On Polar Cones and Differentiability in Reflexive Banach Spaces

Ildar Sadeqi$^{1*}$ and Sima Hassankhali$^2$

Abstract. Let $X$ be a Banach space, $C \subseteq X$ be a closed convex set included in a well-based cone $K$, and also let $\sigma_C$ be the support function which is defined on $C$. In this note, we first study the existence of a bounded base for the cone $K$, then using the obtained results, we find some geometric conditions for the set $C$, so that $\text{int}(\text{dom}\sigma_C) \neq \emptyset$. The latter is a primary condition for subdifferentiability of the support function $\sigma_C$. Eventually, we study Gateaux differentiability of support function $\sigma_C$ on two sets, the polar cone of $K$ and $\text{int}(\text{dom}\sigma_C)$.

1. Introduction

1.1. A short survey on convex cones. The study of convex cones and the geometric structure of their bases in a Banach space has many applications in the theory of optimization, economics and engineering which motivate us to study the subject. We recall $[3, 6, 5, 11]$ and the long list of their references for more details.

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space (unless it is specified) whose dual $X^*$ is endowed with the dual norm, denoted also by $\|\cdot\|$. As usual, having a nonempty subset $C$ of $X$, define:

\begin{align*}
C^+ &:= \{ x^* \in X^* : x^*(x) \geq 0, \forall x \in C \}, \\
C^- &:= -C^+, \\
C^\perp &:= C^+ \cap C^-.
\end{align*}

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* Corresponding author.
Let $K$ be a cone in a Banach space $X$. We say that $K$ is pointed if $K \cap -K = \{0\}$ and it is solid if the interior of $K$, say $\text{int} K$, is nonempty. A convex subset $B$ of $X$ is called a base for $K$ if $0$ is not included in the closure of $B$, say $\text{cl} B$, and $\text{cone} B := \{ tx : x \in B, t \geq 0 \} = K$. Also, a proper convex cone $K$ is called well-based if $K$ has a bounded base. It is known that a cone with a base is necessarily convex and pointed. Also, if $X$ is a real separable normed space, every nontrivial closed pointed convex cone has a base [S, Corollary 3.39]. Note that the separability assumption is essential and cannot be dropped. Krein-Rutman [S], gave an interesting example which shows that the assertion fails in a nonseparable space.

Let $K$ be a cone. The polar of $K$ is defined by

$$K^\# := \{ x^* \in X^* : x^*(x) > 0, \forall x \in K \setminus \{0\} \}.$$

There is a direct connection between existence of a base for a cone $K$ and the structure of $K^\#$. In fact, a convex cone $K$ has a base if and only if $K^\# \neq \emptyset$. Indeed, let $K^\# \neq \emptyset$. For every $x^* \in K^\#$, the set $B_{x^*} := \{ x \in K : x^*(x) = 1 \}$ defines a base on the cone $K$. Conversely, from the Hahn-Banach theorem, we could separate $B$ from $0$ by $x^* \in K^\#$.

Note that the polar cone $K^\#$ could be empty. For example, we consider the space $B([a, b])$ of all functions on the real interval $[a, b]$ endowed with the usual " sup " norm and the standard positive cone:

$$K = \{ f \in B[a, b] : f(t) \geq 0, \forall t \in [a, b] \},$$

then $K^\#$ is empty. See [S] and references therein for more details.

**Remark 1.1.** If a closed convex cone $K$ is pointed, then $\text{int} K^+ = \text{int} K^\#$ and we have the followings:

(a) The polar cone $K^\#$ need not necessarily be the interior of $K^+$. For example, if $K$ is the nonnegative orthant of the sequence space $l_p$, $1 < p < \infty$, then $\text{int} K^+ = \emptyset$ but $K^\#$ is nonempty.

