Sahand Communications in Mathematical Analysis (SCMA) Vol. 12 No. 1 (2018), 59-88
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.88929.466

(-1)-Weak Amenability of Second Dual of Real Banach Algebras

Hamidreza Alihoseini[1] and Davood Alimohammadi[2*]

ABSTRACT. Let $(A, \|\cdot\|)$ be a real Banach algebra, a complex algebra $A_{\mathbb{C}}$ be a complexification of A and $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying a simple condition together with the norm $\|\cdot\|$ on A. In this paper we first show that A^* is a real Banach A^{**}-module if and only if $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module. Next we prove that A^{**} is (-1)-weakly amenable if and only if $(A_{\mathbb{C}})^{**}$ is (-1)-weakly amenable. Finally, we give some examples of real Banach algebras which their second duals of some them are and of others are not (-1)-weakly amenable.

1. Introduction And Preliminaries

The symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. For a Banach space $\mathfrak{X}$ over $\mathbb{F}$ we denote by $\mathfrak{X}^*$ and $\mathfrak{X}^{**}$ the dual space and the second dual space of $\mathfrak{X}$, respectively.

Let B be an algebra over $\mathbb{F}$ and $\mathfrak{X}$ be a B-module over $\mathbb{F}$ with the module operations $(a, x) \longmapsto a \cdot x$, $(a, x) \longmapsto x \cdot a : B \times \mathfrak{X} \longrightarrow \mathfrak{X}$. A linear map $D : B \longrightarrow \mathfrak{X}$ over $\mathbb{F}$ is called an $\mathfrak{X}$-derivation on B over $\mathbb{F}$ if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in B$. For each $x \in \mathfrak{X}$, the map $\delta_x : B \longrightarrow \mathfrak{X}$ defined by $\delta_x(a) = a \cdot x - x \cdot a \quad (a \in B)$, is an $\mathfrak{X}$-derivation on B over $\mathbb{F}$. An $\mathfrak{X}$-derivation D on B is called inner if $D = \delta_x$ for some $x \in \mathfrak{X}$.

Let $(B, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$. A B-module $\mathfrak{X}$ over $\mathbb{F}$ is called a Banach B-module if $\mathfrak{X}$ is a Banach space with a norm $\|\cdot\|$ and

---

2010 *Mathematics Subject Classification.* 46H25, 46H20.
Key words and phrases. Banach algebra, Banach module, Complexification, Derivation, (-1)-Weak amenability.
Received: 27 June 2018, Accepted: 11 October 2018.
* Corresponding author.

there exists a positive constant k such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \quad \|x \cdot a\| \leq k\|a\|\|x\|,$$

for all $a \in B$ and $x \in \mathfrak{X}$. Clearly, B is a Banach B-module over $\mathbb{F}$ with the module operations $a \cdot b = ab$ and $b \cdot a = ba$ for all $a, b \in B$. Let $\mathfrak{X}$ be a Banach B-module over $\mathbb{F}$ with the module operations $(a, x) \mapsto a \cdot x$, $(a, x) \mapsto x \cdot a : B \times \mathfrak{X} \longrightarrow \mathfrak{X}$. Then $\mathfrak{X}^*$ is a Banach B-module over $\mathbb{F}$ with the natural module operations $(\lambda, a) \longmapsto a \cdot \lambda$, $(\lambda, a) \longmapsto \lambda \cdot a : B \times \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ given by

$$(a \cdot \lambda)(x) = \lambda(x \cdot a), \qquad (\lambda \cdot a)(x) = \lambda(a \cdot x), \quad (a \in B, \lambda \in \mathfrak{X}^*, x \in \mathfrak{X}),$$

and with the operator norm $\| \cdot \|_{op}$. In particular, B^* is a Banach B-module over $\mathbb{F}$. We denote by $Z^1_{\mathbb{F}}(B, \mathfrak{X})$ the set of all continuous $\mathfrak{X}$-derivations on B over $\mathbb{F}$. Clearly, $Z^1_{\mathbb{F}}(B, \mathfrak{X})$ is a linear space over $\mathbb{F}$ which contains all inner $\mathfrak{X}$-derivations on B over $\mathbb{F}$. We denote by $N^1_{\mathbb{F}}(B, \mathfrak{X})$ the set of all inner $\mathfrak{X}$-derivations on B over $\mathbb{F}$. Clearly, $N^1_{\mathbb{F}}(B, \mathfrak{X})$ is a linear subspace of $Z^1_{\mathbb{F}}(B, \mathfrak{X})$ over $\mathbb{F}$. We denote by $H^1_{\mathbb{F}}(B, \mathfrak{X})$ the quotient space $Z^1_{\mathbb{F}}(B, \mathfrak{X}) / N^1_{\mathbb{F}}(B, \mathfrak{X})$ which it is called the first cohomology group of B over $\mathbb{F}$ with coefficients in $\mathfrak{X}$.

A Banach algebra B over $\mathbb{F}$ is called amenable if $H^1_{\mathbb{F}}(B, \mathfrak{X}^*) = \{0\}$ for all Banach B-module $\mathfrak{X}$ over $\mathbb{F}$. This concept was first introduced by Johnson in [12]. The notion of weak amenability was first introduced by Bade, Curtis and Dales for commutative Banach algebras in [4] and later defined for Banach algebras, not necessarily commutative, by Johnson in [13]. In fact, a Banach algebra B over $\mathbb{F}$ is called weakly amenable if $H^1_{\mathbb{F}}(B, B^*) = \{0\}$.

Let B be a Banach algebra over $\mathbb{F}$. For each $(\lambda, \Lambda) \in B^* \times B^{**}$ the $\mathbb{F}$-valued functions $\lambda \cdot \Lambda$ and $\Lambda \cdot \lambda$ on B are defined by

$$\begin{aligned} (\lambda \cdot \Lambda)(a) &= \Lambda(a \cdot \lambda), \quad (a \in B), \\ (\Lambda \cdot \lambda)(a) &= \Lambda(\lambda \cdot a), \quad (a \in B). \end{aligned}$$

Then $\lambda \cdot \Lambda \in B^*$, $\|\lambda \cdot \Lambda\|_{op} \leq \|\lambda\|_{op}\|\Lambda\|_{op}$, $\Lambda \cdot \lambda \in B^*$ and $\|\Lambda \cdot \lambda\|_{op} \leq \|\Lambda\|_{op}\|\lambda\|_{op}$. For each $\Lambda, \Gamma \in B^{**}$, the $\mathbb{F}$-valued functions $\Lambda \square \Gamma$ and $\Lambda \triangle \Gamma$ on B^* are defined by

$$\begin{aligned} (\Lambda \square \Gamma)(\lambda) &= \Lambda(\Gamma \cdot \lambda), \quad (\lambda \in B^*), \\ (\Lambda \triangle \Gamma)(\lambda) &= \Gamma(\lambda \cdot \Lambda), \quad (\lambda \in B^*). \end{aligned}$$

Then $\Lambda \square \Gamma \in B^{**}$, $\|\Lambda \square \Gamma\|_{op} \leq \|\Lambda\|_{op}\|\Gamma\|_{op}$, $\Lambda \triangle \Gamma \in B^{**}$ and $\|\Lambda \square \Gamma\|_{op} \leq \|\Lambda\|_{op}\|\Gamma\|_{op}$. Moreover, B^{**} is a Banach algebra over $\mathbb{F}$ with respect to either of the products $\square$ and $\triangle$ and with the operator norm $\| \cdot \|_{op}$. These products are called the first and second Arens products on B^{**}, respectively. The Banach algebra B over $\mathbb{F}$ is called Arens regular if two products $\square$ and $\triangle$ coincide on B^{**}. For the general theory of Arens products, see [3, 7, 18], for example. For the product $\square$ on B^{**} one can show that B^* is a Banach B^{**}-module over $\mathbb{F}$ if and only if the following statements hold:

(i) $(\Lambda \cdot \lambda) \cdot \Gamma = \Lambda \cdot (\lambda \cdot \Gamma)$ for all $(\Lambda, \lambda, \Gamma) \in B^{**} \times B^* \times B^{**}$,
(ii) $\lambda \cdot (\Lambda \square \Gamma) = (\lambda \cdot \Lambda) \cdot \Gamma$ for all $(\lambda, \Lambda, \Gamma) \in B^* \times B^{**} \times B^{**}$,
(iii) $(\Lambda \square \Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda)$ for all $(\Lambda, \Gamma, \lambda) \in B^{**} \times B^{**} \times B^*$.

Definition 1.1. Let $(B, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$ and $\times$ be one of the Arens products $\square$ and $\triangle$ on B^{**}. We say that B^{**} (with the product $\times$) is (-1)-weakly amenable if B^* is a Banach B^{**}-module over $\mathbb{F}$ and $H^1_{\mathbb{F}}(B^{**}, B^*) = \{0\}$.

Medghalchi and Yazdanpanah introduced the concept of (-1)-weak amenability for Banach algebras in [17] and obtained some results in this area. Eshaghi Gordji, Hosseinioun and Valadkhani in [8] gave some examples of complex Banach algebras that their second duals which are and some others which are not (-1)-weakly amenable. Hosseinioun and Valadkhani obtained interesting results in (-1)-weak amenability of complex Banach algebras in [10, 11].

Let E be a real linear space (real algebra, respectively). A complex linear space (complex algebra, respectively) $E_{\mathbb{C}}$ is called a complexification of E if there exists an injective real linear map (real algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ such that $E_{\mathbb{C}} = J(E) \oplus iJ(E)$.

If $\mathfrak{X}$ is a real linear space, then $\mathfrak{X} \times \mathfrak{X}$ with the additive operation and scalar multiplication defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad (x_1, x_2, y_1, y_2 \in \mathfrak{X}),$$
$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x), \quad (\alpha, \beta \in \mathbb{R}, x, y \in \mathfrak{X}),$$

is a complexification of $\mathfrak{X}$ with respect to the injective linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$ defined by $J(x) = (x, 0)$, $x \in \mathfrak{X}$.

If A is a real algebra, then $A \times A$ with the algebra operations

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \quad (a_1, a_2, b_1, b_2 \in A),$$
$$(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \alpha b + \beta a), \quad (\alpha, \beta \in \mathbb{R}, a, b \in A),$$
$$(a_1, b_1)(a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2), \quad (a_1, b_1, a_2, b_2 \in A),$$

is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow A \times A$ defined by $J(a) = (a, 0)$, $a \in A$.

It is known [5, Proposition I.1.13] that if $(E, \|\cdot\|)$ is a real normed algebra (real normed space, respectively), then there exists an algebra norm (a norm, respectively) $|||\cdot|||$ on $E \times E$ satisfying $|||(a, 0)||| = \|a\|$ for all $a \in E$ and

$$\max\{\|a\|, \|b\|\} \le |||(a, b)||| \le 2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$.

Definition 1.2. Let $(E, \|\cdot\|)$ be a real normed linear space (real normed algebra, respectively), let a complex linear space (algebra, respectively) $E_{\mathbb{C}}$ be a complexification of E with respect to an injective real linear map (real algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ and let $|||\cdot|||$ be a norm (an algebra norm, respectively) on $E_{\mathbb{C}}$. We say that $|||\cdot|||$ satisfies the $(*)$ condition if there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \leq k_1 |||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$.

Note that the $(*)$ condition implies that $(E, \|\cdot\|)$ is a Banach space (Banach algebra, respectively) if and only if $(E_{\mathbb{C}}, |||\cdot|||)$ is Banach space (Banach algebra, respectively). Moreover, the existence of a norm (an algebra norm, respectively) $|||\cdot|||$ on $E_{\mathbb{C}}$ satisfying the $(*)$ condition guarantees by [5, Proposition I.1.13].

It is shown [2] that if $(A, \|\cdot\|)$ is a real Banach algebra and if $|||\cdot|||$ is an algebra norm on complex algebra $A \times A$ satisfying

$$\max\{\|a\|, \|b\|\} \leq k_1 |||(a, b)||| \leq k_2 \max\{\|a\|, \|b\|\}$$

for some positive constants k_1 and k_2 and for all $a, b \in A$, then

(i) A is amenable if and only if $A \times A$ is amenable [2, Theorem 2.4].
(ii) A is weakly amenable if and only if $A \times A$ is weakly amenable [2, Theorem 2.5].

In Section 2 we assume that $(A, \|\cdot\|)$ is a real Banach algebra, a complex algebra $A_{\mathbb{C}}$ is the complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, $|||\cdot|||$ is an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition and $(A_{\mathbb{C}})^*$ is the dual space of $(A_{\mathbb{C}}, |||\cdot|||)$. We first show that A is Arens regular if and only if $A_{\mathbb{C}}$ is Arens regular. Next we prove that A^* is a real Banach A^{**}-module if and only if $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module. Moreover, we prove that if A is a real Banach algebra such that A^* is a real Banach A^{**}-module, then A^{**} is (-1)-weakly amenable if and only if $(A_{\mathbb{C}})^{**}$ is (-1)-weakly amenable. Finally, we give some examples of real Banach algebras which their second duals of some them are and of others are not (-1)-weakly amenable.

2. Main Results and Applications

We first give some lemmas which they will use in the sequel to prove of the main results.

Lemma 2.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space, let $\mathfrak{X}_{\mathbb{C}}$ be a complexification of $\mathfrak{X}$ with respect to an injective real linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$,* *let $|||\cdot|||$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ satisfying the $(*)$ condition with respect to positive constants k_1 and k_2 and let $(\mathfrak{X}_{\mathbb{C}})^*$ be the dual space of the complex Banach space $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$.*

(i) *Let $\varphi \in \mathfrak{X}^*$ and define the map $\varphi_C : \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathbb{C}$ by*

$$\varphi_C(J(x) + iJ(y)) = \varphi(x) + i\varphi(y) \qquad (x, y \in \mathfrak{X}).$$

Then $\varphi_C(J(x)) = \varphi(x)$ for all $x \in \mathfrak{X}$, $\varphi_C \in (\mathfrak{X}_{\mathbb{C}})^*$, $\|\varphi_C\|_{op} \leq 2k_1\|\varphi\|_{op}$ and $\|\varphi\|_{op} \leq \frac{k_2}{k_1}\|\varphi_C\|_{op}$.

