-1}(Tx,Ty)+\Delta(Tx,Ty)\right] &< \epsilon\\ &= M^{-1}(x,y,c_0)+\Delta(x,y).\end{aligned}$$

□

Remark 3.4. Let $(X,M,*)$ be a c_0-triangular fuzzy metric space and T be an $\alpha\Delta$-Meir-Keeler contractive mapping. Then

$$\alpha(x,y)[M^{-1}(Tx,Ty,c_0)-1]\leq M^{-1}(x,y,c_0)-1,$$

for all $x,y\in X$ with $\Delta(x,y)=-1$.

Proof. From Remark 3.3, we know that

$$\alpha(x,y)[M^{-1}(Tx,Ty,c_0)-1]< M^{-1}(x,y,c_0)-1,$$

for all $x,y\in X$ with $x\neq y$ and $\Delta(x,y)=-1$. If $x=y$ then $M^{-1}(Tx,Ty,c_0)=1$. Thus,

$$\begin{aligned}\alpha(x,y)[M^{-1}(Tx,Ty,c_0)-1] &= 0\\ &\leq M^{-1}(x,y,c_0)-1,\end{aligned}$$

and these two arguments show that

$$\alpha(x,y)[M^{-1}(Tx,Ty,c_0)-1]\leq M^{-1}(x,y,c_0)-1,$$

for all $x,y\in X$ with $\Delta(x,y)=-1$. □

Now we are ready to prove our first theorem.

Theorem 3.5. *Let $(X,M,*)$ be a complete c_0-triangular fuzzy metric space with c_0 regular and $T:X\to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *T is a triangular α-admissible mapping,*
(ii) *T is $\alpha\Delta$-Meir-Keeler contractive,*
(iii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$,*
(iv) *T is a c_0-continuous mapping,*
(v) *if $\alpha(x,y)\geq 1$ for $x,y\in X$, then, $\Delta(x,y)=-1$.*

Then T has a fixed point $z\in X$ such that $\Delta(z,z)=-1$. Further, if $\alpha(x,y)\geq 1$ for all $x,y\in Fix(T)$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Suppose $\{x_n\}$ is a Picard sequence starting from x_0, i.e., $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a triangular α-admissible mapping then by applying Lemma 1.6 we have,

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n,$$

and then by (v) we have,

$$\Delta(x_m, x_n) = -1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then evidently T has a fixed point. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Therefore by using Remark 3.3 we have,

$$M^{-1}(x_n, x_{n+1}, c_0) - 1 < M^{-1}(x_{n-1}, x_n, c_0) - 1 < \cdots < M^{-1}(x_0, x_1, c_0) - 1.$$

This shows that, the sequence $\left\{c_n := M^{-1}(x_n, x_{n+1}, c_0) - 1\right\}$ is non-increasing. So, the sequence $\{c_n\}$ is convergent to $c \in \mathbb{R}_+$. We will show that $c = 0$. Suppose, to the contrary, that $c > 0$. Hence, we have

$$0 < c < M^{-1}(x_n, x_{n+1}, c_0) - 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Let $\epsilon = c > 0$. Then by hypothesis, there exists $\delta(\varepsilon) > 0$ such that (3.1) holds. On the other hand, by the definition of ϵ, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \epsilon &< c_{n_0} \\ &= M^{-1}(x_{n_0}, x_{n_0+1}, c_0) - 1 \\ &< \epsilon + \delta. \end{aligned}$$

Now by (3.1), we have

$$\begin{aligned} c_{n_0+1} &= M^{-1}(x_{n_0+1}, x_{n_0+2}, c_0) - 1 \\ &\leq \alpha(x_{n_0}, x_{n_0+1})[M^{-1}(x_{n_0+1}, x_{n_0+2}, c_0) - 1] \\ &= \alpha(x_{n_0}, x_{n_0+1})[M^{-1}(Tx_{n_0}, Tx_{n_0+1}, c_0) - 1] \\ &< \epsilon. \end{aligned}$$

That is,

$$c_{n_0+1} < \epsilon,$$

which is a contradiction. Hence $c = 0$. That is,

$$\lim_{n \to \infty} [M^{-1}(x_n, x_{n+1}, c_0) - 1] = 0.$$

For given $\epsilon > 0$, by the hypothesis, there exists a $\delta = \delta(\epsilon) > 0$ such that (3.1) holds. Without loss of generality, we assume that $\delta < \epsilon$. Since $c = 0$ then there exists $N_0 \in \mathbb{N}$ such that

$$c_{n-1} = M^{-1}(x_{n-1}, x_n, c_0) - 1 < \delta, \quad \text{for all } n \geq N_0. \tag{3.2}$$