(b) The interior of $K^+$ could be nonempty. Let $\alpha \in (0, 1)$ and $a^\alpha = (\alpha, \alpha^2, \ldots) \in l_2$ with

$$||a^\alpha||^2 = \frac{\alpha^2}{(1 - \alpha^2)}.$$

For any $0 < \varepsilon < (1 - \alpha^2)^{\frac{1}{2}}$, set

$$K := \{ z \in l_2 : a^\alpha z \geq \varepsilon ||a^\alpha|| \cdot ||z|| \}.$$

Then $\text{int} K^+ \neq \emptyset$.

Part (a) in Remark 1.1 is equivalent to the existence of a bounded base for the cone $K$ as stated in the following theorem.
**Theorem 1.2** ([3] Theorem 2.2). Let $K$ be a nontrivial closed convex cone in $X$. Then $K$ is well-based if and only if $K^\#$ is solid.

In fact, $B_{x^*}$ is a bounded base for the closed convex cone $K$ if and only if $x^* \in \text{int}K^\#$.

**Definition 1.3.** Let $X$ be a Banach space, $X^*$ be its dual and let $K$ be a cone in $X$ (see [6] and references therein).

1. $K$ is said to be acute if there is an open half space $L_{x^*} = \{ x \in X : x^*(x) > 0 \}$, with $x^* \in X^*$, $x^* \neq 0$, such that $dK \subset L_{x^*} \cup \{0\}$.

2. For $x^* \in K^-$ and $\delta > 0$, set $v(x^*, \delta) := \{ x \in K : x^*(x) \geq \delta \}$. Recall that the cone $K$ satisfies property $(\pi)$ (weak property $(\pi)$), if there exists $x^* \in K^-$ such that for all $\delta > 0$ the set $v(x^*, \delta)$ is relatively weakly compact (bounded).

3. A closed convex cone $K$ satisfies angle property if there exist $x^* \in X^* \setminus \{0\}$ and $0 < \varepsilon \leq 1$ such that $K \subset \{ x \in K : x^*(x) \leq \varepsilon ||x^*|| ||x|| \}$.

4. A closed convex cone $K$ is said to be a locally weakly compact cone, if for every bounded set $A$ in $K$, $A$ is relatively weakly compact.

A cone satisfying the (weak) property $(\pi)$ is pointed and one can replace: ‘for all $\delta > 0$’ by ‘there exists $\delta > 0$’. Cesari and Suryanarrayana showed that there exist infinite dimensional spaces including cones, satisfying angle property. It is of interest to know that in a Banach space $X$, a closed convex cone with angle property is acute and hence pointed. Also, a closed convex cone $K$ with property $(\pi)$ is acute (and hence pointed). Furthermore, when $X$ is a reflexive Banach space, angle property implies property $(\pi)$. However, Cesari and Suryanarrayana showed that in the Hilbert space $l_2$, we can find acute cones which neither have property $(\pi)$ nor the angle property. See [6, 7, 11] and references therein, for more details.

**Theorem 1.4.** Let $X$ be a Banach space and $K$ be a closed convex cone. Then,

- $K$ has angle property $\Rightarrow$ $K$ is acute $\Rightarrow$ $K$ is pointed.

Also, when $X$ is reflexive

- $K$ satisfies property $(\pi)$ $\Rightarrow$ $K$ is acute $\Rightarrow$ $K$ is pointed.

In 1978, Cesari and Suryanarrayana illustrated an example to show that acuteness (hence either angle property or property $(\pi)$) is not
satisfied for a half-space. In 1994, Han investigated the relations between cones satisfying angle property and solid cones. The investigation showed that the two classes of cones are dual in some senses. Also, they found a relation between solid cones, acute cones, cones satisfying (weak) property (π) and cones having bounded bases. See, [6, 5] and references therein.

**Theorem 1.5.** Let $K$ be a nontrivial closed convex cone in a Banach space $X$. Then:

- $(c_1)$ $K$ has angle property if and only if $K$ has a closed bounded base if and only if $K$ satisfies the weak property ($\pi$).
- $(c_2)$ $K^-$ ($K$) is solid if and only if $K$ ($K^-$) is well-based.
- $(c_3)$ $K$ is acute if and only if $K$ has a closed convex base.
- $(c_4)$ $K$ has property ($\pi$) if and only if $K$ is relatively weakly compact with weak property ($\pi$).