(ii) *Let $\lambda \in (\mathfrak{X}_{\mathbb{C}})^*$ and define the map $\lambda_R : \mathfrak{X} \longrightarrow \mathbb{R}$ by*

$$\lambda_R(x) = \operatorname{Re} \lambda(J(x)) \qquad (x \in \mathfrak{X}).$$

Then $\lambda_R \in \mathfrak{X}^*$ and $\|\lambda_R\|_{op} \leq \frac{k_2}{k_1}\|\lambda\|_{op}$.

(iii) *Let $\lambda \in (\mathfrak{X}_{\mathbb{C}})^*$ and define the map $\lambda_I : \mathfrak{X} \longrightarrow \mathbb{R}$ by*

$$\lambda_I(x) = \operatorname{Im} \lambda(J(x)) \qquad (x \in \mathfrak{X}).$$

Then $\lambda_I \in \mathfrak{X}^*$ and $\|\lambda_I\|_{op} \leq \frac{k_2}{k_1}\|\lambda\|_{op}$.

Proof. Let $x \in X$. Then

$$\begin{aligned}\varphi_C(J(x)) &= \varphi_C(J(x) + iJ(0)) \\ &= \varphi(x) + i\varphi(0) \\ &= \varphi(x) + i0 \\ &= \varphi(x).\end{aligned}$$

It is easy to see that φ_C is a complex linear functional on $\mathfrak{X}_{\mathbb{C}}$. Since

$$\begin{aligned}|\varphi_C(J(x) + iJ(y))| &= |\varphi(x) + i\varphi(y)| \\ &\leq |\varphi(x)| + |\varphi(y)| \\ &\leq 2\|\varphi\|_{op} \max\{\|x\|, \|y\|\} \\ &\leq 2k_1\|\varphi\|_{op}|||J(x) + iJ(y)|||\end{aligned}$$

for all $x, y \in \mathfrak{X}$, we deduce that $\varphi_C \in (\mathfrak{X}_{\mathbb{C}})^*$ and $\|\varphi_C\|_{op} \leq 2k_1\|\varphi\|_{op}$. On the other hand, we have

$$\begin{aligned}|\varphi(x)| &= |\varphi_C(J(x))| \\ &\leq \|\varphi_C\|_{op}|||J(x)||| \\ &\leq \|\varphi_C\|_{op}\frac{k_2}{k_1}\|x\|,\end{aligned}$$

for all $x \in \mathfrak{X}$. Hence, $\|\varphi\|_{op} \leq \frac{k_2}{k_1}\|\varphi_C\|_{op}$. Therefore, (i) holds.

Clearly, λ_R is a real linear functional on $\mathfrak{X}$. Since

$$\begin{aligned}|\lambda_R(x)| &= |\operatorname{Re} \lambda(J(x))| \\ &\leq |\lambda(J(x))| \\ &\leq \|\lambda\|_{op}|||J(x)|||\end{aligned}$$

$$\leq \|\lambda\|_{op}\frac{k_2}{k_1}\|x\|,$$

for all $x \in \mathfrak{X}$, we deduce that $\lambda_R \in \mathfrak{X}^*$ and $\|\lambda_R\|_{op} \leq \frac{k_2}{k_1}\|\lambda\|$. Hence, (ii) holds.

It is easy to see that λ_I is a real linear functional on $\mathfrak{X}$. Moreover, for each $x \in \mathfrak{X}$ we have

$$\begin{aligned}|\lambda_I(x)| &= |\operatorname{Im}\lambda(J(x))| \\ &\leq |\lambda(J(x))| \\ &\leq \|\lambda\|_{op}|||J(x)||| \\ &\leq \|\lambda\|_{op}\frac{k_2}{k_1}\|x\|.\end{aligned}$$

Hence, $\lambda_I \in \mathfrak{X}^*$ and $\|\lambda_I\|_{op} \leq \frac{k_2}{k_1}\|\lambda\|_{op}$. Therefore, (iii) holds. $\square$

Lemma 2.2. *Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space, let $\mathfrak{X}_{\mathbb{C}}$ be a complexification of $\mathfrak{X}$ with respect to an injective real linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$, let $|||\cdot|||$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ satisfying $(*)$ condition with respect to positive constants k_1 and k_2 and let $(\mathfrak{X}_{\mathbb{C}})^*$ be the dual space of the complex Banach space $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$. Define the map $J_1 : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ by*

$$J_1(\varphi) = \varphi_C, \qquad (\varphi \in \mathfrak{X}^*). \tag{2.1}$$

Then:

(i) *$J_1(\varphi)(J(x)+iJ(y)) = \varphi(x)+i\varphi(y)$ for all $\varphi \in \mathfrak{X}^*$ and $x, y \in \mathfrak{X}$.*
(ii) *J_1 is a real linear map from $\mathfrak{X}^*$ into $(\mathfrak{X}_{\mathbb{C}})^*$.*
(iii) *If $\lambda \in (\mathfrak{X}_{\mathbb{C}})^*$, then $\lambda = J_1(\lambda_R) + iJ_1(\lambda_I)$.*
(iv) *J_1 is injective and $(\mathfrak{X}_{\mathbb{C}})^* = J_1(\mathfrak{X}^*) \oplus iJ_1(\mathfrak{X}^*)$.*
(v) *$(\mathfrak{X}_{\mathbb{C}})^*$ is a complexification of $\mathfrak{X}^*$ with respect to the map $J_1 : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ defined by (2.1) and*

$$\begin{aligned}\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} &\leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op} \\ &\leq 4k_2 \max\{\|\varphi\|_{op}, \|\psi\|_{op}\},\end{aligned}$$

for all $\varphi, \psi \in \mathfrak{X}^*$.

Proof. By part (i) of Lemma 2.1, J_1 is well-defined. Let $\varphi \in \mathfrak{X}^*$ and $x, y \in \mathfrak{X}$. Then, by part (i) of Lemma 2.1, we have

$$\begin{aligned}J_1(\varphi)(J(x) + iJ(y)) &= \varphi_C(J(x) + iJ(y)) \\ &= \varphi_C(J(x)) + i\varphi_C(J(y)) \\ &= \varphi(x) + i\varphi(y).\end{aligned}$$

Hence, (i) holds.

It is easy to see that $(\varphi+\psi)_C = \varphi_C + \psi_C$ for all $\varphi, \psi \in \mathfrak{X}^*$ and $(\alpha\varphi)_C = \alpha\varphi_C$ for all $\alpha \in \mathbb{R}$ and $\varphi \in \mathfrak{X}^*$. Hence, (ii) holds.

Let $\lambda \in (\mathfrak{X}_{\mathbb{C}})^*$. By parts (ii) and (iii) of Lemma 2.1, $\lambda_R, \lambda_I \in \mathfrak{X}^*$. Since

$$\begin{aligned}\lambda(J(x)+iJ(y)) =& \lambda(J(x)) + i\lambda(J(y)))\\ =& (\operatorname{Re}\lambda(J(x)) + i\operatorname{Im}\lambda(J(x)))\\ &+ i\,(\operatorname{Re}\lambda(J(y)) + i\operatorname{Im}\lambda(J(y)))\\ =& (\lambda_R(x) + i\lambda_I(x)) + i\,(\lambda_R(y) + i\lambda_I(y))\\ =& (\lambda_R(x) + i\lambda_R(y)) + i\,(\lambda_I(x) + i\lambda_I(y))\\ =& (\lambda_R)_C\,(J(x)+iJ(y)) + i(\lambda_I)_C\,(J(x)+iJ(y))\\ =& (((\lambda_R)_C) + i((\lambda_I)_C))\,(J(x)+iJ(y))\\ =& (J_1(\lambda_R) + iJ_1\,(\lambda_I)(J(x)+iJ(y))\,,\end{aligned}$$

for all $x, y \in \mathfrak{X}$, we have $\lambda = J_1(\lambda_R) + iJ_1(\lambda_I)$. Hence, (iii) holds.

Let $\varphi \in \mathfrak{X}^*$ and $J_1(\varphi) = 0$. Then $\varphi_C = 0$ and so $\varphi_C(J(x)) = 0$ for all $x \in \mathfrak{X}$. This implies that $\varphi(x) = 0$ for all $x \in \mathfrak{X}$ by part (ii) of Lemma 2.1. Hence, $\varphi = 0$ and so J_1 is injective.

By the definition of the map $J_1 : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ and (iii), we conclude that

$$(\mathfrak{X}_{\mathbb{C}})^* = J_1(\mathfrak{X}^*) + iJ_1(\mathfrak{X}^*). \tag{2.2}$$

Let $\lambda \in J_1(\mathfrak{X}^*) \cap iJ_1(\mathfrak{X}^*)$. Then there exist $\varphi, \psi \in \mathfrak{X}^*$ such that $\lambda = J_1(\varphi) = iJ_1(\psi)$. This implies that $\varphi(x) = i\psi(x)$ for all $x \in \mathfrak{X}$ and so $\varphi(x) = 0$ for all $x \in \mathfrak{X}$ since φ and ψ are real-valued functions on $\mathfrak{X}$. Hence, $\varphi = 0$ and so $\lambda = J_1(\varphi) = 0$. Thus

$$J_1(\mathfrak{X}^*) \cap iJ_1(\mathfrak{X}^*) = \{0\}. \tag{2.3}$$

From (2.2) and (2.3) we have $(\mathfrak{X}_{\mathbb{C}})^* = J_1(\mathfrak{X}^*) \oplus iJ_1(\mathfrak{X}^*)$. Therefore, (iv) holds.

Applying (ii) and (iv), we deduce that $(\mathfrak{X}_{\mathbb{C}})^*$ is a complexification of $\mathfrak{X}^*$ with respect to the injective real linear map $J_1 : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ which is defined by (2.1).

Let $\varphi, \psi \in \mathfrak{X}^*$. Since

$$\begin{aligned}|\varphi(x)| &\leq |\varphi(x) + i\psi(x)|\\ &= |J_1(\varphi)(J(x)) + iJ_1(\psi)(J(x))|\\ &= |(J_1(\varphi) + iJ_1(\psi))(J(x))|\\ &\leq \|J_1(\varphi) + iJ_1(\varphi)\|_{op} \||J(x)\||\\ &\leq \|J_1(\varphi) + iJ_1(\psi)\|_{op} \frac{k_2}{k_1}\|x\|,\end{aligned}$$

for all $x \in \mathfrak{X}$, we deduce that $\|\varphi\|_{op} \leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op}$. Similarly, we have $\|\psi\|_{op} \leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op}$. Hence,

$$\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} \leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op}. \tag{2.4}$$

Since

$$\begin{aligned}
&|(J_1(\varphi) + iJ_1(\psi))(J(x) + iJ(y))| \\
&\qquad = |J_1(\varphi)(J(x) + iJ(y)) + iJ_1(\psi)(J(x) + iJ(y))| \\
&\qquad = |(\varphi(x) + i\varphi(y)) + i(\psi(x) + i\psi(y))| \\
&\qquad \leq |\varphi(x)| + |\varphi(y)| + |\psi(x)| + |\psi(y)| \\
&\qquad \leq \|\varphi\|_{op}\|x\| + \|\varphi\|_{op}\|y\| + \|\psi\|_{op}\|x\| + \|\psi\|_{op}\|y\| \\
&\qquad \leq 2\|\varphi\|_{op}\max\{\|x\|, \|y\|\} + 2\|\psi\|_{op}\max\{\|x\|, \|y\|\} \\
&\qquad \leq 4k_1 |||J(x) + iJ(y)||| \max\{\|\varphi\|_{op}, \|\psi\|_{op}\}
\end{aligned}$$

for all $x, y \in \mathfrak{X}$, we deduce that

$$\|J_1(\varphi) + iJ_1(\psi)\|_{op} \leq 4k_1 \max\{\|\varphi\|_{op}, \|\psi\|_{op}\}. \tag{2.5}$$

From (2.4) and (2.5) we have

$$\begin{aligned}
\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} &\leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op} \\
&\leq 4k_2 \max\{\|\varphi\|_{op}, \|\psi\|_{op}\}.
\end{aligned}$$

Hence, (v) holds. □

Lemma 2.3. *Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space, let $\mathfrak{X}_{\mathbb{C}}$ be a complexification of $\mathfrak{X}$ with respect to an injective real linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$, let $|||\cdot|||$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ satisfying $(*)$ condition with positive constants k_1 and k_2 and let $(\mathfrak{X}_{\mathbb{C}})^*$ be the dual space of $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$. Define the map $J_2 : \mathfrak{X}^{**} \longrightarrow (\mathfrak{X}_{\mathbb{C}})^{**}$ by*

$$J_2(\Phi) = \Phi_C \qquad (\Phi \in \mathfrak{X}^{**}). \tag{2.6}$$

Then:

(i) *$J_2(\Phi)(J_1(\varphi) + iJ_1(\psi)) = \Phi(\varphi) + i\Phi(\psi)$ for all $\Phi \in \mathfrak{X}^{**}$ and $\varphi, \psi \in \mathfrak{X}^*$.*

(ii) *J_2 is a real linear map from $\mathfrak{X}^{**}$ into $(\mathfrak{X}_{\mathbb{C}})^{**}$.*

(iii) *If $\Lambda \in (\mathfrak{X}_{\mathbb{C}})^{**}$, then the maps $\Lambda_R, \Lambda_I : \mathfrak{X}^* \longrightarrow \mathbb{R}$ defined by*

$$\begin{aligned}
\Lambda_R(\varphi) &= \operatorname{Re}\Lambda(J_1(\varphi)) \quad (\varphi \in \mathfrak{X}^*), \\
\Lambda_I(\varphi) &= \operatorname{Im}\Lambda(J_1(\varphi)) \quad (\varphi \in \mathfrak{X}^*),
\end{aligned}$$

belong to $\mathfrak{X}^{**}$ and

$$\Lambda = J_2(\Lambda_R) + iJ_2(\Lambda_I).$$

(iv) *J_2 is injective and $(\mathfrak{X}_{\mathbb{C}})^{**} = J_2(\mathfrak{X}^{**}) \oplus iJ_2(\mathfrak{X}^{**})$.*

(v) *$(\mathfrak{X}_{\mathbb{C}})^{**}$ is a complexification of $\mathfrak{X}^{**}$ with respect to the map $J_2 : \mathfrak{X}^{**} \longrightarrow (\mathfrak{X}_{\mathbb{C}})^{**}$ defined by (2.6) and*

$$\begin{aligned}\max\{\|\Phi\|_{op}, \|\Psi\|_{op}\} &\leq 4k_1\|J_2(\Phi) + iJ_2(\Psi)\|_{op}\\ &\leq 16k_2 \max\{\|\Phi\|_{op}, \|\Psi\|_{op}\},\end{aligned}$$

for all $\Phi, \Psi \in \mathfrak{X}^{**}$.