We will prove that for any fixed $k \geq N_0$,

$$M^{-1}(x_k, x_{k+l}, c_0) - 1 \leq \epsilon, \quad \text{for all } l \in \mathbb{N}, \tag{3.3}$$

Note that (3.3), by (3.2), holds for $l = 1$. Suppose the condition (3.3) is satisfied for some $m \in \mathbb{N}$. That is,

$$M^{-1}(x_k, x_{k+m}, c_0) - 1 \leq \epsilon, \quad \text{for some } m \in \mathbb{N}. \tag{3.4}$$

For $l = m + 1$, by (3.2) and (3.4), we get

$$\begin{aligned} M^{-1}(x_{k-1}, x_{k+m}, c_0) - 1 &= M^{-1}(x_{k-1}, x_{k+m}, c_0) - 1 \\ &\leq M^{-1}(x_{k-1}, x_k, c_0) - 1 \\ &\quad + M^{-1}(x_k, x_{k+m}, c_0) - 1 \\ &< \epsilon + \delta. \end{aligned} \tag{3.5}$$

Now if

$$M^{-1}(x_{k-1}, x_{k+m}, c_0) + \Delta(x_{k-1}, x_{k+m}) = M^{-1}(x_{k-1}, x_{k+m}, c_0) - 1 \geq \epsilon,$$

then by (3.1) and (3.5) we get,

$$\begin{aligned} M^{-1}(x_k, x_{k+m+1}, c_0) - 1 &= M^{-1}(Tx_{k-1}, Tx_{k+m}, c_0) - 1 \\ &\leq \alpha(x_{k-1}, x_{k+m})[M^{-1}(Tx_{k-1}, Tx_{k+m}, c_0) - 1] \\ &= \alpha(x_{k-1}, x_{k+m}) \\ &\quad \times \left[M^{-1}(Tx_{k-1}, Tx_{k+m}, c_0) + \Delta(Tx_{k-1}, Tx_{k+m})\right] \\ &< \epsilon, \end{aligned}$$

which implies

$$M^{-1}(x_k, x_{k+m+1}, c_0) - 1 < \epsilon,$$

and hence (3.3) holds.

If $M^{-1}(x_{k-1}, x_{k+m}, c_0) - 1 < \epsilon$, then by applying Remark 3.4 we have,

$$\begin{aligned} M^{-1}(x_k, x_{k+m+1}, c_0) - 1 &= M^{-1}(Tx_{k-1}, Tx_{k+m}, c_0) - 1 \\ &\leq M^{-1}(x_{k-1}, x_{k+m}, c_0) - 1 \\ &< \epsilon. \end{aligned}$$

Consequently, (3.3) holds for $l = m + 1$. Hence,

$$M^{-1}(x_k, x_{k+l}, c_0) - 1 \leq \epsilon, \quad \text{for all } l \in \mathbb{N}.$$

Thus, we have proved that $\{x_n\}$ is a c_0-Cauchy sequence. The completeness of X ensures that there exists $x^* \in X$ such that $M(x_n, x^*, c_0) \to 1$ as $n \to +\infty$. Now, since T is a c_0-continuous mapping then

$$M(x_{n+1}, Tx^*, c_0) = M(Tx_n, Tx^*, c_0) \to 1 \text{ as } n \to +\infty.$$

Equivalently we can write, $M^{-1}(x_{n+1}, Tx^*, c_0) \to 1$ as $n \to +\infty$. From

$$M^{-1}(x^*, Tx^*, c_0) - 1 \leq M^{-1}(x^*, x_{n+1}, c_0) - 1 + M^{-1}(x_{n+1}, Tx^*, c_0) - 1,$$

taking limit as $n \to +\infty$, we get

$$M^{-1}(x^*, Tx^*, c_0) = 1 \quad \text{or} \quad M(x^*, Tx^*, c_0) = 1,$$

and hence $x^* = Tx^*$, because M is c_0-regular. Thus, T has a fixed point. Let $\alpha(x, y) \geq 1$ for $x, y \in Fix(T)$. So, $\Delta(x, y) = -1$.