From $(b_4)$ of Definition 1.3, every convex cone $K$ in a reflexive Banach space is relatively weakly compact. This remark together with $(c_1)$ and $(c_4)$ of Theorem 1.5, imply that in reflexive Banach spaces, a closed convex cone $K$ has angle property if and only if it has property ($\pi$). Hence, in reflexive Banach spaces, $K^-$ is solid if and only if $K$ satisfies angle property [6, Theorem 1.2].

### 1.2. Convex functions.

Let $U \subset X$ be an open subset of a Banach space $X$ and $f : U \to \mathbb{R}$ be a real valued function. We say that $f$ is Gateaux differentiable at $x \in U$, if for every $h \in X$, $f'(x)(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$, exists in $\mathbb{R}$ and is a linear continuous function in $h$ (i.e $f'(x) \in X^*$).

The functional $f'(x)$ is then called the Gateaux derivative or Gateaux differential of $f$ at $x$.

Recall that $\text{dom}f$ of a function $f : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is the set $\{x \in X : f(x) < \infty\}$ and $f$ is proper if $\text{dom}f \neq \emptyset$ and $f(x) \neq -\infty$ for each $x \in X$. The subdifferential of a proper function $f$ at $x \in \text{dom}f$ ($f(x) \neq -\infty$) is defined by

$$\partial f(x) := \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x), \forall y \in X\},$$

and $\partial f(x) = \emptyset$ for $x \in X \setminus \text{dom}f$. Of course, the domain of $\partial f$ is $\text{dom}\partial f = \{x \in X : \partial f(x) \neq \emptyset\} \setminus \text{dom}f$. 
Let $C$ be a nonempty subset of the Banach space $X$. The support function of $C$ is an extended real valued function on $X^*$ defined by

$$\sigma_C : X^* \to \bar{\mathbb{R}}, \quad \sigma_C(x^*) := \sup_{C^*} x^*.$$  

It is well-known that for a nonempty subset $C$ of the Banach space $X$, we have $\sigma_C = \sigma_{\text{conv} C} = \sigma_{\text{cl} C} = \sigma_{\text{cl}(\text{conv} C)}$, where $\text{conv} C$ is the convex hull of $C$. So we could assume that $C$ is a nonempty, closed and convex set (unless otherwise is specified). Moreover, when the Banach space $X$ is reflexive, we have

$$(1.1) \quad \partial \sigma_C(x^*) = \{ u \in C : x^*(u) = \sigma_C(x^*) \},$$

and $\partial \sigma_C(0) = C \quad (\square, \square)$.

Rest of the paper is organized as follows. In Section 2, we find some results related to the solidness of polar cones, specially the polar of recession cone of a closed convex set $C$, and we apply the nonemptiness of $\text{int}(\text{dom} \sigma_C)$. In Section 3, assuming that $C$ is a subset of a closed well-based convex cone $K$, we study Gateaux differentiability of $\sigma_C$ on both $K^\#$ and int(dom$\sigma_C$).

## 2. CONDITIONS IN WHICH int(dom$\sigma_C) \neq \emptyset$

In Theorem 1.2, it is shown that a nontrivial closed convex cone is well-based if and only if its polar is solid. Here, we show that in reflexive Banach spaces, solidness of the polar cone is equivalent to the existence of a weakly compact base for the cone.

**Theorem 2.1.** Let $X$ be a reflexive Banach space and $K$ be a closed convex cone. Then $K^\#$ is solid if and only if $K$ has a weakly compact base $B_{x^*}$, for some $x^* \in K^\#$.

