(-1)-Weak Amenability of Second Dual of Real Banach Algebras

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Abstract. Let \((A, \norm{\cdot})\) be a real Banach algebra, a complex algebra \(A_C\) be a complexification of \(A\) and \(\norm{\cdot}_1\) be an algebra norm on \(A_C\) satisfying a simple condition together with the norm \(\norm{\cdot}\) on \(A\). In this paper we first show that \(A\) is a real Banach \(A\)-module if and only if \((A_C)\) is a complex Banach \((A_C)\)-module. Next we prove that \(A\) is \((-1)\)-weakly amenable if and only if \((A_C)\) is \((-1)\)-weakly amenable. Finally, we give some examples of real Banach algebras which their second duals of some them are and of others are not \((-1)\)-weakly amenable.

1. Introduction and Preliminaries

The symbol \(\mathbb{F}\) denotes a field that can be either \(\mathbb{R}\) or \(\mathbb{C}\). For a Banach space \(\mathcal{X}\) over \(\mathbb{F}\) we denote by \(\mathcal{X}^*\) and \(\mathcal{X}^{**}\) the dual space and the second dual space of \(\mathcal{X}\), respectively.

Let \(B\) be an algebra over \(\mathbb{F}\) and \(\mathcal{X}\) be a \(B\)-module over \(\mathbb{F}\) with the module operations \((a, x) \mapsto a \cdot x, (a, x) \mapsto x \cdot a : B \times \mathcal{X} \to \mathcal{X}\). A linear map \(D : B \to \mathcal{X}\) over \(\mathbb{F}\) is called an \(\mathcal{X}\)-derivation on \(B\) over \(\mathbb{F}\) if \(D(ab) = D(a) \cdot b + a \cdot D(b)\) for all \(a, b \in B\). For each \(x \in \mathcal{X}\), the map \(\delta_x : B \to \mathcal{X}\) defined by \(\delta_x(a) = a \cdot x - x \cdot a\) \((a \in B)\), is an \(\mathcal{X}\)-derivation on \(B\) over \(\mathbb{F}\). An \(\mathcal{X}\)-derivation \(D\) on \(B\) is called inner if \(D = \delta_x\) for some \(x \in \mathcal{X}\).

Let \((B, \norm{\cdot})\) be a Banach algebra over \(\mathbb{F}\). A \(B\)-module \(\mathcal{X}\) over \(\mathbb{F}\) is called a Banach \(B\)-module if \(\mathcal{X}\) is a Banach space with a norm \(\norm{\cdot}\) and
Then respectively. The Banach algebra $B$ is a Banach $B$-module over $\mathbb{F}$ with the module operations $a \cdot b = ab$ and $b \cdot a = ba$ for all $a, b \in B$. Let $\mathcal{X}$ be a Banach $B$-module over $\mathbb{F}$ with the module operations $(a, x) \mapsto a \cdot x$, $(a, x) \mapsto x \cdot a : B \times \mathcal{X} \to \mathcal{X}$. Then $\mathcal{X}^*$ is a Banach $B$-module over $\mathbb{F}$ with the natural module operations $(\lambda, a) \mapsto a \cdot \lambda$, $(\lambda, a) \mapsto \lambda \cdot a : B \times \mathcal{X}^* \to \mathcal{X}^*$ given by

$$(a \cdot \lambda)(x) = \lambda(a \cdot x), \quad (\lambda \cdot a)(x) = \lambda(a \cdot x), \quad (a \in B, \lambda \in \mathcal{X}^*, x \in \mathcal{X}),$$

and with the operator norm $\| \cdot \|_{op}$. In particular, $B^*$ is a Banach $B$-module over $\mathbb{F}$. We denote by $Z^2_{op}(B, \mathcal{X})$ the set of all continuous $\mathcal{X}$-derivations on $B$ over $\mathbb{F}$. Clearly, $Z^2_{op}(B, \mathcal{X})$ is a linear space over $\mathbb{F}$ which contains all inner $\mathcal{X}$-derivations on $B$ over $\mathbb{F}$. We denote by $N^1_{op}(B, \mathcal{X})$ the set of all inner $\mathcal{X}$-derivations on $B$ over $\mathbb{F}$. Clearly, $N^1_{op}(B, \mathcal{X})$ is a linear subspace of $Z^2_{op}(B, \mathcal{X})$ over $\mathbb{F}$. We denote by $H^1_{op}(B, \mathcal{X})$ the quotient space $Z^2_{op}(B, \mathcal{X})/N^1_{op}(B, \mathcal{X})$ which it is called the first cohomology group of $B$ over $\mathbb{F}$ with coefficients in $\mathcal{X}$.

A Banach algebra $B$ over $\mathbb{F}$ is called amenable if $H^1_{op}(B, \mathcal{X}^*) = \{0\}$ for all Banach $B$-module $\mathcal{X}$ over $\mathbb{F}$. This concept was first introduced by Johnson in [12]. The notion of weak amenability was first introduced by Bade, Curtis and Dales for commutative Banach algebras in [13] and later defined for Banach algebras, not necessarily commutative, by Johnson in [13]. In fact, a Banach algebra $B$ over $\mathbb{F}$ is called weakly amenable if $H^1_{op}(B, B^*) = \{0\}$.