Now if $x \neq y$, then from Remark 3.3 we have,

$$\begin{aligned} M^{-1}(x, y, c_0) - 1 &= M^{-1}(Tx, Ty, c_0) - 1 \\ &< M^{-1}(x, y, c_0) - 1, \end{aligned}$$

which is a contradiction. So, $x = y$. That is, T has a unique fixed point when $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

Also, since Δ is c_0-continuous and

$$\Delta(x_n, x_m) = -1, \quad \Delta(x^*, x^*) = -1.$$

□

For a self-mapping which is not continuous we have the following result.

Theorem 3.6. *Let $(X, M, *)$ be a complete c_0-triangular fuzzy metric space with c_0 regular let $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *T is a triangular α-admissible mapping,*
(ii) *T is $\alpha\Delta$-Meir-Keeler contractive,*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iv) *if $\{x_n\}$ is a sequence in X where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$,*
(v) *if $\alpha(x, y) \geq 1$ for $x, y \in X$, then $\Delta(x, y) = -1$.*

Then T has a unique fixed point $z \in X$ such that $\Delta(z, z) = -1$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. As in the proof of Theorem 3.5, we deduce that a Picard sequence $\{x_n\}$ starting from x_0 is c_0-Cauchy and so converges to a point $x^* \in X$ and

$$\alpha(x_n, x_m) \geq 1, \qquad \Delta(x_n, x_m) = -1, \quad \text{for all } n < m.$$

By (v), $\alpha(x_n, z) \geq 1$. Then, $\Delta(x_n, z) = -1$. Therefore, by Remark 3.4, we have

$$\begin{aligned} M^{-1}(x_{n+1}, Tx^*, c_0) - 1 &= M^{-1}(Tx_n, Tx^*, c_0) - 1 \\ &\leq M^{-1}(x_n, x^*, c_0) - 1, \end{aligned}$$

for all $n \geq 0$. Then

$$\lim_{n\to+\infty} [M^{-1}(x_{n+1}, Tx^*, c_0) - 1] = 0.$$

That is,

$$\lim_{n\to+\infty} M^{-1}(x_{n+1}, Tx^*, c_0) = 1,$$

and hence

$$\begin{aligned} M^{-1}(x^*, Tx^*, c_0) - 1 &\leq \lim_{n\to+\infty} [M^{-1}(x^*, x_{n+1}, c_0) - 1 \\ &\quad + M^{-1}(x_{n+1}, Tx^*, c_0) - 1] \\ &= 0. \end{aligned}$$

Thus, we get $x^* = Tx^*$. The other statements follow as in the proof of Theorem 3.5. $\square$

Example 3.7. Let $c_0 > 0$ and $X = \mathbb{R}$ be endowed with the c_0-complete c_0-triangular fuzzy metric

$$M(x, y, c_0) = \begin{cases} \frac{c_0}{c_0+|x|+|y|}, & \text{if } x \neq y, \\ 1, & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Define $T : X \to X$, $\alpha : X \times X \to [0, +\infty)$ and $\Delta : X \times X \to [-1, \infty)$ by

$$Tx = \begin{cases} x^2, & \text{if } x \in (-\infty, 0), \\ \frac{1}{8}x^2, & \text{if } x \in [0, 1], \\ 3x, & \text{if } x \in (1, 2), \\ x^{104}, & \text{if } x \in [2, +\infty), \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\Delta(x,y)=\begin{cases}-1, & \text{if } x,y\in[0,1],\\ 1, & \text{otherwise.}\end{cases}$$

Let $\alpha(x,y)\geq 1$, then $x,y\in[0,1]$. On the other hand, $Tw\in[0,1]$ for all $w\in[0,1]$. Thus $\alpha(Tx,Ty)\geq 1$. Also, let $\alpha(x,y)\geq 1$ and $\alpha(y,z)\geq 1$. So, $x,y,z\in[0,1]$. i.e., $\alpha(x,z)\geq 1$. Then T is a triangular α-admissible mapping. Further, if $\alpha(x,y)\geq 1$ then $x,y\in[0,1]$. Therefore, $\Delta(x,y)=-1$.

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ with $x_n\to x$ as $n\to+\infty$, then $x_n\in[0,1]$ for all $n\in\mathbb{N}$ and so $x\in[0,1]$. This ensures that $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$. Clearly, $\alpha(0,T0)\geq 1$.