(vi) *$J_2 \circ \pi_{\mathfrak{X}} = \pi_{\mathfrak{X}_{\mathbb{C}}} \circ J$, whenever $\pi_Y : Y \longrightarrow Y^{**}$ is the natural embedding Y in Y^{**} defined by*

$$\pi_Y(y)(\lambda) = \lambda(y) \qquad (y \in Y,\, \lambda \in Y^*).$$

(vii) *$\mathfrak{X}$ is reflexive if and only if $\mathfrak{X}_{\mathbb{C}}$ is reflexive.*

Proof. By Lemma 2.2, we deduce that the map $J_1 : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ defined by (2.1) is an injective real linear map, the complex linear space $(\mathfrak{X}_{\mathbb{C}})^*$ is a complexification of $\mathfrak{X}^*$ with respect to J_1,

$$\lambda = J_1(\lambda_R) + iJ_1(\lambda_I) \qquad (\lambda \in (\mathfrak{X}_{\mathbb{C}})^*),$$

$$\begin{aligned}\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} &\leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op}\\ &\leq 4k_2 \max\{\|\varphi\|_{op}, \| \ psi\|_{op}\},\end{aligned}$$

for all $\varphi, \psi \in \mathfrak{X}^*$, and

$$J_1(\varphi)(J(x) + iJ(y)) = \varphi(x) + i\varphi(y)$$

for all $\varphi \in \mathfrak{X}^*$ and $x, y \in \mathfrak{X}$. Hence, by the definition of J_2, we deduce that (i), (ii), (iii), (iv) and (v) hold.

To prove (vi), suppose that $x \in \mathfrak{X}$. Then for each $\lambda \in (\mathfrak{X}_{\mathbb{C}})^*$ we have

$$\begin{aligned}((\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(x))(\lambda) &= (\pi_{\mathfrak{X}_{\mathbb{C}}}(J(x)))(\lambda)\\ &= \lambda(J(x))\\ &= (J_1(\lambda_R) + iJ_1(\lambda_I))(J(x))\\ &= (J_1(\lambda_R))(J(x)) + i(J_1(\lambda_I))(J(x))\\ &= \lambda_R(x) + i\lambda_I(x)\\ &= \pi_{\mathfrak{X}}(x)(\lambda_R) + i\pi_{\mathfrak{X}}(x)(\lambda_I)\\ &= J_2(\pi_{\mathfrak{X}})(x)(J_1(\lambda_R)) + iJ_2(\pi_{\mathfrak{X}})(x))(J_1(\lambda_I))\\ &= J_2(\pi_{\mathfrak{X}})(x)(J_1(\lambda_R) + iJ_1(\lambda_I))\\ &= (J_2 \circ \pi_{\mathfrak{X}})(x)(\lambda).\end{aligned}$$

This implies that

$$(\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(x) = (J_2 \circ \pi_{\mathfrak{X}})(x). \tag{2.7}$$

Since (2.7) holds for all $x \in \mathfrak{X}$, we deduce that $\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J = J_2 \circ \pi_{\mathfrak{X}}$. Hence (vi) holds.

To prove (vii) we first assume that $\mathfrak{X}$ is reflexive. Then $\pi_{\mathfrak{X}}(\mathfrak{X}) = \mathfrak{X}^{**}$. Let $\Lambda \in (\mathfrak{X}_{\mathbb{C}})^{**}$. By part (iii) we have

$$\Lambda = J_2(\Lambda_R) + iJ_2(\Lambda_I).$$

Since $\Lambda_R, \Lambda_I \in \mathfrak{X}^{**}$, there exist $x, y \in \mathfrak{X}$ such that $\pi_{\mathfrak{X}}(x) = \Lambda_R$ and $\pi_{\mathfrak{X}}(y) = \Lambda_I$. Hence, by part (vi) we have

$$\begin{aligned}\Lambda &= J_2(\pi_{\mathfrak{X}}(x)) + iJ_2(\pi_{\mathfrak{X}}(y))\\ &= (J_2 \circ \pi_{\mathfrak{X}})(x) + i(J_2 \circ \pi_{\mathfrak{X}})(y)\\ &= (\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(x) + i(\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(y)\\ &= \pi_{\mathfrak{X}_{\mathbb{C}}}(J(x) + iJ(y)),\end{aligned}$$

and so $\Lambda \in \pi_{\mathfrak{X}_{\mathbb{C}}}(\mathfrak{X}_{\mathbb{C}})$. Therefore, $\pi_{\mathfrak{X}_{\mathbb{C}}}$ is surjective and so $\mathfrak{X}_{\mathbb{C}}$ is reflexive.

We now assume that $\mathfrak{X}_{\mathbb{C}}$ is reflexive. Then $\pi_{\mathfrak{X}_{\mathbb{C}}}(\mathfrak{X}_{\mathbb{C}}) = (\mathfrak{X}_{\mathbb{C}})^{**}$. Let $\Phi \in \mathfrak{X}^{**}$. Then $J_2(\Phi) \in (\mathfrak{X}_{\mathbb{C}})^{**}$ and so there exist $x, y \in \mathfrak{X}$ such that

$$J_2(\Phi) = \pi_{\mathfrak{X}_{\mathbb{C}}}(J(x) + iJ(y)).$$

Hence, by part (vi) we have

$$\begin{aligned}J_2(\Phi) + iJ_2(0) &= J_2(\Phi)\\ &= (\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(x) + i(\pi_{\mathfrak{X}_{\mathbb{C}}} \circ J)(y)\\ &= (J_2 \circ \pi_{\mathfrak{X}})(x) + i(J_2 \circ \pi_{\mathfrak{X}})(y)\\ &= J_2(\pi_{\mathfrak{X}}(x)) + iJ_2(\pi_{\mathfrak{X}}(y)).\end{aligned}$$

This implies that $J_2(\Phi) = J_2(\pi_{\mathfrak{X}}(x))$ since $(\mathfrak{X}_{\mathbb{C}})^{**} = J_2(\mathfrak{X}^{**}) \oplus iJ_2(\mathfrak{X}^{**})$. Therefore, $\Phi = \pi_{\mathfrak{X}}(x)$ since J_2 is injective. Hence, $\pi_{\mathfrak{X}}$ is surjective and so $\mathfrak{X}$ is reflexive. Thus, (vii) holds. $\square$

Lemma 2.4. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition and let $(A_{\mathbb{C}})^*$ be the dual space of $(A_{\mathbb{C}}, |||\cdot|||)$.*

(i) *If $a \in A$ and $\varphi \in A^*$, then*

$$J_1(a \cdot \varphi) = J(a) \cdot J_1(\varphi), \quad J_1(\varphi \cdot a) = J_1(\varphi) \cdot J(a).$$

(ii) *If $\varphi \in A^*$ and $\Phi \in A^{**}$, then*

$$J_1(\varphi \cdot \Phi) = J_1(\varphi) \cdot J_2(\Phi), \quad J_1(\Phi \cdot \varphi) = J_2(\Phi) \cdot J_1(\varphi).$$

(iii) *If $\Phi, \Psi \in A^{**}$, then*

$$J_2(\Phi \Box \Psi) = J_2(\Phi) \Box J_2(\Psi), \quad J_2(\Phi \triangle \Psi) = J_2(\Phi) \triangle J_2(\Psi).$$

(iv) *If* $\Lambda \in (A_{\mathbb{C}})^{**}$ *and* $\lambda \in (A_{\mathbb{C}})^{*}$, *then*

$$\Lambda \cdot \lambda = J_1(\Lambda_R \cdot \lambda_R - \Lambda_I \cdot \lambda_I) + iJ_1(\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R),$$

$$\lambda \cdot \Lambda = J_1(\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) + iJ_1(\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R)$$

Proof. Let $a \in A$ and $\varphi \in A^*$. Then, by Lemma 2.3, we have

$$\begin{aligned} J_1(a \cdot \varphi)(J(b)) &= (a \cdot \varphi)(b) \\ &= \varphi(ba) \\ &= J_1(\varphi)(J(ba)) \\ &= J_1(\varphi)(J(b)J(a)) \\ &= (J(a) \cdot J_1(\varphi))(J(b)), \end{aligned}$$

for all $b \in A$. This implies that

$$\begin{aligned} J_1(a \cdot \varphi)(J(b) + iJ(c)) &= J_1(a \cdot \varphi)(J(b)) + iJ_1(a \cdot \varphi)(J(c)) \\ &= (J(a) \cdot J_1(\varphi))(J(b)) + i(J(a) \cdot J_1(\varphi))(J(c)) \\ &= (J(a) \cdot J_1(\varphi))(J(b) + iJ(c)), \end{aligned}$$

for all $b, c \in A$. Hence,

$$J_1(a \cdot \varphi) = J(a) \cdot J_1(\varphi).$$

Similarly, we can show that

$$J_1(\varphi \cdot a) = J_1(\varphi) \cdot J(a).$$

Therefore, (i) holds.

Let $\varphi \in A^*$ and $\Phi \in A^{**}$. Then, by (i), we have

$$\begin{aligned} J_1(\varphi \cdot \Phi)(J(a)) &= J_1(\varphi \cdot \Phi)(a) \\ &= \Phi(a \cdot \varphi) \\ &= J_2(\Phi)(J_1(a \cdot \varphi)) \\ &= J_2(\Phi)(J(a) \cdot J_1(\varphi)) \\ &= (J_1(\varphi) \cdot J_2(\Phi))(J(a)) \end{aligned}$$

for all $a \in A$. This implies that

$$\begin{aligned} J_1(\varphi \cdot \Phi)(J(a) + iJ(b)) =& J_1(\varphi \cdot \Phi)(J(a)) + iJ_1(\varphi \cdot \Phi)(J(b)) \\ =& (J_1(\varphi) \cdot J_2(\Phi))(J(a)) \\ &+ i(J_1(\varphi) \cdot J_2(\Phi))(J(b)) \\ =& (J_1(\varphi) \cdot J_2(\Phi))(J(a) + iJ(b)), \end{aligned}$$

for all $a, b \in A$. Hence,

$$J_1(\varphi \cdot \Phi) = J_1(\varphi) \cdot J_2(\Phi).$$

Similarly, we can show that

$$J_1(\Phi \cdot \varphi) = J_2(\Phi) \cdot J_1(\varphi).$$

Therefore, (ii) holds.

Let $\Phi, \Psi \in A^{**}$. Then, by (ii), we have

$$\begin{aligned} J_2(\Phi \Box \Psi)(J_1(\varphi)) &= (\Phi \Box \Psi)(\varphi) = \Phi(\Psi \cdot \varphi) \\ &= J_2(\Phi)(J_1(\Psi \cdot \varphi)) \\ &= J_2(\Phi)(J_2(\Psi) \cdot J_1(\varphi)) \\ &= (J_2(\Phi) \Box J_2(\Psi))(J_1(\varphi)), \end{aligned}$$

for all $\varphi \in A^*$. This implies that

$$\begin{aligned} &J_2(\Phi \Box \Psi)(J_1(\varphi) + iJ_1(\psi)) \\ &\quad = J_2(\Phi \Box \Psi)(J_1(\varphi)) + iJ_2(\Phi \Box \Psi)(J_1(\psi)) \\ &\quad = (J_2(\Phi) \Box J_2(\Psi)(J_1(\varphi)) + i(J_2(\Phi) \Box J_2(\Psi))(J_1(\psi)) \\ &\quad = (J_2(\Phi) \Box J_2(\Psi))(J_1(\varphi) + iJ_1(\psi)), \end{aligned}$$

for all $\varphi, \psi \in A^*$. Hence,

$$J_2(\Phi \Box \Psi) = J_2(\Phi) \Box J_2(\Psi).$$

Similarly, we can show that

$$J_2(\Phi \triangle \Psi) = J_2(\Phi) \triangle J_2(\Psi).$$

Therefore, (iii) holds.