**Proof.** Let $\text{int} K^\# \neq \emptyset$. By Theorem 1.2, for each $x^* \in \text{int} K^\#$, the set $B_{x^*}$ is a bounded base for the cone $K$ and by [10, Theorem 4], $K^\# = \text{int} K^\#$. We show that for each $x^* \in K^\#$, the base $B_{x^*}$ is weakly compact. By the contrary, let $x^*_0 \in K^\#$ where $B_{x^*_0}$ is not weakly compact. By James theorem [14, Theorem 3.130], there exists $y^* \in X^*$ such that

$$y^*(x) > \inf\{y^*(b) : b \in B_{x^*_0}\}, \quad \forall x \in B_{x^*_0}.$$  

Let $l := \inf\{y^*(b) : b \in B_{x^*_0}\}$. When $l > 0$, define $x^*_1 := \frac{y^*}{l} - x_0$ (for $l = 0$, define $x^*_1 := y^*$ and for $l < 0$, define $x^*_1 := x_0 - \frac{y^*}{l}$). So, we have $x^*_1 \in K^\# \backslash \text{int} K^\#$. Indeed, if $x^*_1 \in \text{int} K^\#$, there exists $r > 0$ such that $x^*_1 + r B_{X^*} \subset K^\#$ ($B_{X^*}$ is the unit ball of $X^*$). Therefore, for each $k \in K$, we get

$$x^*_1(k) = \frac{y^*(k)}{l} - x^*_0(k) \geq r\|k\|.$$
From definition of $l$, there exists $(b_n) \subset B_{x_0^*}$ so that $y^*(b_n) \to l$. So, for each $n \in \mathbb{N}$, $r||b_n|| \geq 0$ and $b_n \to 0$. But this makes a contradiction since $0 \notin B_{x_0^*}$. Therefore, $x_1^* \in K^* \cap \text{int}K^*$ which contradicts $K^* = \text{int}K^*$. So, for each $x^* \in K^*$, the base $B_{x^*}$ is weakly compact. 

Recall that recession cone of a subset $C$ of a Banach space $X$ is defined by

$$C_\infty := \{w \in X : x + tw \in C, \forall t \geq 0, \forall x \in C\}.$$

It is clear that if $C$ is closed, then $C_\infty$ is closed, and if $C$ is closed and convex, we have $C_\infty = \bigcap_{t \geq 0} t(C - a)$, where $a \in C$. Note that in finite dimensional Banach spaces, the closed convex set $C$ is bounded if and only if $C_\infty = \{0\}$. But, the latter is not correct for infinite case in general. For example, let $X = l_2$ and $C = \{x \in l_2 : |x_i| \leq 1, \forall i \in \mathbb{N}\}$. It is clear that $X$ is a reflexive Banach space and $C$ is a closed convex unbounded subset of $X$. We show that $C_\infty = \{0\}$. It is obvious that $0 \in C_\infty$. Let $v \in C_\infty$ where $v \neq 0$. Then $c + tv \in C$ for each $t \geq 0$. Hence, for a sufficiently large $t$, we may have $|c_i + tv_i| > 1$ for some $i \in \mathbb{N}$ which is a contradiction with definition of $C$.

**Theorem 2.2.** Let $X$ be a reflexive Banach space and $K$ be a closed convex cone. Then the following statements are established.

$(h_1)$ $\text{int}(\text{dom} \sigma_K) \neq \emptyset$ if and only if $K^\#$ is solid.

$(h_2)$ $\text{int}(\text{dom} \sigma_C) \neq \emptyset$ where $C$ is a closed convex subset of $K$ with $C_\infty \neq \{0\}$.

**Proof.** For $(h_1)$, since $K_\infty = K$, the “if” part is a consequence of $(h_1)$. For the other part, let $K^\#$ be solid. Then for each $(x_n) \subset K$ with $||x_n|| \to \infty$ and $(||x_n||^{-1} x_n) \rightharpoonup u$, one has $u \neq 0$. Otherwise, let $u = 0$. Since the polar of $K$ is solid, each sequence of $K$ which weakly converges to zero is norm convergence. Hence, $||x_n||^{-1} x_n$ is norm convergence to zero, which is a contradiction. To prove $(h_2)$, it is sufficient to point out that when $C \subset K$, we have $\text{dom} \sigma_K \leq \text{dom} \sigma_C$ and $K^\# \subset C_\infty$. 