Let $\alpha(x,y)=1$, $\epsilon\leq M^{-1}(x,y,c_0)+\Delta(x,y)<\epsilon+\delta$ where $\epsilon,\delta>0$. Then we have $x,y\in[0,1]$ and

$$\frac{1}{c_0}(|x|+|y|)<\epsilon+\delta.$$

Hence we have,

$$\begin{aligned}M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)&=\frac{1}{c_0}\left(|Tx|+|Ty|\right)\\&=\frac{1}{8c_0}\left(|x|^2+|y|^2\right)\\&\leq\frac{1}{8c_0}\left(|x|+|y|\right)\\&<\frac{1}{8}(\epsilon+\delta).\end{aligned}$$

Otherwise, $\alpha(x,y)=0$ and then evidently,

$$\begin{aligned}\alpha(x,y)[M^{-1}(Tx,Ty)+\Delta(Tx,Ty)]&=0\\&<\frac{1}{8}(\epsilon+\delta).\end{aligned}$$

Now, by taking $\delta\leq 7\epsilon$ (for all $\epsilon>0$) the condition (3.1) holds. That is, T is an $\alpha\Delta$-Meir-Keeler contractive mapping. Thus all conditions of Theorem 3.6 hold and T has a fixed point (here $x=0$ is a fixed point of T).

If in Theorem 3.6 we take $\Delta(x,y)=-1$ for all $x,y\in X$, then we have the following Corollary.

Corollary 3.8. *Let $(X,M,*)$ be a c_0-complete c_0-triangular fuzzy metric space with ω regular and $T:X\to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *T is a triangular α-admissible mapping,*

(ii) *for each* $\epsilon > 0$ *and* $x,\ y \in X$ *there exists* $\delta > 0$ *such that*

$$\epsilon \leq M^{-1}(x, y, c_0) - 1 < \epsilon + \delta(\epsilon),$$

then

$$\alpha(x, y)[M^{-1}(Tx, Ty, c_0) - 1] < \epsilon,$$

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

(iv) *if* $\{x_n\}$ *is a sequence in* X *whence* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *with* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$.

Then T *has a fixed point. Further, if* $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T)$, *then* T *has a unique fixed point.*

If in the above Corollary we take $\alpha(x, y) = 1$ for all $x, y \in X$, then we have the following Corollary.

Corollary 3.9. *Let* $(X, M, *)$ *be a* c_0*-complete* c_0*-triangular fuzzy metric space with* c_0 *regular and* $T : X \to X$ *be a self-mapping. Assume that for each* $\epsilon > 0$ *and* $x,\ y \in X$ *there exists* $\delta > 0$ *such that*

$$\epsilon \leq M^{-1}(x, y, c_0) - 1 < \epsilon + \delta(\epsilon)$$

then

$$M^{-1}(Tx, Ty, c_0) - 1 < \epsilon.$$

Then T *has a unique fixed point.*

4. Some Meir-Keeler Type Fixed Point Results in c_0-triangular Fuzzy Metric Space Endowed With a Partial Order

The existence of fixed points in partially ordered metric spaces has been considered in [10]. Let X is a nonempty set. If X be a c_0-triangular fuzzy metric space and $(X, \preceq)$ is a partially ordered set, then X is called a partially ordered c_0-triangular fuzzy metric space. Two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$. A mapping $T : X \to X$ is said to be non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

In this section, we will show that many Meir-Keeler type fixed point results in c_0-triangular fuzzy metric spaces endowed with a partial order $\preceq$ can be deduced easily from our presented theorems.

Definition 4.1. Let $(X, \preceq)$ be a partially ordered c_0-triangular fuzzy metric space and T be a self-mapping on X. We say that T is partially Δ-Meir-Keeler contractive if for each $\epsilon > 0$ and $x,\ y \in X$ with $x \preceq y$ there exists $\delta > 0$ such that

$$\epsilon \leq M^{-1}(x, y, c_0) + \Delta(x, y) < \epsilon + \delta,$$

then

$$M^{-1}(T(x), T(y), c_0) + \Delta(Tx, Ty) < \epsilon.$$

Theorem 4.2. *Let $(X, M, * \preceq)$ be a c_0-complete partially ordered c_0-triangular fuzzy metric space with c_0 regular and $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *T is c_0-continuous,*
(iii) *T is an increasing mapping,*
(iv) *T is a partially Δ-Meir-Keeler contractive mapping,*
(v) *if $x \preceq y$ then $\Delta(x, y) = -1$.*