Let $\Lambda \in (A_{\mathbb{C}})^{**}$ and $\lambda \in (A_{\mathbb{C}})^*$. Then by part (iii) of Lemma 2.3 and part (iii) of Lemma 2.2, we have

$$\Lambda = J_2(\Lambda_R) + iJ_2(\Lambda_I), \quad \lambda = J_1(\lambda_R) + iJ_1(\lambda_I). \tag{2.8}$$

Applying (2.8) and (ii), we get

$$\begin{aligned} \Lambda \cdot \lambda = &(J_2(\Lambda_R + iJ_2(\Lambda_I)) \cdot (J_1(\lambda_R) + iJ_1(\lambda_I) \\ = &(J_2(\Lambda_R) \cdot J_1(\lambda_R) - J_2(\Lambda_I) \cdot J_1(\lambda_I)) \\ &+ i(J_2(\Lambda_R) \cdot J_1(\lambda_I) + J_2(\Lambda_I) \cdot J_1(\lambda_R)) \\ = &(J_1(\Lambda_R \cdot \lambda_R) - J_1(\Lambda_I \cdot \lambda_I)) + i(J_1(\Lambda_R \cdot \lambda_I) + J_1(\Lambda_I \cdot \lambda_R)) \\ = &J_1(\Lambda_R \cdot \lambda_R - \Lambda_I \lambda_I) + iJ_1(\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R). \end{aligned}$$

Similarly, we can show that

$$\lambda \cdot \Lambda = J_1(\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) + iJ_1(\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R).$$

Hence, (iv) holds. $\Box$

Theorem 2.5. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition and let $(A_{\mathbb{C}})^*$ be the dual space of $(A_{\mathbb{C}}, |||\cdot|||)$. Then A is Arens regular if and only if $A_{\mathbb{C}}$ is Arens regular.*

Proof. We first assume that A is Arens regular. Then

$$\Phi\Box\Psi = \Phi\triangle\Psi, \tag{2.9}$$

for all $\Phi, \Psi \in A^{**}$. Let $\Lambda, \Gamma \in (A_{\mathbb{C}})^{**}$. Then, by part (iii) of Lemma 2.3, we have $\Lambda_R, \Lambda_I, \Gamma_R, \Gamma_I \in A^{**}$ and

$$\Lambda = J_2(\Lambda_R) + iJ_2(\lambda_I), \qquad \Gamma = J_2(\Gamma_R) + iJ_2(\Gamma_I). \tag{2.10}$$

Since (2.9) holds for all $\Phi, \Psi \in A^{**}$, we have

$$\begin{aligned} &\Lambda_R\Box\Gamma_R = \Lambda_R\triangle\Gamma_R, \qquad \Lambda_R\Box\Gamma_I = \Lambda_R\triangle\Gamma_I, \\ &\Lambda_I\Box\Gamma_R = \Lambda_I\triangle\Gamma_R, \qquad \Lambda_I\Box\Gamma_I = \Lambda_I\triangle\Gamma_I. \end{aligned} \tag{2.11}$$

By Lemma 2.4 and according to (2.10) and (2.11), we get

$$\begin{aligned} \Lambda\Box\Gamma =& (J_2(\Lambda_R) + iJ_2(\Lambda_I)) \,\Box\, (J_2(\Gamma_R) + iJ_2(\Gamma_I)) \\ =& (J_2(\Lambda_R)\Box J_2(\Gamma_R) - (J_2(\Lambda_I)\Box J_2(\Gamma_I))) \\ & + i\,((J_2(\Lambda_R)\Box J_2(\Gamma_I)) + (J_2(\Lambda_I\Box J_2(\Gamma_R)) \\ =& (J_2(\Lambda_R\Box\Gamma_R) - J_2(\Lambda_I\Box\Gamma_I)) \\ & + i\,(J_2(\Lambda_R\Box\Gamma_I) + J_2(\Lambda_I\Box\Gamma_R)) \\ =& (J_2(\Lambda_R\triangle\Gamma_R) - J_2(\Lambda_I\triangle\Gamma_I)) \\ & + i\,(J_2(\Lambda_R\triangle\Gamma_I) + J_2(\Lambda_I\triangle\Gamma_R)) \\ =& (J_2(\Lambda_R)\triangle J_2(\Gamma_R) - J_2(\Lambda_I)\triangle J_2(\Gamma_I)) \\ & + i\,(J_2(\Lambda_R)\triangle J_2(\Gamma_I) + J_2(\Lambda_I)\Box J_2(\Gamma_R)) \\ =& (J_2(\Lambda_R) + iJ_2(\Lambda_I)) \,\triangle\, (J_2(\Gamma_R) + iJ_2(\Gamma_I)) \\ =& \Lambda\triangle\Gamma. \end{aligned}$$

Therefore, $(A_{\mathbb{C}})^{**}$ is Arens regular.

We now assume that $A_{\mathbb{C}}$ is Arens regular. Then

$$\Lambda\Box\Gamma = \Lambda\triangle\Gamma, \tag{2.12}$$

for all $\Lambda, \Gamma \in (A_{\mathbb{C}})^{**}$. Let $\Phi, \Psi \in A^{**}$. Then, by Lemma 2.3, we have $J_2(\Phi), J_2(\Psi) \in (A_{\mathbb{C}})^{**}$ and so by (2.12) we have

$$J_2(\Phi)\Box J_2(\Psi) = J_2(\Phi)\triangle J_2(\Psi). \tag{2.13}$$

Moreover,

$$J_2(\Phi\Box\Psi) = J_2(\Phi)\Box J_2(\Psi), \quad J_2(\Phi)\triangle J_2(\Psi) = J_2(\Phi\triangle\Psi), \tag{2.14}$$

by part (iii) of Lemma 2.4. From (2.13) and (2.14) we get $J_2(\Phi\Box\Psi) = J_2(\Phi\triangle\Psi)$. This implies that $\Phi\Box\Psi = \Phi\triangle\Psi$, since J_2 is injective. Therefore, A is Arens regular. $\Box$

Theorem 2.6. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition and let $(A_{\mathbb{C}})^*$ be the dual space of $(A_{\mathbb{C}}, |||\cdot|||)$. Then A^* is a real Banach A^{**}-module if and only if $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module.*

Proof. We prove the result for the first Arens product $\Box$ on A^{**} and $(A_{\mathbb{C}})^{**}$. Similarly, on can show that the result hold for the second Arens product $\triangle$ on A^{**} and $(A_{\mathbb{C}})^{**}$.

We first assume that A^* is a real Banach A^{**}-module. Then

$$(\Phi \cdot \varphi) \cdot \Psi = \Phi \cdot (\varphi \cdot \Psi), \tag{2.15}$$

$$\varphi \cdot (\Phi\Box\Psi) = (\varphi \cdot \Phi) \cdot \Psi, \tag{2.16}$$

$$(\Phi\Box\Psi) \cdot \varphi = \Phi \cdot (\Psi \cdot \varphi), \tag{2.17}$$

for all $(\varphi, \Phi, \Psi) \in A^* \times A^{**} \times A^{**}$. Let $(\Lambda, \lambda, \Gamma) \in (A_{\mathbb{C}})^{**} \times (A_{\mathbb{C}})^* \times (A_{\mathbb{C}})^{**}$. Then $\Lambda_R, \Lambda_I \in A^{**}$, $\lambda_R, \lambda_I \in A^*$ and $\Gamma_R, \Gamma_I \in A^{**}$. Applying part (iv) of Lemma 2.4 and (2.15), we get

$$\begin{aligned}
&(\Lambda \cdot \lambda) \cdot \Gamma \\
=&(J_1(\Lambda_R \cdot \lambda_R - \Lambda_I \cdot \lambda_I) + iJ_1(\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R)) \cdot \Gamma \\
=&J_1((\Lambda_R \cdot \lambda_R - \Lambda_I \cdot \lambda_I) \cdot \Gamma_R - (\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R) \cdot \Gamma_I) \\
&+ iJ_1((\Lambda_R \cdot \lambda_R - \Lambda_I \cdot \lambda_I) \cdot \Gamma_I + (\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R) \cdot \Gamma_R) \\
=&J_1((\Lambda_R \cdot \lambda_R) \cdot \Gamma_R - (\Lambda_I \cdot \lambda_I) \cdot \Gamma_R - (\Lambda_R \cdot \lambda_I) \cdot \Gamma_I - (\Lambda_I \cdot \lambda_R) \cdot \Gamma_I) \\
&+ iJ_1((\Lambda_R \cdot \lambda_R) \cdot \Gamma_I - (\Lambda_I \cdot \lambda_I)) \cdot \Gamma_I + (\Lambda_R \cdot \lambda_I) \cdot \Gamma_R + (\Lambda_I \cdot \lambda_R) \cdot \Gamma_R) \\
=&J_1(\Lambda_R \cdot (\lambda_R \cdot \Gamma_R) - \Lambda_I \cdot (\lambda_I \cdot \Gamma_R) - \Lambda_R \cdot (\lambda_I \cdot \Gamma_I) - \Lambda_I \cdot (\lambda_R \cdot \Gamma_I)) \\
&+ iJ_1(\Lambda_R \cdot (\lambda_R \cdot \Gamma_I) - \Lambda_I \cdot (\lambda_I \cdot \Gamma_I) + \Lambda_R \cdot (\lambda_I \cdot \Gamma_R) + \Lambda_I \cdot (\lambda_R \cdot \Gamma_R)) \\
=&J_1(\Lambda_R \cdot (\lambda_R \cdot \Gamma_R - \lambda_I \cdot \Gamma_I) - \Lambda_I \cdot (\lambda_I \cdot \Gamma_R + \lambda_R \cdot \Gamma_I) \\
&+ iJ_1(\Lambda_I \cdot (\lambda_R \cdot \Gamma_I - \lambda_I \cdot \Gamma_R) + \Lambda_I \cdot (\lambda_R \cdot \Gamma_R - \lambda_I \cdot \Gamma_I)) \\
=&\Lambda \cdot (J_1(\lambda_R \cdot \Gamma_R - \lambda_I \cdot \lambda_I) + iJ_1(\lambda_R \cdot \Gamma_I + \lambda_I \cdot \Gamma_R)) \\
=&\Lambda \cdot (\lambda \cdot \Gamma).
\end{aligned}$$

Applying part (ii) of Lemma 2.4 and (2.16), we get

$$\begin{aligned}
&\lambda \cdot (\Lambda\Box\Gamma) \\
=&(J_1(\lambda_R) + iJ_1(\lambda_I))
\end{aligned}$$

$$
\begin{aligned}
&\cdot (J_2(\Lambda_R\Box\Gamma_R - \Lambda_I\Box\Gamma_I) + iJ_2(\Lambda_R\Box\Gamma_I + \Lambda_I\Box\Gamma_R))\\
=&J_1(\lambda_R \cdot (\Lambda_R\Box\Gamma_R - \Lambda_I\Box\Gamma_I) - \lambda_I \cdot (\Lambda_R\Box\Gamma_I + \Lambda_I\Box\Gamma_R))\\
&+ iJ_1(\lambda_R \cdot (\Lambda_R\Box\Gamma_I + \Lambda_I\Box\Gamma_R) + \lambda_I \cdot (\Lambda_R\Box\Gamma_R - \Lambda_I\Box\Gamma_I))\\
=&J_1(\lambda_R \cdot (\Lambda_R\Box\Gamma_R) - \Lambda_R \cdot (\Lambda_I\Box\Gamma_I) - \lambda_I \cdot (\Lambda_R\Box\Gamma_I) - \lambda_I \cdot (\Lambda_I\Box\Gamma_R))\\
&+ iJ_1(\lambda_R \cdot (\Lambda_R\Box\Gamma_I) + \lambda_R \cdot (\Lambda_I\Box\Gamma_R) + \lambda_I \cdot (\Lambda_R\Box\Gamma_R) - \lambda_I \cdot (\Lambda_I\Box\Gamma_I))\\
=&J_1((\lambda_R \cdot \Lambda_R) \cdot \Gamma_R - (\lambda_R \cdot \Lambda_I) \cdot \Gamma_I - (\lambda_I \cdot \Lambda_R) \cdot \Gamma_I - (\lambda_I \cdot \Lambda_I) \cdot \Gamma_R)\\
&+ iJ_1((\lambda_R \cdot \Lambda_R) \cdot \Gamma_I + (\lambda_R \cdot \Lambda_I) \cdot \Gamma_R + (\lambda_I \cdot \Lambda_R) \cdot \Gamma_R - (\lambda_I \cdot \Lambda_I) \cdot \Gamma_I)\\
=&J_1((\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) \cdot \Gamma_R - (\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R) \cdot \Gamma_I)\\
&+ iJ_1((\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) \cdot \Gamma_I + (\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R) \cdot \Gamma_R)\\
=&(J_1(\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) + iJ_1(\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R))\\
&\cdot (J_2(\Gamma_R) + iJ_2(\Gamma_I))\\
=&(\lambda \cdot \Lambda) \cdot \Gamma.
\end{aligned}
$$

Similarly, applying part (ii) of Lemma 2.4 and (2.17) we get

$$(\Lambda\Box\Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda).$$

Therefore, $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module.

We now assume that $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module. Then

$$(\Lambda \cdot \lambda) \cdot \Gamma = \Lambda \cdot (\lambda \cdot \Gamma), \tag{2.18}$$

$$\lambda \cdot (\Lambda\Box\Gamma) = (\lambda \cdot \Lambda) \cdot \Gamma, \tag{2.19}$$

$$(\Lambda\Box\Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda), \tag{2.20}$$

for all $(\Lambda, \lambda, \Gamma) \in (A_{\mathbb{C}})^{**} \times (A_{\mathbb{C}})^* \times (A_{\mathbb{C}})^{**}$. Let $(\Phi, \varphi, \Psi) \in A^{**} \times A^* \times A^{**}$. Then $(J_2(\Phi), J_1(\varphi), J_2(\Psi)) \in (A_{\mathbb{C}})^{**} \times (A_{\mathbb{C}})^* \times (A_{\mathbb{C}})^{**}$. By (2.18), we have

$$(J_2(\Phi) \cdot J_1(\varphi)) \cdot J_2(\Psi) = J_2(\Phi) \cdot (J_1(\varphi) \cdot J_2(\Psi)). \tag{2.21}$$

Applying part (ii) of Lemma 2.4 and (2.21), we get

$$
\begin{aligned}
J_1((\Phi \cdot \varphi) \cdot \Psi) &= J_1(\Phi \cdot \varphi) \cdot J_2(\Psi)\\
&= (J_2(\Phi) \cdot J_1(\varphi)) \cdot J_2(\Psi)\\
&= J_2(\Phi) \cdot (J_1(\varphi) \cdot J_2(\Psi))\\
&= J_2(\Phi) \cdot J_1(\varphi \cdot \Psi)\\
&= J_1(\Phi \cdot (\varphi \cdot \Psi)).
\end{aligned}
$$

This implies that $(\Phi \cdot \varphi) \cdot \Psi = \Phi \cdot (\varphi \cdot \Psi)$, since J_1 is injective.