Let $C$ be a nonempty closed subset of the Banach space $X$ including a half-line. Define $P := cl(\text{conv} C)$ and the set-valued function $W_C : X^* \to X$ by

$$W_C(x^*) := \{u \in C : x^*(u) = \sigma_C(x^*)\} = C \cap \partial \sigma_C(x^*).$$

The following theorem is holds.

**Theorem 2.3.** Let $X$ be a reflexive Banach space and $C$ be a nonempty closed subset of $X$ which is included in a closed convex well-based cone. Then the following assertions hold:
\((i_1)\) \(\operatorname{int}(\operatorname{dom} \sigma_C) = \operatorname{int}P^-\) and \(\sigma_C\) is continuous on \(\operatorname{int}P^-\).

\((i_2)\) \(\partial \sigma_C(x^*)\) is nonempty and weakly compact for every \(x^* \in \operatorname{int}P^-\).

Also, \(\partial \sigma_C(x^*)\) is singleton if and only if \(\sigma_C\) is Gateaux differentiable at \(x^*\).

\((i_3)\) \(\operatorname{dom} W_C \subset \operatorname{dom} \sigma_C \subset P^-\). Also, \(W_C(x^*)\) is nonempty and \(w\)-compact for every \(x^* \in \operatorname{int}P^-\).

\((i_4)\) \(\operatorname{int}(\operatorname{dom} \sigma_C) = \operatorname{int}(\operatorname{dom} \sigma_C) \setminus \{0\} = \operatorname{int}(\operatorname{dom} \sigma_C) \setminus (\operatorname{lin}_0 C)^\perp\).

\textbf{Proof.} \((i_1)\) It is clear that

\[
\operatorname{cl}(\operatorname{dom} \sigma_C) = (\operatorname{cl}(\operatorname{conv} C))^- = P^-;
\]

(for convex sets, the weak and norm closure coincide). Theorem \([12]\) implies that \(\operatorname{int}(\operatorname{dom} \sigma_C) \neq \emptyset\). So, from \([5]\, \text{Lemma 12}\), convexity of \(\operatorname{dom} \sigma_C\) implies that

\[
\operatorname{int}(\operatorname{dom} \sigma_C) = \operatorname{int}(\operatorname{cl}(\operatorname{dom} \sigma_C)).
\]

Hence, \(\operatorname{int}(\operatorname{dom} \sigma_C) = \operatorname{int}P^-\). Also, from \([2, \text{Proposition 4.1.5}]\), the support function \(\sigma_C\) is continuous on \(x^* \in \operatorname{dom} \sigma_C\) if and only if \(x^* \in \operatorname{int}(\operatorname{dom} \sigma_C)\).

\((i_2)\) The subdifferential of a proper convex function is nonempty, convex and \(w\)-compact at any point of continuity from its domain \([11, \text{Theorem 7.13}]\). The second allegation comes from Smulyan lemma \([4, \text{Theorem 7.17}]\).

\((i_3)\) Fix \(u_0 \in C\) and \(x^* \in \operatorname{int}P^-\). Define

\[
C_0 := \{ u \in C : x^*(u) \geq x^*(u_0) \}.
\]

It is clear that \(C_0\) is nonempty, closed and \(\sigma_{C_0}(x^*) = \sigma_C(x^*)\).