Then T has a fixed point $z \in X$ such that $\Delta(z, z) = 0$. Further, if $x \preceq y$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

At first we show that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geq 1$. Then $x \preceq y$. By (iii), we get $Tx \preceq Ty$. That is, $\alpha(Tx, Ty) \geq 1$. Also, let $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. So, $x \preceq y$ and $y \preceq z$. Then $x \preceq z$. i.e. $\alpha(x, z) \geq 1$. So T is a triangular α-admissible mapping. By (i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. That is, $\alpha(x_0, Tx_0) \geq 1$. Further if $\alpha(x, y) \geq 1$, then $x \preceq y$. Hence by (v) we get $\Delta(x, y) = -1$. Assume that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\alpha(x, y) \geq 1$ and

$$\epsilon \leq M^{-1}(x, y, c_0) + \Delta(x, y) < \epsilon + \delta(\epsilon).$$

This implies $x \preceq y$ and

$$\epsilon \leq M^{-1}(x, y, c_0) + \Delta(x, y) < \epsilon + \delta(\epsilon).$$

Now, since T is a partially Δ-Meir-Keeler contractive mapping,

$$M^{-1}(T(x), T(y), c_0) + \Delta(Tx, Ty) < \epsilon.$$

That is, T is an $\alpha\Delta$-Meir-Keeler contractive mapping. Therefore conditions (i)-(v) of Theorem 3.5 hold and T has a fixed point. Also, if $x \preceq y$ for all $x, y \in Fix(T)$ then, $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$. Hence in this case T has a unique fixed point. □

Theorem 4.3. *Let $(X, M, * \preceq)$ be a c_0-complete partially ordered c_0-triangular fuzzy metric space with ω regular and $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *T is an increasing mapping,*
(iii) *T is a partially Δ-Meir-Keeler contractive mapping,*
(iv) *if $x \preceq y$ then $\Delta(x, y) = -1$,*

(v) *if* $\{x_n\}$ *be an increasing sequence in* X *with* $x_n \to x$ *as* $n \to \infty$, *then,* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point* $z \in X$ *such that* $\Delta(z,z) = -1$. *Further, if* $x \preceq y$ *for all* $x, y \in Fix(T)$, *then* T *has a unique fixed point.*

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty)$ as in the proof of Theorem 5.3. Similar to the proof of Theorem 5.3, we can prove that the conditions (i)-(iii) and (v) of Theorem 3.6 are satisfied. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By (v) we get $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, conditions (i)-(v) of Theorem 3.6 hold and T has a fixed point. Also, if $x \preceq y$ for all $x, y \in Fix(T)$ then, $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$. Hence in this case T has a unique fixed point. □

5. Some Integral Type Contractions

Let Φ denotes the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

- every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,
- for any $\phi \in \Phi$ and any $\epsilon > 0$,
$$\int_0^\epsilon \phi(\tau)d\tau > 0.$$

Following arguments similar to those in Theorem 3.5 and 3.6, we can prove the following Theorems.

Theorem 5.1. *Let* $(X, M, *, \preceq)$ *be a* c_0*-complete* c_0*-triangular fuzzy metric space with* c_0 *regular and* $T : X \to X$ *be a self-mapping. Assume that the following assertions hold:*

(i) T *is a triangular* α*-admissible mapping,*

(ii) *for each* $\epsilon > 0$ *and* $x,\ y \in X$ *there exists* $\delta > 0$ *such that*
$$\int_0^{M^{-1}(x,y,c_0)+\Delta(x,y)} \phi(\tau)d\tau < \epsilon + \delta,$$
implies
$$\alpha(x, y) \int_0^{M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)} \phi(\tau)d\tau < \epsilon,$$

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,

(iv) T *is a* c_0*-continuous mapping,*

(v) *if* $x \preceq y$ *then* $\Delta(x, y) = -1$.