By (2.19), we have

$$J_1(\varphi) \cdot (J_2(\Phi) \cdot J_2(\Psi)) = (J_1(\varphi) \cdot J_2(\Phi)) \cdot J_2(\Psi). \tag{2.22}$$

Applying part (iii) of Lemma 2.4 and (2.22), we get

$$\begin{aligned} J_1(\varphi\cdot(\Phi\Box\Psi)) &= J_1(\varphi)\cdot J_2(\Phi\Box\Psi) \\ &= J_1(\varphi)\cdot(J_2(\Phi)\Box J_2(\Psi)) \\ &= (J_1(\varphi)\cdot J_2(\Phi))\cdot J_2(\Psi) \\ &= J_1(\varphi\cdot\Phi)\cdot J_2(\Psi) \\ &= J_1((\varphi\cdot\Phi)\cdot\Psi). \end{aligned}$$

This implies that $\varphi\cdot(\Phi\Box\Psi)=(\varphi\cdot\Phi)\cdot\Psi$, since J_1 is injective.

Similarly, we can show that

$$(\Phi\Box\Psi)\cdot\varphi=\Phi\cdot(\Psi\cdot\varphi).$$

Therefore, A^* is a real Banach A^{**}-module. □

Applying Theorem 2.6 and [11, Example 2], we give an example of a real Banach algebra A for which A^* is not a real Banach A^{**}-module.

Example 2.7. Let $\mathbb{Z}$ be the set of all integer numbers and $l^1(\mathbb{Z})$ denote the complex Banach algebra consisting of all sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $\mathbb{C}$ for which $\sum_{n=-\infty}^{\infty}|a_n|<\infty$ with convolution product $*$ defined by

$$a*b=\{c_n\}_{n=-\infty}^{\infty},\quad a=\{a_n\}_{n=-\infty}^{\infty},b=\{b_n\}_{n=-\infty}^{\infty}\in l^1(\mathbb{Z}),$$

where $c_n=\sum_{j=-\infty}^{\infty}a_{n-j}b_j$ for all $n\in\mathbb{Z}$ and with the l^1-norm $\|\cdot\|_1$ defined by

$$\|a\|_1=\sum_{n=-\infty}^{\infty}|a_n|,\quad a=\{a_n\}_{n=-\infty}^{\infty}\in l^1(\mathbb{Z}).$$

It is shown [11, Example 2] that $(l^1(\mathbb{Z}))^*$ is not a complex Banach $(l^1(\mathbb{Z}))^{**}$-module.

Let $\tau:\mathbb{Z}\longrightarrow\mathbb{Z}$ be a bijection additive map. Define

$$l^1(\mathbb{Z},\tau)=\left\{\{a_n\}_{n=-\infty}^{\infty}\in l^1(\mathbb{Z}):a_{\tau(n)}=\overline{a_n}\ (n\in\mathbb{Z})\right\}.$$

It is easy to see that $l^1(\mathbb{Z},\tau)$ is a real closed subalgebra of $l^1(\mathbb{Z})$ and

$$l^1(\mathbb{Z})=l^1(\mathbb{Z},\tau)\oplus il^1(\mathbb{Z},\tau).$$

Hence, $l^1(\mathbb{Z},\tau)$ is a real Banach algebra with the algebra norm $\|\cdot\|_1$ and $l^1(\mathbb{Z})$ is the complexification of $l^1(\mathbb{Z},\tau)$ with respect to the injective real algebra homomorphism $J:l^1(\mathbb{Z},\tau)\longrightarrow l^1(\mathbb{Z})$ defined by

$$J(a)=a,\quad a=\{a_n\}_{n=-\infty}^{\infty}\in l^1(\mathbb{Z},\tau).$$

Since $\|a-ib\|_1=\|a+ib\|_1$ for all $a=\{a_n\}_{n=-\infty}^{\infty},b=\{b_n\}_{n=-\infty}^{\infty}\in l^1(\mathbb{Z},\tau)$, we deduce that

$$\max\{\|a\|_1,\|b\|_1\}\leq\|a+ib\|_1\leq 2\max\{\|a\|_1,\|b\|_1\}$$

for all $a = \{a_n\}_{n=-\infty}^{\infty}, b = \{b_n\}_{n=-\infty}^{\infty} \in l^1(\mathbb{Z}, \tau)$. Therefore, $(l^1(\mathbb{Z}, \tau))^*$ is not a real Banach $(l^1(\mathbb{Z}, \tau))^{**}$-module by Theorem 2.6.

Note that the map $\tau : \mathbb{Z} \longrightarrow \mathbb{Z}$ is a bijection additive map if and only if either $\tau(n) = n$ for all $n \in \mathbb{Z}$ or $\tau(n) = -n$ for all $n \in \mathbb{Z}$.

We now discuss the relationship between the (-1)-weak amenability of A^{**} and (-1)-weak amenability of $(A_{\mathbb{C}})^{**}$. For this purpose we need the following lemma.

Lemma 2.8. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying $(*)$ condition and let $(A_{\mathbb{C}})^{**}$ be the second dual of $(A_{\mathbb{C}}, |||\cdot|||)$. Suppose that A^* is a real Banach A^{**}-module. Then:*

(i) *If $d \in Z^1_{\mathbb{R}}(A^{**}, A^*)$ and $\Phi \in A^{**}$, then $J_1(d(\Phi)) \in (A_{\mathbb{C}})^*$.*

(ii) *If $d \in Z^1_{\mathbb{R}}(A^{**}, A^*)$ then $\Delta_d \in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$, where the map $\Delta_d : (A_{\mathbb{C}})^{**} \longrightarrow (A_{\mathbb{C}})^*$ is defined by*

$$\Delta_d(J_2(\Phi) + iJ_2(\Psi)) = J_1(d(\Phi)) + iJ_1(d(\Psi)), \quad \Phi, \Psi \in A^{**}. \tag{2.23}$$

(iii) *The map $J_Z : Z^1_{\mathbb{R}}(A^{**}, A^*) \longrightarrow Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$ defined by*

$$J_Z(d) = \Delta_d, \quad d \in Z^1_{\mathbb{R}}(A^{**}, A^*) \tag{2.24}$$

is an injective real linear map.

(iv) *The complex linear space $Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$ is a complexification of the real linear space $Z^1_{\mathbb{R}}(A^{**}, A^*)$ with respect to the injective linear map J_Z.*

(v) *If $\varphi \in A^*$, then $J_Z(\delta_\varphi) = \delta_{J_1(\varphi)}$.*

(vi) *If $\lambda \in (A_{\mathbb{C}})^*$, then $\delta_\lambda = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I})$.*

(vii) *$H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\}$ if and only if $H^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*) = \{0\}$.*

Proof. Let $d \in Z^1_{\mathbb{R}}(A^{**}, A^*)$ and $\Phi \in A^{**}$. Then $d(\Phi) \in A^*$ and so $J_1(d(\Phi)) \in (A_{\mathbb{C}})^*$ by Lemma 2.2. Hence, (i) holds.

Let $d \in Z^1_{\mathbb{R}}(A^{**}, A^*)$ and define $\Delta_d : (A_{\mathbb{C}})^{**} \longrightarrow (A_{\mathbb{C}})^*$ by (2.23). Then Δ_d is well-defined by (i). It is easy to see that Δ_d is a complex linear map from $(A_{\mathbb{C}})^{**}$ to $(A_{\mathbb{C}})^*$. Since $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying $(*)$ condition, there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \le k_1 |||J(a) + iJ(b)||| \le k_2 \max\{\|a\|, \|b\|\}$$

for all $a, b \in A$. Applying part (v) of Lemma 2.2 and part (v) of Lemma 2.3, we get

$$\begin{aligned}
\|\Delta_d(J_2(\Phi) + iJ_2(\Psi))\|_{op} &= \|J_1(d(\Phi) + iJ_1(d(\Psi)))\|_{op} \\
&\le 4k_1 \max\{\|d(\Phi)\|_{op}, \|d(\Psi)\|_{op}\} \\
&\le 4k_1 \|d\|_{op} \max\{\|\Phi\|_{op}, \|\Psi\|_{op}\}
\end{aligned}$$

$$\begin{aligned}&\leq 4k_1\|d\|_{op}\|d\|_{op}4k_1\|J_2(\Phi)+iJ_2(\Psi)\|_{op}\\&=16k_1^2\|d\|_{op}\|J_2(\Phi)+iJ_2(\Psi)\|_{op},\end{aligned}$$

for all $\Phi,\Psi\in A^{**}$. Therefore, Δ_d is a bounded complex linear operator and

$$\|\Delta_d\|_{op}\leq 16k_1^2\|d\|_{op}.$$

By Theorem 2.6, $(A_{\mathbb{C}})^*$ is complex Banach $(A_{\mathbb{C}})^{**}$-module. Since $d\in Z^1_{\mathbb{R}}(A^{**},A^*)$, by Lemma 2.4, for all $\Phi,\Psi\in A^{**}$ we have

$$\begin{aligned}\Delta_d(J_2(\Phi)\Box J_2(\Psi))&=\Delta_d(J_2(\Phi\Box\Psi))\\&=J_1(d(\Phi\Box\Psi))\\&=J_1(d(\Phi)\cdot\Psi+\Phi\cdot d(\Psi))\\&=J_1(d(\Phi)\cdot\Psi)+J_1(\Phi\cdot d(\Psi))\\&=J_1(d(\Phi))\cdot J_2(\Psi)+J_2(\Phi)\cdot J_1(d(\Psi))\\&=\Delta_d(J_2(\Phi))J_2(\Psi))+J_2(\Phi)\cdot\Delta_d(J_2(\Psi)).\end{aligned}$$

This implies that for all $\Phi,\Psi,\Phi',\Psi'\in A^{**}$ we have

$$\begin{aligned}\Delta_d(J_2(\Phi)+&iJ_2(\Psi))\Box(J_2(\Phi'+iJ_2(\Psi')))\\&=\Delta_d((J_2(\Phi)\Box J_2(\Phi'))-(J_2(\Psi)\Box J_2(\Psi'))\\&\quad+i((J_2(\Phi)\Box J_2(\Psi'))+(J_2(\Psi)\Box J_2(\Phi')))\\&=\Delta_d(J_2(\Phi)\Box J_2(\Phi'))-\Delta_d(J_2(\Psi)\Box J_2(\Psi'))\\&\quad+i\Delta_d(J_2(\Phi)\Box J_2(\Psi'))+i\Delta_d(J_2(\Psi)\Box J_2(\Phi'))\\&=(\Delta_d(J_2(\Phi))\cdot J_2(\Phi')+J_2(\Phi)\cdot\Delta_d(J_2(\Phi'))\\&\quad-\Delta_d(J_2(\Psi))\cdot J_2(\Psi')-J_2(\Psi)\cdot\Delta_d(J_2(\Psi')))\\&\quad+i(\Delta_d(J_2(\Phi))\cdot J_2(\Psi')+J_2(\Phi)\cdot\Delta_d(J_2(\Psi')))\\&\quad+i\left(\Delta_d(J_2(\Psi))\cdot J_2(\Phi')+J_2(\Psi)\cdot\Delta_d(J_2(\Phi'))\right)\\&=(\Delta_d(J_2(\Phi))+i\Delta_d(J_2(\Psi)))\cdot\left(J_2(\Phi')+iJ_2(\Psi')\right)\\&\quad+(J_2(\Phi)+iJ_2(\Psi))\cdot\left(\Delta_d(J_2(\Phi'))+i\Delta_d(J_2(\Psi'))\right)\\&=\Delta_d(J_2(\Phi)+iJ_2(\Psi))\cdot\left(J_2(\Phi')+iJ_2(\Psi')\right)\\&\quad+(J_2(\Phi)+iJ_2(\Psi))\cdot\Delta_d(J_2(\Phi')+iJ_2(\Psi')).\end{aligned}$$

Therefore, $\Delta_d\in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**},(A_{\mathbb{C}})^*)$. Hence, (ii) holds.

It is clear that the map $J_Z:Z^1_{\mathbb{R}}(A^{**},A^*)\longrightarrow Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**},(A_{\mathbb{C}})^*)$, defined by (2.24), is a real linear map. Let $d\in Z^1_{\mathbb{R}}(A^{**},A^*)$ and $J_Z(d)=0$. Then $\Delta_d=0$ and so for each $\Phi\in A^{**}$ we have

$$0=\Delta_d(J_2(\Phi))=J_1(d(\Phi)).$$

This implies that $d(\Phi) = 0$ for all $\Phi \in A^{**}$, since J_1 is injective. Hence, $d = 0$ and so J_Z is injective.