We show that \(C_0\) is bounded. By the contrary, assume that there exists \((x_n) \subset C_0\) with \(\|x_n\| \to \infty\). We may assume that \(x_n||x_n^{-1}\) weakly converges to \(v\). Note that from \((g_1)\) of Theorem \([2, \text{Theorem 7.13}]\), we have \(v \neq 0\) and \(v \in P\setminus\{0\}\). Since \(x^*(x_n) \geq x^*(u_0)\) for every \(n \in \mathbb{N}\), we get the contradiction \(0 > x^*(v) \geq 0\). Hence, \(C_0\) is bounded, and so weakly compact. Now, by James theorem \([11, \text{Theorem 3.130}]\), \(x^*\) attains its supermum on \(C_0\) at \(v \in C_0\) and \(\sigma_C(x^*) = x^*(v)\). It follows that \(v \in W_C(x^*)\). So, \(W_C(x^*)\) is nonempty for each \(x^* \in \operatorname{int}P^-\). Moreover, the fact that the intersection of a closed set and a weakly compact set is weakly compact completes the proof.

\((i_4)\) This is a consequence of \((i_1)\) and definitions of \(\operatorname{int}P^-\) and \((\operatorname{lin}_0 C)^\perp\).

\hfill \square

\textbf{Remark 2.4.} Note that \(\partial \sigma_C(0) = C\) and \(\sigma_C\) is Gateaux differentiable on 0 if and only if \(C\) is singleton. Also, \(x^* \in X^*\) is constant on \(C\) if and only if \(x^*\) belongs to \((\operatorname{lin}_0 C)^\perp\). Hence, when \(C\) is not singleton, \(\sigma_C\) is
not Gateaux differentiable on \((\text{lin}_0 C)^\perp\). But, according to \((i_4)\), under our assumptions, \(\text{int}(\text{dom}\sigma_C) \cap (\text{lin}_0 C)^\perp = \emptyset\) and we can speak about differentiability of \(\sigma_C\) on \(\text{int}(\text{dom}\sigma_C)\).

**Corollary 2.5.** Let \(X\) be a reflexive Banach space and \(C\) be a closed convex set that \(C_\infty \neq \{0\}\). The followings are equivalent:

1. \(C_\infty^# \neq \emptyset\) and for every sequence \((x_n) \in C\) with \(\|x_n\| \to \infty\) and \(\|x_n\|^{-1} x_n \overset{w}{\to} u\), one has \(u \neq 0\).
2. \(\text{int}(\text{dom}\sigma_C) \neq \emptyset\).
3. \(C_\infty\) is well-based (\(C_\infty^#\) is solid).
4. There exists \(x^* \in K^#\) such that \(B_{x^*}\) is a weakly compact base for the cone \(K\).
5. \(C_\infty\) has property \((\pi)\) (weak-property \((\pi)\)).
6. \(C_\infty\) has angle property.

### 3. Differentiability of \(\sigma_C\)

In this section, let \(X\) be a reflexive Banach space and \(C\) be a nonempty closed subset of \(X\) which is included in a closed convex well-based cone \(K\) (unless otherwise is stated).

**Theorem 3.1.** The support function \(\sigma_C\) is Gateaux differentiable on \(\text{int}(\text{dom}\sigma_C)\) if and only if

\[
\forall x, y \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C)), x \neq y, \forall \lambda \in (0, 1) : \\
\lambda x + (1 - \lambda) y \notin \partial\sigma_C(\text{int}(\text{dom}\sigma_C)).
\]

**Proof.** Let (i) hold. Consider \(x^* \in \text{int}(\text{dom}\sigma_C)\) where \(\sigma_C\) is not Gateaux differentiable on \(x^*\). From \((i_2)\) of Theorem 2.3, \(\sigma_C\) is subdifferentiable on \(\text{int}(\text{dom}\sigma_C)\) and there exist \(x, y \in \partial\sigma_C\) with \(x \neq y\). Therefore, the convexity of \(\partial\sigma_C\) implies that \(\lambda x + (1 - \lambda) y \notin \partial\sigma_C(\text{int}(\text{dom}\sigma_C))\) for all \(\lambda \in (0, 1)\), which is a contradiction.