Then T *has a fixed point* $z \in X$ *such that* $\Delta(z, z) = -1$. *Further, if* $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T)$, *then* T *has a unique fixed point.*

Theorem 5.2. *Let $(X, M, *, \preceq)$ be a c_0-complete c_0-triangular fuzzy metric space with c_0 regular and $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *T is a triangular α-admissible mapping,*

(ii) *for each $\epsilon > 0$ and $x,\ y \in X$ there exists $\delta(\epsilon) > 0$ such that*

$$\int_0^{M^{-1}(x,y,c_0)+\Delta(x,y)} \phi(\tau)d\tau < \epsilon + \delta(\epsilon)$$

implies

$$\alpha(x, y) \int_0^{M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)} \phi(\tau)d\tau < \epsilon,$$

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iv) *if $\{x_n\}$ is a sequence in X whence $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point $z \in X$ such that $\Delta(z, z) = -1$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Theorem 5.3. *Let $(X, M, *, \preceq)$ be a c_0-complete partially ordered c_0-triangular fuzzy metric space with c_0 regular and $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*

(ii) *T is c_0-continuous,*

(iii) *T is an increasing mapping,*

(iv) *for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$\int_0^{M^{-1}(x,y,c_0)+\Delta(x,y)} \phi(\tau)d\tau < \epsilon + \delta(\epsilon),$$

implies

$$\int_0^{M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)} \phi(\tau)d\tau < \epsilon,$$

for any $x,\ y \in X$ with $x \preceq y$.

Then T has a fixed point $z \in X$ such that $\Delta(z, z) = -1$. Further, if $x \preceq y$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Theorem 5.4. *Let $(X, M, *, \preceq)$ be a c_0-complete partially ordered c_0-triangular fuzzy metric space with c_0 regular and $T : X \to X$ be a self-mapping. Assume that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*

(ii) *T is c_0-continuous,*

(iii) *T is an increasing mapping,*

(iv) *for each* $\epsilon > 0$ *and* $x,\ y \in X$ *with* $x \preceq y$ *there exists* $\delta > 0$ *such that*

$$\int_0^{M^{-1}(x,y,c_0)+\Delta(x,y)} \phi(\tau)d\tau < \epsilon + \delta(\epsilon),$$

implies

$$\int_0^{M^{-1}(Tx,Ty,c_0)+\Delta(Tx,Ty)} \phi(\tau)d\tau < \epsilon,$$

(v) *if* $\{x_n\}$ *be an increasing sequence in* X *with* $x_n \to x$ *as* $n \to \infty$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$,

(vi) *if* $x \preceq y$ *then* $\Delta(x,y) = -1$.

Then T *has a fixed point* $z \in X$ *such that* $\Delta(z,z) = -1$. *Further, if* $x \preceq y$ *for all* $x, y \in Fix(T)$, *then* T *has a unique fixed point.*

Acknowledgment. The author wish to thank Editor-in-Chief and Referee(s) for their valuable comments and suggestions, which were very useful to improve the paper.

References

1. C. Di Bari and C. Vetro, *A fixed point theorem for a family of mappings in a fuzzy metric space*, Rend. Circ. Mat. Palermo, 52 (2003), pp. 315-321.
2. C. Di Bari and C. Vetro, *Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space*, J. Fuzzy Math., 13 (2005), pp. 973-982.
3. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), pp. 395-399.
4. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27 (1988), pp. 385-389.
5. J. Jachymski, *Equivalent condition and the Meir-Keeler type theorems* , J. Math. Anal. Appl., 194 (1995), pp. 293-303.
6. E. Karapinar, P. Kumam, and P. Salimi, *On* $\alpha-\psi$*-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., 1 (2013), pp. 1-12.
7. I. Kramosil and J. Michlek, *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), pp. 336-344.
8. A. Meir and E. Keeler, *A theorem on contraction mapping*, J. Math. Anal. Appl., 28 (1969), pp. 326-329.
9. S. Park and B.E. Rhoades, *Meir-Keeler type contractive condition*, Math. Japon., 26 (1981), pp. 13-20.
10. A.C.M. Ran and M.C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132 (2004), pp. 1435-1443.
11. B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for* $\alpha-\psi$*-contractive type mappings*, Nonlinear Anal., 75 (2012), pp. 2154-2165.
12. Y.Shen, D.Qiu, and W.Chenc, *Fixed point theorems in fuzzy metric spaces*, Appl. Math. Letters, 25 (2012), pp. 138-141.
13. L.A. Zadeh, *Fuzzy sets*, Information & Control, 8 (1965), pp. 338-353.

Department of Mathematics, Payame Noor University, p.o.box.19395-3697, Tehran, Iran.

E-mail address: masoomeh.hezarjaribi@pnu.ac.ir