Assume that $D \in \mathbb{Z}^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$. Define the maps $D_R, D_I : A^{**} \longrightarrow A^*$ by

$$D_R(\Phi) = (D(J_2(\Phi)))_R, \quad (\Phi \in A^{**}), \tag{2.25}$$

$$D_I(\Phi) = (D(J_2(\Phi)))_I, \quad (\Phi \in A^{**}). \tag{2.26}$$

By Lemma 2.1, D_R is well-defined. It is easy to see that D_R is a real linear map from A^{**} to A^*. Applying part (iii) of Lemma 2.1 and part (v) of Lemma 2.3, we have

$$\begin{aligned}
\|D_R(\Phi)\|_{op} &= \|(D(J_2(\Phi)))_R\|_{op} \\
&\leq \frac{k_2}{k_1}\|D(J_2(\Phi))\|_{op} \\
&\leq \frac{k_2}{k_1}\|D\|_{op}\|J_2(\Phi)\|_{op} \\
&\leq \frac{k_2}{k_1}\|D\|_{op}\frac{4k_2}{k_1}\|\Phi\|_{op} \\
&= \frac{4k_2^2}{k_1^2}\|D\|_{op}\|\Phi\|_{op}
\end{aligned}$$

for all $\Phi \in A^{**}$. Hence, D_R is a bounded real linear operator and

$$\|D_R\|_{op} \leq \frac{4k_2^2}{k_1^2}\|D\|_{op}.$$

On the other hand, for all $\Phi, \Psi \in A^{**}$ we have

$$\begin{aligned}
D_R(\Phi \Box \Psi) &= (D(J_2(\Phi \Box \Psi)))_R \\
&= (D(J_2(\Phi) \Box J_2(\Psi))_R \\
&= (D(J_2(\Phi)) \cdot J_2(\Psi) + J_2(\Phi) \cdot D(J_2(\Psi)))_R \\
&= (D(J_2(\Phi)) \cdot J_2(\Psi))_R + (J_2(\Phi) \cdot D(J_2(\Psi)))_R \\
&= (D(J_2(\Phi)))_R \cdot \Psi + \Phi \cdot D(J_2(\Psi)))_R \\
&= D_R(\Phi) \cdot \Psi + \Phi \cdot D_R(\Psi).
\end{aligned}$$

Therefore, D_R is a real A^*-derivation on A^{**} and so $D_R \in Z^1_{\mathbb{R}}(A^{**}, A^*)$.

Similarly, we can show that $D_I \in Z^1_{\mathbb{R}}(A^{**}, A^*)$.

Now we show that

$$D = J_Z(D_R) + iJ_Z(D_I). \tag{2.27}$$

Let $\Phi \in A^{**}$. For each $a \in A$ we have

$$\begin{aligned}
D(J_2(\Phi))(J(a)) =& \mathrm{Re}\, D(J_2(\Phi))(J(a)) + i\mathrm{Im}\, D(J_2(\Phi))(J(a)) \\
=& D_R(\Phi)(a) + iD_I(\Phi)(a)
\end{aligned}$$

$$\begin{aligned}&=J_1(D_R(\Phi))(J(a))+iJ_1(D_I(\Phi))(J(a))\\&=(J_1(D_R(\Phi))+iJ_1(D_I(\Phi))(J(a))\\&=(J_Z(D_R)(J_2(\Phi))+iJ_Z(D_I)(J_2(\Phi)))(J(a))\\&=((J_Z(D_R)+iJ_Z(D_I))(J_2(\Phi)))(J(a)).\end{aligned}$$

This implies that

$$D(J_2(\Phi))=(J_Z(D_R)+iJ_Z(D_I))(J_2(\Phi)), \tag{2.28}$$

since $D(J_2(\Phi))$ and $(J_Z(D_R)+iJ_Z(D_I))(J_2(\Phi))$ are complex linear mappings from $A_{\mathbb{C}}$ to $\mathbb{C}$. Since D and $J_Z(D_R)+iJ_Z(D_I)$ are complex linear mappings from $(A_{\mathbb{C}})^{**}$ to $(A_{\mathbb{C}})^{*}$ and (2.28) holds for each $\Phi\in A^{**}$, we deduce that

$$D(J_2(\Phi)+iJ_2(\Psi))=(J_Z(D_R)+iJ_Z(D_I))(J_2(\Phi)+iJ_2(\Psi))$$

for all $\Phi,\Psi\in A^{**}$. Hence, (2.27) holds. Since (2.27) holds for all $D\in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**},(A_{\mathbb{C}})^{*})$, we have

$$Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**},(A_{\mathbb{C}})^{*})=J_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))+iJ_Z(Z^1_{\mathbb{R}}(A^{**},A^{*})). \tag{2.29}$$

Let $D\in J_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))\cap iJ_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))$. Then there exist two functions $d_1,d_2\in Z^1_{\mathbb{R}}(A^{**},A^{*})$ such that $D=J_Z(d_1)=iJ_Z(d_2)$. Hence, for each $\Phi\in A^{**}$ we have

$$\begin{aligned}J_1(d_1(\Phi))&=(J_Z(d_1))(J_2(\Phi))\\&=(iJ_Z(d_2))(J_2(\Phi))\\&=i(J_Z(d_2))(J_2(\Phi))\\&=iJ_1(d_2(\Phi)),\end{aligned}$$

and so $J_1(d_1(\Phi))=0$, since $J_1(A^*)\cap iJ_1(A^*)=\{0\}$. This implies that $d_1(\Phi)=0$ for all $\Phi\in A^{**}$, since J_1 is injective. Hence, $d_1=0$ and so $D=J_Z(d_1)=0$. Therefore,

$$J_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))\cap iJ_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))=\{0\}. \tag{2.30}$$

From (2.29) and (2.30) we obtain

$$Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**},(A_{\mathbb{C}})^{*})=J_Z(Z^1_{\mathbb{R}}(A^{**},A^{*}))\oplus iJ_Z(Z^1_{\mathbb{R}}(A^{**},A^{*})).$$

Therefore, (iv) holds.

Let $\varphi\in A^*$. Since

$$\begin{aligned}J_Z(\delta_\varphi)(J_2(\Phi)+iJ_2(\Psi))&=J_1(\delta_\varphi(\Phi))+iJ_1(\delta_\varphi(\Psi))\\&=J_1(\Phi\cdot\varphi-\varphi\cdot\Phi)+iJ_1(\Psi\cdot\varphi-\varphi\cdot\Psi)\\&=(J_1(\Phi\cdot\varphi)-J_1(\varphi\cdot\Phi))\\&\quad+i\,(J_1(\Psi\cdot\varphi)-J_1(\varphi\cdot\Psi))\\&=(J_2(\Phi)\cdot J_1(\varphi)-J_1(\varphi)\cdot J_2(\Phi))\end{aligned}$$

$$
\begin{aligned}
&+ i\left(J_2(\Psi)\cdot J_1(\varphi) - J_1(\varphi)\cdot J_2(\Psi)\right)\\
=&\left(J_2(\Phi) + iJ_2(\Psi)\right)\cdot J_1(\varphi)\\
&- J_1(\varphi)\cdot\left(J_2(\Phi) + iJ_2(\Psi)\right)\\
=&\delta_{J_1(\varphi)}\left(J_2(\Phi) + iJ_2(\Psi)\right)
\end{aligned}
$$

for all $\Phi, \Psi \in A^{**}$, we deduce that $J_Z(\delta_\varphi) = \delta_{J_1(\varphi)}$. Hence (v) holds.

Let $\lambda \in (A_{\mathbb{C}})^*$. By parts (ii) and (iii) of Lemma 2.1 and part (iii) of Lemma 2.2, we have $\lambda_R, \lambda_I \in A^*$ and

$$\lambda = J_1(\lambda_R) + iJ_1(\lambda_I). \tag{2.31}$$

Since $J_Z(\delta_{\lambda_R})$, $\delta_{J_1(\lambda_R)} \in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$ and

$$
\begin{aligned}
J_Z(\delta_{\lambda_R})(J_2(\Phi)) &= J_1(\delta_{\lambda_R}(\Phi))\\
&= J_1(\Phi\cdot\lambda_R - \lambda_R\cdot\Phi)\\
&= J_1(\Phi\cdot\lambda_R) - J_1(\lambda_R\cdot\Phi)\\
&= J_2(\Phi)\cdot J_1(\lambda_R) - J_1(\lambda_R)\cdot J_2(\Phi)\\
&= \delta_{J_1(\lambda_R)}(J_2(\Phi))
\end{aligned}
$$

for all $\Phi \in A^{**}$, we conclude that

$$J_Z(\delta_{\lambda_R})(J_2(\Phi) + iJ_2(\Psi)) = \delta_{J_1(\delta_R)}(J_2(\Phi) + iJ_2(\Psi))$$

for all $\Phi, \Psi \in A^{**}$. Hence,

$$J_Z(\delta_{\lambda_R}) = \delta_{J_1(\lambda_R)}. \tag{2.32}$$

Similar to the argument above we can obtian

$$J_Z(\delta_{\lambda_I}) = \delta_{J_1(\lambda_I)}. \tag{2.33}$$

Applying (2.32), (2.33) and (2.31), we get

$$
\begin{aligned}
J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}) &= \delta_{J_1(\lambda_R)} + i\delta_{J_1(\lambda_I)}\\
&= \delta_{J_1(\lambda_R)+iJ_1(\lambda_I)}\\
&= \delta_\lambda.
\end{aligned}
$$

Hence, (vi) holds.

To prove (vii), we first assume that

$$H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\}. \tag{2.34}$$

Let $D \in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$. By (iv), there exist unique elements $d, d' \in Z^1_{\mathbb{R}}(A^{**}, A^*)$ such that

$$D = J_Z(d) + iJ_Z(d'). \tag{2.35}$$

By (2.34), there exist $\varphi, \varphi' \in A^*$ such that

$$d = \delta_\varphi, \qquad d' = \delta_{\varphi'}. \tag{2.36}$$

Set $\lambda = J_1(\varphi) + iJ_1(\varphi')$. Then $\lambda \in (A_{\mathbb{C}})^*$ and

$$\varphi = \lambda_R, \qquad \varphi' = \lambda_I. \tag{2.37}$$

From (2.35), (2.36) and (2.37) we obtain

$$D = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}). \tag{2.38}$$

Since $\lambda \in (A_{\mathbb{C}})^*$, we deduce that

$$\delta_\lambda = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}), \tag{2.39}$$

by (vi). From (2.38) and (2.39), we have $D = \delta_\lambda$ and so

$$H^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*) = \{0\}.$$

We now assume that

$$H^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*) = \{0\}. \tag{2.40}$$

Let $d \in Z^1_{\mathbb{R}}(A^{**}, A^*)$. Then $J_Z(d) \in Z^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*)$. By (2.40), there exists $\lambda \in (A_{\mathbb{C}})^*$ such that $J_Z(d) = \delta_\lambda$, and so by (vi) we have

$$J_Z(d) + iJ_Z(0) = J_Z(d) = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}). \tag{2.41}$$

Applying (2.41) and (iv), we deduce that $J_Z(d) = J_Z(\delta_{\lambda_R})$ and so $d = \delta_{\lambda_R}$, since J_Z is injective. Therefore, $H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\}$ and so (vii) holds. $\square$

Theorem 2.9. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexifiction of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition, and let $(A_{\mathbb{C}})^*$ be the dual space of $(A_{\mathbb{C}}, |||\cdot|||)$. Then A^{**} is (-1)-weakly amenable if and only if $(A_{\mathbb{C}})^{**}$ is (-1)-weakly amenable.*

Proof. We first assume that A^{**} is (-1)-weakly amenable. Then A^* is a real Banach A^{**}-module and $H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\}$. Hence, $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module by Theorem 2.6 and $H^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*) = \{0\}$ by part (vii) of Lemma 2.8. Therefore, $(A_{\mathbb{C}})^{**}$ is (-1)-weakly amenable.

We now assume that $(A_{\mathbb{C}})^{**}$ is (-1)-weakly amenable. Then $(A_{\mathbb{C}})^*$ is a complex Banach $(A_{\mathbb{C}})^{**}$-module and $H^1_{\mathbb{C}}((A_{\mathbb{C}})^{**}, (A_{\mathbb{C}})^*) = \{0\}$. Hence, A^* is a real Banach A^{**}-module by Theorem 2.6 and so we conclude that $H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\}$ by part (vii) of Lemma 2.8. Therefore, A^{**} is (-1)-weakly amenable. $\square$

Here, as applications of Theorem 2.9, we give some examples of real Banach algebras which their second duals of some them are and of others are not (-1)-weakly amenable.

Example 2.10. Let $A = \mathbb{R}$ with the zero multiplication.Then A is a real Banach algebra with the Euclidean norm $|\cdot|$. Set $A_{\mathbb{C}} = \mathbb{C}$ with the zero multiplication. Clearly, $A_{\mathbb{C}}$ is a complex Banach algebra with Euclidean norm $|\cdot|$ and $A_{\mathbb{C}} = A \oplus iA$. Hence, $A_{\mathbb{C}}$ is a complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = a \quad (a \in \mathbb{R})$. Moreover,

$$\max\{|a|, |b|\} \leq |a + ib| \leq 2 \max\{|a|, |b|\},$$

for all $a, b \in A$. It is known [11, Example 2.2] that $(A_{\mathbb{C}})^{**}$ is not (-1)-weakly amenable. Therefore, A^{**} is not (-1)-weakly amenable by Theorem 2.9.

Example 2.11. Let S be a discrete semigroup. We denote by $l^1(S)$ the set of all complex-valued functions f on S for which $\sum_{s\in S} |f(s)| < \infty$. Then $l^1(S)$ is a self-adjoint complex Banach algebra with the convolution product $*$ defined by

$$(f * g)(r) = \sum_{s,t\in S, st=r} f(s)g(t), \quad f, g \in l^1(S),$$

and with the algebra norm $\|\cdot\|_1$ defined by

$$\|f\|_1 = \sum_{s\in S} |f(s)|, \quad f \in l^1(S).$$

Let $\tau : S \longrightarrow S$ be a self-map of S satisfying $\tau(st) = \tau(s)\tau(t)$ for all $s, t \in S$ and $\tau(\tau(s)) = s$ for all $s \in S$. It is easy to see $\bar{f} \circ \tau \in l^1(S)$ for all $f \in l^1(S)$. Define

$$l^1(S, \tau) = \{f \in l^1(S) : \bar{f} \circ \tau = f\}.$$

Then $l^1(S, \tau)$ is a real closed subalgebra of $l^1(S)$ and

$$l^1(S) = l^1(S, \tau) \oplus il^1(S, \tau).$$

Hence, $l^1(S)$ is the complexification of $l^1(S, \tau)$ with respect to the injective real algebra homomorphism $J : l^1(S, \tau) \longrightarrow l^1(S)$ defined by $J(f) = f \quad (f \in l^1(S, \tau))$. Since $\|f - ig\|_1 = \|f + ig\|_1$ for all $f, g \in l^1(S, \tau)$, we deduce that

$$\max\{\|f\|_1, \|g\|_1\} \leq \|f + ig\|_1 \leq 2 \max\{\|f\|_1, \|g\|_1\},$$

for all $f, g \in l^1(S, \tau)$. It is known [11, Example 2.3] that if $S^2 \neq S$ then $(l^1(S))^{**}$ is not (-1)-weakly amenable. Therefore, if $S^2 \neq S$ then $(l^1(S, \tau))^{**}$ is not (-1)-weakly amenable by Theorem 2.9.