Now, let \(\sigma_C\) be Gateaux differentiable on \(\text{int}(\text{dom}\sigma_C)\). By the contrary, let \(x_0, y_0 \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C))\), where \(x_0 \neq y_0\) and \(\lambda_0 \in (0, 1)\) such that \(z := \lambda_0 x_0 + (1 - \lambda_0) y_0 \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C))\). So, there exists \(x^* \in \text{int}(\text{dom}\sigma_C)\) such that \(z \in \partial\sigma_C(x^*)\) which implies that

\[
x^*(z) = \sigma_C(x^*) \geq \lambda_0 x^*(x_0) + (1 - \lambda_0) x^*(y_0) = x^*(z).
\]

It means that \(x^*(x_0) = x^*(y_0) = \sigma_C(x^*)\) and \(x_0, y_0 \in \partial\sigma_C(x^*)\). But, by \((i_2)\) of Theorem 2.3, \(\partial\sigma_C(x^*)\) is singleton for each \(x^* \in \text{int}(\text{dom}\sigma_C)\), which is a contradiction. \(\square\)

**Theorem 3.2.** \(\sigma_C\) is Gateaux differentiable on \(\text{int}(\text{dom}\sigma_C)\) if
(3.2) \[ \forall x, y \in C, x \neq y, \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)y \in H, \]

where \( H := C + (C_\infty \setminus \{0\}) \).

Proof. First, we show that \( \partial \sigma_C(\text{int}(\text{dom} \ C)) \) is a subset of \( C \setminus H \). Letting \( y \in \partial \sigma_C(\text{int}(\text{dom} \ C)) \cap H \) (by the contrary), there exist \( y^* \in \text{int}(\text{dom} \ C) \), \( u \in C \) and \( v \in C_\infty \setminus \{0\} \) such that \( y \in \partial \sigma_C(y^*) \) and \( y = u + v \). So,

\[ \sigma_C(y^*) > y^*(u) + y^*(v) = y^*(y) = \sigma_C(y^*), \]

which is not possible. Now, it is easy to show that (3.2) implies the following condition:

(3.3) \[ \forall x, y \in C \setminus H, x \neq y, \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)y \notin C \setminus H. \]

In fact, the implications (3.2) \( \Rightarrow \) (3.3) \( \Rightarrow \) (3.1) hold. Therefore, (3.2) and (3.3) together imply that \( \sigma_C \) is Gateaux differentiable on \( \text{int}(\text{dom} \ C) \). \( \square \)

Definition 3.3. Let \( K \) be a closed convex pointed cone and \( C \) be a nonempty set in \( X \). The set of Pareto minimal points, \( S \)-properly points, Borwien properly minimal points, and Henig global properly minimal points of \( C \) with respect to \( K \) are shown by,

\[ \text{Min}(C, K) := \{ u \in C : C \cap (u + K) = \{u\} \} = C \setminus (C + (K \setminus \{0\})), \]
\[ S - \text{PMin}(C, K) := \{ u \in C : \exists y^* \in K^\#, \forall y \in C, y^*(u) \leq y^*(y) \}, \]
\[ \text{Bo} - \text{Min}(C, K) := \{ u \in C : \text{clcone}(C - u) \cap (-K) = \{0\} \}, \]

and

\[ \text{GHe} - \text{PMin}(C, K) := \left\{ u \in C : \exists \text{ a proper convex cone } P \text{ with } \right\} \]
\[ K \setminus \{0\} \subset \text{int}P \text{ such that } (C - u) \cap (-\text{int}P) = \emptyset \right\}, \]

respectively.

Remark 3.4. Let \( K \) be a closed convex well-based cone and \( C \subseteq K \) be closed and convex with \( C_\infty \neq \{0\} \).