Example 2.12. Let $\mathbb{N}^{<\omega} = \cup_{k\in\mathbb{N}}\mathbb{N}^k$ and let P be the set of all elements $p = (p_1, \ldots, p_k) \in \mathbb{N}^{<\omega}$ such that $k \geq 2$ and $p_j < p_{j+1}$ for all $j \in$

$\{1,\ldots,k-1\}$. For a sequence $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{F}$ and for $p=(p_1,\ldots p_k)\in P$, define $N(\alpha,p)$ by

$$2(N(\alpha,p))^2=\left(\sum_{j=1}^{k-1}|\alpha_{p_{j+1}}-\alpha_{p_j}|^2\right)+|\alpha_{p_n}-\alpha_{p_1}|^2.$$

For each sequence $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{F}$, we set

$$N(\alpha)=\sup\{N(\alpha,p):p\in P\}.$$

Then $N(\alpha)\in[0,\infty]$ for all sequence $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{F}$. Define

$$\mathfrak{J}_{\mathbb{F}}=\{\alpha=\{\alpha_n\}_{n=1}^{\infty}:\alpha\in\mathbb{F},N(\alpha)<\infty\}.$$

Then $\mathfrak{J}_{\mathbb{F}}$ is a closed subalgebra of Banach algebra $(l^{\infty}_{\mathbb{F}}(\mathbb{N}),\|\cdot\|_{\infty})$ over $\mathbb{F}$, where $l^{\infty}_{\mathbb{F}}(\mathbb{N})$ is the set of all sequence $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{F}$ for which $\sup\{|\alpha_n|:n\in\mathbb{N}\}<\infty$ and $\|\cdot\|_{\infty}$ is the algebra norm on $l^{\infty}_{\mathbb{F}}(\mathbb{N})$ over $\mathbb{F}$ defined by

$$\|\alpha\|_{\infty}=\sup\{|\alpha_n|:n\in\mathbb{N}\},\quad(\alpha=\{\alpha_n\}_{n=1}^{\infty}\in l^{\infty}_{\mathbb{F}}(\mathbb{N})).$$

$\mathfrak{J}_{\mathbb{F}}$ is called the James algebra over $\mathbb{F}$. It is clear that $\mathfrak{J}_{\mathbb{R}}$ is a real subalgebra of $\mathfrak{J}_{\mathbb{C}}$ and $\mathfrak{J}_{\mathbb{C}}=\mathfrak{J}_{\mathbb{R}}\oplus i\mathfrak{J}_{\mathbb{R}}$. Hence, $\mathfrak{J}_{\mathbb{C}}$ is a complexification of $\mathfrak{J}_{\mathbb{R}}$ with the injective real algebra homomorphism $J:\mathfrak{J}_{\mathbb{R}}\longrightarrow\mathfrak{J}_{\mathbb{C}}$ defined by $J(\alpha)=\alpha\qquad(\alpha\in\mathfrak{J}_{\mathbb{R}})$. It is easy to see that

$$\max\{\|\alpha\|_{\infty},\|\beta\|_{\infty}\}\leq\|\alpha+i\beta\|_{\infty}\leq 2\max\{\|\alpha\|_{\infty},\|\beta\|_{\infty}\},$$

for all $\alpha=\{\alpha_n\}_{n=1}^{\infty}$, $\beta=\{\beta\}_{n=1}^{\infty}\in\mathfrak{J}_{\mathbb{R}}$.

By [6, Theorem 4.1.45], we have some properties of $\mathfrak{J}_{\mathbb{C}}$ as:

(i) $\mathfrak{J}_{\mathbb{C}}$ is Arens regular,
(ii) $\mathfrak{J}_{\mathbb{C}}$ is weakly amenable,
(iii) $\mathfrak{J}_{\mathbb{C}}$ is not amenable.

It is shown [8, Example 2.2] that $(\mathfrak{J}_{\mathbb{C}})^{**}$ is (-1)-weakly amenable. Therefore, we deduce that $\mathfrak{J}_{\mathbb{R}}$ is weakly amenable by [2, Theorem 2.5], $\mathfrak{J}_{\mathbb{R}}$ is not amenable by [2, Theorem 2.4], $\mathfrak{J}_{\mathbb{R}}$ is Arens regular by Theorem 2.5 and $(\mathfrak{J}_{\mathbb{R}})^{**}$ is (-1)-weakly amenable by Theorem 2.9.

Example 2.13. Let $1<p<\infty$ and let $l^p(\mathbb{Z})$ denote the set of all sequences $\alpha=\{\alpha_n\}_{n=-\infty}^{\infty}$ in $\mathbb{C}$ for which $\sum_{n=-\infty}^{\infty}|\alpha_n|^p<\infty$. Then $l^p(\mathbb{Z})$ with the pointwise addition and scalar multiplication is a complex Banach space with the norm $\|\cdot\|_p$ defined by

$$\|\alpha\|_p=\left(\sum_{n=-\infty}^{\infty}|\alpha_n|^p\right)^{\frac{1}{p}},\qquad(\alpha=\{\alpha_n\}_{n=-\infty}^{\infty}\in l^p(\mathbb{Z})).$$

Moreover, $l^p(\mathbb{Z})$ with the pointwise multiplication becomes a complex algebra and $\|\cdot\|_p$ is a complete algebra norm on $l^p(\mathbb{Z})$. Hence, $(l^p(\mathbb{Z}),\|\cdot$

$\|_p)$ is a complex Banach algebra. For each $m \in \mathbb{Z}$ we have $e_m \in l^p(\mathbb{Z})$ and $e_m e_m = e_m$ whenever $e_m = \{e_{m,n}\}_{n=-\infty}^{\infty}$ and

$$e_{m,n} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \qquad (n \in \mathbb{Z}).$$

Moreover, $l^p(\mathbb{Z})$ generates by $\{e_m : m \in \mathbb{Z}\}$. Hence, $l^p(\mathbb{Z})$ is weakly amenable by [6, Proposition 2.8.72(i)]. Therefore, $(l^p(\mathbb{Z}))^{**}$ is (-1)-weakly amenable since $l^p(\mathbb{Z})$ is reflexive.

Let $\tau : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a bijection additive map. Define

$$l^p(\mathbb{Z}, \tau) = \left\{\alpha = \{\alpha_n\}_{n=-\infty}^{\infty} \in l^p(\mathbb{Z}) : \alpha_{\tau(n)} = \overline{\alpha_n}, \quad (n \in \mathbb{Z})\right\}.$$

It is easy to see that $l^p(\mathbb{Z}, \tau)$ is closed real subalgebra of $l^p(\mathbb{Z})$ and $l^p(\mathbb{Z}) = l^p(\mathbb{Z}, \tau) \oplus i l^p(\mathbb{Z}, \tau)$. Hence, $(l^p(\mathbb{Z}, \tau), \|\cdot\|_p)$ is a real Banach algebra and $l^p(\mathbb{Z})$ is a complexification of $l^p(\mathbb{Z}, \tau)$ with respect to the injective real algebra homomorphism $J : l^p(\mathbb{Z}, \tau) \longrightarrow l^p(\mathbb{Z})$ defined by $J(\alpha) = \alpha \quad (\alpha \in l^p(\mathbb{Z}, \tau))$. Since $\|\alpha - i\beta\|_p = \|\alpha + i\beta\|_p$ for all $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}$, $\beta = \{\beta_n\}_{n=-\infty}^{\infty} \in l^p(\mathbb{Z}, \tau)$, we deduce that

$$\max\{\|\alpha\|_p, \|\beta\|_p\} \leq \|\alpha + i\beta\|_p \leq 2\max\{\|\alpha\|_p, \|\beta\|_p\}$$

for all $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}, \beta = \{\beta_n\}_{n=-\infty}^{\infty} \in l^p(\mathbb{Z}, \tau)$. Therefore, $l^p(\mathbb{Z}, \tau)$ is reflexive by the reflexivity of $l^p(\mathbb{Z})$ and part (vii) of Lemma 2.3, $l^p(\mathbb{Z}, \tau)$ is weakly amenable by [2, Theorem 2.5] and $(l^p(\mathbb{Z}), \tau)^{**}$ is (-1)-weakly amenable by Theorem 2.9.

Example 2.14. Let X be a compact Hausdorff space. We denote by $C_{\mathbb{F}}(X)$ the algebra of all $\mathbb{F}$-valued continuous functions on X over $\mathbb{F}$. Then $C_{\mathbb{F}}(X)$ is a Banach algebra over $\mathbb{F}$ with the uniform norm $\|\cdot\|_X$ defined by

$$\|f\|_X = \sup\{|f(x)| : x \in X\}, \quad (f \in C(X)).$$

We write $C(X)$ instead of $C_{\mathbb{C}}(X)$.

A self-map $\tau : X \longrightarrow X$ is called a topological involution on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Clearly, $\bar{f} \circ \tau \in C(X)$ for all $f \in C(X)$. Define

$$C(X, \tau) = \{f \in C(X) : \bar{f} \circ \tau = f\}.$$

Then $C(X, \tau)$ is a real closed subalgebra of $C(X), 1_X \in C(X, \tau)$ and $i1_X \notin C(X, \tau)$, where 1_X is the constant function on X with value 1. Moreover, $C(X) = C(X, \tau) \oplus iC(X, \tau)$. Hence, $C(X)$ is a complexification of $C(X, \tau)$ with respect to the injective real algebra homomorphism $J : C(X, \tau) \longrightarrow C(X)$ defined by $J(f) = f, \quad (f \in C(X, \tau))$. Since $\|f - ig\|_X = \|f + ig\|_X$ for all $f, g \in C(X, \tau)$, we deduce that

$$\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2\max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C(X, \tau)$. Real Banach algebra $C(X, \tau)$ was first defined by Kulkarni and Limaye in [14]. For further general facts about $C(X, \tau)$ and certain real subalgebras we refer to [15].

Clearly, $C(X)$ is a complex C^*-algebra with the natural algebra involution $f \mapsto \bar{f} : C(X) \longrightarrow C(X)$. Hence, $C(X)$ is Arens regular and, by [11, Corollary 3.7], $(C(X))^{**}$ is (-1)-weakly amenable. Therefore, if τ is a topological involution on X then $C(X, \tau)$ is Arens regular by Theorem 2.5 and $(C(X, \tau))^{**}$ is (-1)-weakly amenable by theorem 2.9.

Example 2.15. Let (X, d) be an infinite compact metric space and let $\alpha \in (0, 1]$. We denote by $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ the set of all $\mathbb{F}-$valued functions f on X for which

$$p_{(X,d^\alpha)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X,\, x \neq y\right\} < \infty.$$

Clearly, $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ is a subalgebra of $C_{\mathbb{F}}(X)$ and $1_X \in \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$. Moreover, $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ is a Banach algebra over $\mathbb{F}$ with the α-Lipschitz norm $\|\cdot\|_{\mathrm{Lip(X,d^\alpha)}}$ defined by

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f), \quad (f \in \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)).$$

$\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ is called the Lipschits algebra of order α on (X, d) over $\mathbb{F}$. This algebra was first introduced by Sherbert in [19]. We write $\mathrm{Lip}(X, d^\alpha)$ instead of $\mathrm{Lip}_{\mathbb{C}}(X, d^\alpha)$.

Let (X, d) be a metric space. A Lipschitz mapping on (X, d) is a self-map $\tau : X \longrightarrow X$ for which there exist a positive constant M such that $d(\tau(x), \tau(y)) \leq Md(x, y)$ for all $x, y \in X$. For a Lipschitz mapping $\tau : X \longrightarrow X$ on (X, d), the constant Lipschitz of τ is denoted by $p(\tau)$ and defined by

$$p(\tau) = \sup\left\{\frac{d(\tau(x), \tau(y))}{d(x, y)} : x, y \in X,\, x \neq y\right\}.$$

A self-map $\tau : X \longrightarrow X$ is called a Lipschitz involution on (X, d) if τ is a Lipschitz mapping and $\tau(\tau(x)) = x$ for all $x \in X$.

Let (X, d) be a compact metric space, let $\alpha \in (0, 1]$ and let $\tau : X \longrightarrow X$ be a Lipschitz involution on (X, d). It is easy to see that $\bar{f} \circ \tau \in \mathrm{Lip}(X, d^\alpha)$ for all $f \in \mathrm{Lip}(X, d^\alpha)$. Define

$$\mathrm{Lip}(X, d^\alpha, \tau) = \left\{f \in \mathrm{Lip}(X, d^\alpha) : \bar{f} \circ \tau = f\right\}.$$

Then $\mathrm{Lip}(X, d^\alpha, \tau)$ is a real closed subalgebra of $\mathrm{Lip}(X, d^\alpha)$, containing 1_X, $i1_X \notin \mathrm{Lip}(X, d^\alpha, \tau)$ and

$$\mathrm{Lip}(X, d^\alpha) = \mathrm{Lip}(X, d^\alpha, \tau) \oplus i\mathrm{Lip}(X, d^\alpha, \tau).$$

Hence, $\left(\mathrm{Lip}(X, d^\alpha, \tau), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}\right)$ is a real Banach algebra and the complex algebra $\mathrm{Lip}(X, d^\alpha)$ is a complexification of $\mathrm{Lip}(X, d^\alpha, \tau)$ with respect to the injective real algebra homomorphism $J : \mathrm{Lip}(X, d^\alpha, \tau) \longrightarrow \mathrm{Lip}(X, d^\alpha)$ by $J(f) = f \quad (f \in \mathrm{Lip}(X, d^\alpha, \tau))$. Moreover,

$$\begin{aligned}\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} &\leq C\|f + ig\|_{\mathrm{Lip}(X,d^\alpha)} \\ &\leq 2C \max\left\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\right\}\end{aligned}$$

for all $f, g \in \mathrm{Lip}(X, d^\alpha, \tau)$, where $C = (p(\tau))^\alpha$ (see [1]).

By [20, Theorem 9.2], $\mathrm{Lip}(X, d^\alpha)$ has a nonzero continuous point derivation. Hence, $(\mathrm{Lip}(X, d^\alpha))^{**}$ is not (-1)-weakly amenable by [11, Theorem 2.6]. Therefore, if $\tau : X \longrightarrow X$ is a Lipschitz involution on (X, d) then $(\mathrm{Lip}(X, d^\alpha, \tau))^{**}$ is not (-1)-weakly amenable by Theorem 2.9.

Example 2.16. Let (X, d) be a compact metric space, let K be a nonempty compact subset of X and let $\alpha \in (0, 1]$. We denote by $\mathrm{Lip}(X, K, d^\alpha)$ the set of all $f \in C(X)$ for which $f|_K \in \mathrm{Lip}(K, d^\alpha)$. Then $\mathrm{Lip}(X, K, d^\alpha)$ is a complex subalgebra of $C(X)$ and $\mathrm{Lip}(X, d^\alpha)$ is a complex subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$. Moreover, $\mathrm{Lip}(X, K, d^\alpha) = C(X)$ if K is finite and $\mathrm{Lip}(X, K, d^\alpha) = \mathrm{Lip}(X, d^\alpha)$ if $X \setminus K$ is finite.

Furthermore, $\mathrm{Lip}(X, K, d^\alpha)$ is a complex Banach algebra with the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)}$ defined by

$$\|f\|_{\mathrm{Lip}(X,K,d^\alpha)} = \|f\|_X + p_{(K,d^\alpha)}(f), \quad f \in \mathrm{Lip}(X, K, d^\alpha).$$

$\mathrm{Lip}(X, K, d^\alpha)$ is called extended Lipschitz algebra of order α on (X, d) with respect to K. This algebra was first studied in [9].

By [16, Theorem 3.3], $\mathrm{Lip}(X, K, d^\alpha)$ has a nonzero continuous point derivation if $\mathrm{int}(K) \cap K' \neq \emptyset$ where $\mathrm{int}(K)$ is the set of all interior points of K and K' is the set of all limit points of K in (X, d). Therefore, if $\mathrm{int}(K) \cap K' \neq \emptyset$ then $(\mathrm{Lip}(X, K, d^\alpha))^{**}$ is not (-1)-weakly amenable by [12, Theorem 2.6].

Let (X, d) be a compact metric space, let K be compact subset of X, let $\alpha \in (0, 1]$ and let τ be a Lipschitz involution on (X, d) such that $\tau(K) = K$. Clearly, $\bar{f} \circ \tau \in \mathrm{Lip}(X, K, d^\alpha)$ for all $f \in \mathrm{Lip}(X, K, d^\alpha)$. Define

$$\mathrm{Lip}(X, K, d^\alpha, \tau) = \{f \in \mathrm{Lip}(X, K, d^\alpha) : \bar{f} \circ \tau = f\}.$$

It is easy to see that $\mathrm{Lip}(X, K, d^\alpha, \tau)$ is a real closed subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$, $1_X \in \mathrm{Lip}(X, K, d^\alpha, \tau)$ and

$$\mathrm{Lip}(X, K, d^\alpha) = \mathrm{Lip}(X, K, d^\alpha, \tau) \oplus i\mathrm{Lip}(X, K, d^\alpha, \tau).$$

Hence, $(\mathrm{Lip}(X, K, d^\alpha, \tau), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a real Banach algebra and $\mathrm{Lip}(X, K, d^\alpha)$ is a complexification of $\mathrm{Lip}(X, K, d^\alpha, \tau)$ with the injective real algebra homomorphism $J : \mathrm{Lip}(X, K, d^\alpha, \tau) \longrightarrow \mathrm{Lip}(X, K, d^\alpha)$

defined by $J(f) = f \quad (f \in \mathrm{Lip}(X, K, d^\alpha, \tau)$. Moreover,

$$\begin{aligned}\max\{\|f\|_B, \|g\|_B\} &\leq C\|f + ig\|_B \\ &\leq 2C \max\{\|f\|_B, \|g\|_B\},\end{aligned}$$

for all $f, g \in \mathrm{Lip}(X, K, d^\alpha, \tau)$ where $B = \mathrm{Lip}(X, K, d^\alpha)$ and $C = (p(\tau))^\alpha$. Therefore, if $\mathrm{int}(K) \cap K' \neq \emptyset$ and $\tau : X \longrightarrow X$ is a Lipschitz involution on (X, d) with $\tau(K) = K$, then $\mathrm{Lip}(X, K, d^\alpha, \tau)$ is not weakly amenable by [2, Theorem 2.5] and $(\mathrm{Lip}(X, K, d^\alpha, \tau))^{**}$ is not (-1)-weakly amenable by Theorem 2.9.

Example 2.17. Let (X, d) be an infinite compact metric space and $\alpha \in (0, 1)$. We denote by $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ the set of all $f \in \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ for which $\lim_{d(x,y)\to 0} \dfrac{|f(x) - f(y)|}{d^\alpha(x, y)} = 0$, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\dfrac{|f(x) - f(y)|}{d^\alpha(x, y)} < \varepsilon$ whenever $x, y \in X$ with $0 < d(x, y) < \delta$. Then $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ is a closed subalgebra of $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ over $\mathbb{F}$, and $1_X \in \mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$. Hence, $(\mathrm{lip}_{\mathbb{F}}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ is a Banach algebra over $\mathbb{F}$. This algebra is called the little Lipschitz algebra of order α on (X, d) over $\mathbb{F}$ and was first introduced by Sherbert in [20]. We write $\mathrm{lip}(X, d^\alpha)$ instead of $\mathrm{lip}_{\mathbb{C}}(X, d^\alpha)$.

Let (X, d) be an infinite compact metric space, let $\alpha \in (0, 1)$ and let $B = \mathrm{lip}(X, d^\alpha)$. For each $x \in X$ the map $e_{B,x} : B \longrightarrow \mathbb{C}$ defined by

$$e_{B,x}(f) = f(x), \quad f \in B,$$

belongs to B^*. Moreover, $\|e_{B,x} - e_{B,y}\|_{op} \leq d^\alpha(x, y)$ for all $x, y \in X$ and so the map $E_{B,X} : X \longrightarrow B^*$ defined by

$$E_{B,X}(x) = e_{B,x}, \quad x \in X,$$

is a continuous function from (X, d) to $(B^*, \|\cdot\|_{op})$. We know [4, Theorem 3.5] that the map $\eta : B^{**} \longrightarrow \mathrm{Lip}(X, d^\alpha)$ defined by

$$\eta(\Lambda) = \Lambda \circ E_{B,X}, \quad \Lambda \in B^{**},$$

is a complex linear isometry from $(B^{**}, \|\cdot\|_{op})$ onto $(\mathrm{Lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$. It is shown [4, Theorem 3.8] that B is Arens regular and η is an algebra homomorphism. This implies that B^* is a complex Banach B^{**}-module.

Let $\tau : X \longrightarrow X$ be a Lipschitz involution on (X, d). It is easy to see that $\bar{f} \circ \tau \in B$ for all $f \in B$. Define

$$\mathrm{lip}(X, d^\alpha, \tau) = \{f \in B = \mathrm{lip}(X, d^\alpha) : \bar{f} \circ \tau = f\}.$$

Then $\mathrm{lip}(X, d^\alpha, \tau)$ is a real closed subalgebra of B and

$$\mathrm{lip}(X, d^\alpha) = \mathrm{lip}(X, d^\alpha, \tau) \oplus i\mathrm{lip}(X, d^\alpha, \tau).$$

Therefore, $(\operatorname{lip}(X, d^\alpha, \tau), \|\cdot\|_{\operatorname{lip}(X,d^\alpha)})$ is a real Banach algebra and the complex algebra $\operatorname{lip}(X, d^\alpha)$ is a complexification of $\operatorname{lip}(X, d^\alpha, \tau)$ with respect to the injective real algebra homomorphism $J : \operatorname{lip}(X, d^\alpha, \tau) \longrightarrow \operatorname{lip}(X, d^\alpha)$ defined by $J(f) = f \quad (f \in \operatorname{lip}(X, d^\alpha, \tau))$. Moreover,

$$\begin{aligned}\max\{\|f\|_{\operatorname{Lip}(X,d^\alpha)}, \|g\|_{\operatorname{Lip}(X,d^\alpha)}\} &\leq C\|f + ig\|_{\operatorname{Lip}(X,d^\alpha)} \\ &\leq 2C \max\{\|f\|_{\operatorname{Lip}(X,d^\alpha)}, \|g\|_{\operatorname{Lip}(X,d^\alpha)}\},\end{aligned}$$

for all $f, g \in \operatorname{lip}(X, d^\alpha, \tau)$ where $C = (p(\tau))^\alpha$ (see [1]).

By Theorem 2.5, we deduce that $\operatorname{lip}(X, d^\alpha, \tau)$ is Arens regular.

Let $\mathbb{T} = \{Z \in \mathbb{C} : |z| = 1\}$, let d be the Euclidean metric on $\mathbb{T}$ and let $\alpha \in (\frac{1}{2}, 1)$. By [8, Theorem 2.2], $(\operatorname{lip}(\mathbb{T}, d^\alpha))^{**}$ is not (-1)-weakly amenable. Therefore, if $\tau : \mathbb{T} \longrightarrow \mathbb{T}$ be a Lipschitz involution on $\mathbb{T}$ then $(\operatorname{lip}(\mathbb{T}, d^\alpha, \tau))^{**}$ is not (-1)-weakly amenable by Theorem 2.9.

Note that the map $\tau : \mathbb{T} \longrightarrow \mathbb{T}$ defined by one of the following:

$$\begin{aligned}&\tau(z) = z \quad (z \in \mathbb{T}), && \tau(z) = -z, \quad (z \in \mathbb{T}),\\ &\tau(z) = \bar{z} \quad (z \in \mathbb{T}), && \tau(z) = -\bar{z}, \quad (z \in \mathbb{T}),\\ &\tau(z) = i\bar{z} \quad (z \in \mathbb{T}), && \tau(z) = -i\bar{z}, \quad (z \in \mathbb{T}),\end{aligned}$$

is a Lipschitz involution on $(\mathbb{T}, d)$.

Acknowledgment. The authors would like to thank the referees for their useful comments and suggestions.

References

1. D. Alimohammadi and A. Ebadian, *Hedberg's theorem in real Lipschitz algebras*, Indian J. Pure Appl. Math., 32 (2010), pp. 1470-1493.
2. D. Alimohammadi and T.G. Honary, *Contractibility, amenability and weak amenability of real Banach algebras*, J. Aanalysis, (9)(2001), pp. 69-88.
3. R. Arens, *The adjoint of a bilinear operation*, Proc. Math. Amer. Soc., 2 (1951), pp. 839-848.
4. W.G. Bade, P.C. Curtis, and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc., 55 (1987), pp. 359-377.
5. F.F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer Verlag, New York, 1973.
6. H.G. Dales, *Banach Algebras and Automatic Continuity*, Oxford University Press, 2000.
7. J. Duncan and S.A.R. Hosseinioun, *The second dual of a Banach algebra*, Proc. Roy. Soc. Edinburg Sect. A., 84 (1979), pp. 309-325.

8. M. Eshaghi Gordji, S.A.R. Hosseinioun, and A. Valadkhani, *On (-1)-weak amenability of Banach algebras*, Math. Reports, 15 (65), (2013), pp. 271-279.
9. T.G. Honary and S. Moradi, *On the maximal ideal space of extended analytic Lipschitz algebras*, Quaestiones Mathematicae, 30 (2007), pp. 349-353.
10. S.A.R. Hosseinioun and A. Valadkhani, *(-1)-Weak amenability of unitized Banach algebras*, Europ. J. Pure Appl. Math., 9 (2016), pp. 231-239.
11. S.A.R. Hosseinioun and A. Valadkhani, *Weak and (-1)-weak amenability of second dual of Banach algebras*, Int. J. Nonlinear Anal. Appl., 7 (2016), pp. 39-48.
12. B.E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., 127 (1972).
13. B.E. Johnson, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Proc. International conference on Harmonic Analysis, Luxembourg, (Lecture note in Math. Springer-Verlag), 1359 (1987), pp. 191-198.
14. S.H. Kulkarni and B.V. Limaye, *Gleason parts of real function algebras*, Canad. J. Math., 33 (1981), pp. 181-200.
15. S.H. Kulkarni and B.V. Limaye, *Real Function Algebras*, Marcel Dekker, Inc. New York, 1992.
16. M. Mayghani and D. Alimohammadi, *The Structure of ideals, point derivations, amenability and weak amenability of extended Lipschitz algebras*, Int. J. Nonlinear Anal. Appl., 2017, pp. 389-404.
17. A. Medghalchi and T. Yazdanpanah, *Problems concerning n-weak amenability of a Banach algebra*, Czechoslovak Math. J., 55 (2005), pp. 863-876.
18. T.W. Palmer, *Banach Algebras, the General Theory of *-Algebras, Vol. 1: Algebras and Banach Algebras*, Cambridge University Press, Cambridge, 1994.
19. D.R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math., (13) (1963), pp. 1387-1399.
20. D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc, 111 (1964), pp. 240-272.

---

[1] Department of Mathematics, Faculty of Science, University of Arak, 38156-8-8349, Arak, Iran.
E-mail address: `hr_alihoseini@yahoo.com`

[2] Department of Mathematics, Faculty of Science, University of Arak, 38156-8-8349, Arak, Iran.
E-mail address: `d-alimohammadi@araku.ac.ir`