(i) \( S - \text{PMin}(C, K) = \partial \sigma_C(-K^\#) = \bigcup_{x^* \in K^\#} \partial \sigma_C(-x^*). \)
Since \( C + K \) is closed, we get the following results (see [1] for more details)

\[
S - P\text{Min}(C, K) = S - P\text{Min}(C + K, K) \\
= GH e - P\text{Min}(C, K) \\
= Bo - \text{Min}(C, K) \\
\subset \text{Min}(C, K).
\]

**Theorem 3.5.** \( \sigma_C \) is Gateaux differentiable on \(-K\#\) if and only if \( \sigma_{C+K} \) is Gateaux differentiable on \( \text{int}(\text{dom}\sigma_{C+K}) \).

**Proof.** By the assumption, we get \( C_\infty \subset K \) and \((C + K)_\infty = K_\infty = K\). Moreover, \( \sigma_{C+K} = \sigma_C + \sigma_K = \sigma_C + \iota_K \). Since the space is reflexive, we have

\[-K\# = \text{int}K^- \subset \text{dom}\sigma_{C+K} = K^- \cap \text{dom}\sigma_C \subset K^- \cap C_\infty = K^-.
\]

By taking the interior of the both sides, we have \( \text{int}(\text{dom}\sigma_{C+K}) = -K\# \) and

\[
\partial\sigma_{C+K}[\text{int}(\text{dom}\sigma_{C+K})] = \partial\sigma_{C+K}(-K\#) \\
= S - P\text{Min}(C + K, K) \\
= S - P\text{Min}(C, K) \\
= \partial\sigma_C(-K\#).
\]

So \( \partial\sigma_C \) is singleton on \(-K\#\) if and only if \( \partial\sigma_{C+K} \) is singleton on \( \text{int}(\text{dom}\sigma_{C+K}) \) which means that \( \sigma_C \) is Gateaux differentiable on \(-K\#\) if and only if \( \sigma_{C+K} \) is Gateaux differentiable on \( \text{int}(\text{dom}\sigma_{C+K}) \). \( \Box \)

**Theorem 3.6.** \( \sigma_C \) is Gateaux differentiable on \(-K\#\) if and only if

\[
\forall x, y \in S - P\text{Min}(C, K), \ x \neq y, \\
\forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)y \notin S - P\text{Min}(C, K).
\]

**Proof.** By Theorem 3.5, \( \sigma_C \) is Gateaux differentiable on \(-K\#\) if and only if \( \sigma_{C+K} \) is Gateaux differentiable on \( \text{int}(\text{dom}\sigma_{C+K}) \). Now, one obtains the result by using Theorem 3.1 for \( \text{int}(\text{dom}\sigma_{C+K}) \). \( \Box \)

**Corollary 3.7.** \((h_1)\)** Since \( S - P\text{Min}(C, K) \) is a subset of \( \text{Min}(C, K) \), considering the following condition,

\[
\forall x, y \in \text{Min}(C, K), \ x \neq y, \\
\forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)y \notin \text{Min}(C, K),
\]

we have the implication \((j_2) \Rightarrow (j_4)\). Hence, \((j_2) \Rightarrow (j_4)\) implies that \( \sigma_C \) is differentiable on \(-K\#\). Moreover, from the equalities in \((j_2) \) of Remark 3.4, we could replace \( S \)-properly minimal points in...
ON POLAR CONES AND DIFFERENTIABILITY IN REFLEXIVE ... 23

condition \( \mathcal{B}_2 \), by Borwien properly minimal points and Henig global properly minimal points of \( C \) with respect to \( K \).

\((h_2)\) By taking \( K := C_{\infty} \) in Theorem \( \mathcal{B}_2 \), we get \( K^\# = \operatorname{int}(\operatorname{dom}\sigma_C) \).
So, Theorem \( \mathcal{B}_2 \) is a consequence of Theorem \( \mathcal{B}_1 \).

References


1 Department of Mathematics, Faculty of Science, Sahand University of Technology, Tabriz, Iran.  
E-mail address: esadeqi@sut.ac.ir

2 Department of Mathematics, Faculty of Science, Sahand University of Technology, Tabriz, Iran.  
E-mail address: s_hassankhali@sut.ac.ir