Let $B$ be a Banach algebra over $\mathbb{F}$. For each $(\lambda, \Lambda) \in B^* \times B^{**}$ the $\mathbb{F}$-valued functions $\lambda \cdot \Lambda$ and $\Lambda \cdot \lambda$ on $B$ are defined by

$$(\lambda \cdot \Lambda)(a) = \Lambda(a \cdot \lambda), \quad (a \in B),$$

$$(\Lambda \cdot \lambda)(a) = \Lambda(\lambda \cdot a), \quad (a \in B).$$

Then $\lambda \cdot \Lambda \in B^*$, $\|\lambda \cdot \Lambda\|_{op} \leq \|\lambda\|_{op} \|\Lambda\|_{op}$, $\Lambda \cdot \lambda \in B^*$ and $\|\Lambda \cdot \lambda\|_{op} \leq \|\Lambda\|_{op} \|\lambda\|_{op}$ for each $\Lambda, \Gamma \in B^{**}$, the $\mathbb{F}$-valued functions $\Lambda \Box \Gamma$ and $\Lambda \rhd \Gamma$ on $B^*$ are defined by

$$(\Lambda \Box \Gamma)(\lambda) = \Lambda(\Gamma \cdot \lambda), \quad (\lambda \in B^*),$$

$$(\Lambda \rhd \Gamma)(\lambda) = \Gamma(\lambda \cdot \Lambda), \quad (\lambda \in B^*).$$

Then $\Lambda \Box \Gamma \in B^{**}$, $\|\Lambda \Box \Gamma\|_{op} \leq \|\Lambda\|_{op} \|\Gamma\|_{op}$, $\Lambda \rhd \Gamma \in B^{**}$ and $\|\Lambda \rhd \Gamma\|_{op} \leq \|\Lambda\|_{op} \|\Gamma\|_{op}$. Moreover, $B^{**}$ is a Banach algebra over $\mathbb{F}$ with respect to either of the products $\Box$ and $\rhd$, and with the operator norm $\| \cdot \|_{op}$.

These products are called the first and second Arens products on $B^{**}$, respectively. The Banach algebra $B$ over $\mathbb{F}$ is called Arens regular if two products $\Box$ and $\rhd$ coincide on $B^{**}$. For the general theory of Arens
products, see [3, 7, 18], for example. For the product □ on $B^{**}$ one can show that $B^*$ is a Banach $B^{**}$-module over $F$ if and only if the following statements hold:

(i) $(\Lambda \cdot \lambda) \cdot \Gamma = \Lambda \cdot (\lambda \cdot \Gamma)$ for all $(\Lambda, \lambda, \Gamma) \in B^{**} \times B^* \times B^{**}$,

(ii) $\lambda \cdot (\Lambda \square \Gamma) = (\Lambda \cdot \lambda) \cdot \Gamma$ for all $(\Lambda, \lambda, \Gamma) \in B^* \times B^{**} \times B^{**}$,

(iii) $(\Lambda \square \Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda)$ for all $(\Lambda, \Gamma, \lambda) \in B^{**} \times B^{**} \times B^*$.

**Definition 1.1.** Let $(B, \| \cdot \|)$ be a Banach algebra over $F$ and $\times$ be one of the Arens products □ and $\triangle$ on $B^{**}$. We say that $B^{**}$ (with the product $\times$) is $(-1)$-weakly amenable if $B^*$ is a Banach $B^{**}$-module over $F$ and $H^{1}_E(B^{**}, B^*) = \{0\}$.

Medghalchi and Yazdanpanah introduced the concept of $(-1)$-weak amenability for Banach algebras in [17] and obtained some results in this area. Eshaghi Gordji, Hosseinioun and Valadkhani in [8] gave some examples of complex Banach algebras that their second duals which are and some others which are not $(-1)$-weakly amenable. Hosseinioun and Valadkhani obtained interesting results in $(-1)$-weak amenability of complex Banach algebras in [11, 13].

Let $E$ be a real linear space (real algebra, respectively). A complex linear space (complex algebra, respectively) $E_C$ is called a complexification of $E$ if there exists an injective real linear map (real algebra homomorphism, respectively) $J : E \rightarrow E_C$ such that $E_C = J(E) \oplus iJ(E)$.

If $X$ is a real linear space, then $X \times X$ with the additive operation and scalar multiplication defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad (x_1, x_2, y_1, y_2 \in X),$$

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x), \quad (\alpha, \beta \in \mathbb{R}, x, y \in X),$$

is a complexification of $X$ with respect to the injective linear map $J : X \rightarrow X \times X$ defined by $J(x) = (x, 0), x \in X$.

If $A$ is a real algebra, then $A \times A$ with the algebra operations

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \quad (a_1, a_2, b_1, b_2 \in A),$$

$$(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \alpha b + \beta a), \quad (\alpha, \beta \in \mathbb{R}, a, b \in A),$$

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2), \quad (a_1, b_1, a_2, b_2 \in A),$$

is a complexification of $A$ with the injective real algebra homomorphism $J : A \rightarrow A \times A$ defined by $J(a) = (a, 0), a \in A$.

It is known [3, Proposition I.1.13] that if $(E, \| \cdot \|)$ is a real normed algebra (real normed space, respectively), then there exists an algebra norm (a norm, respectively) $\| \cdot \|$ on $E \times E$ satisfying $\|(a, 0)\| = ||a||$ for all $a \in E$ and

$$\max\{||a||, ||b||\} \leq ||(a, b)|| \leq 2 \max\{||a||, ||b||\},$$

for all $a, b \in E$. 

Definition 1.2. Let \((E, \| \cdot \|)\) be a real normed linear space (real normed algebra, respectively), let a complex linear space (algebra, respectively) \(E_C\) be a complexification of \(E\) with respect to an injective real linear map (real algebra homomorphism, respectively) \(J : E \rightarrow E_C\) and let \(|| \cdot ||\) be a norm (an algebra norm, respectively) on \(E_C\). We say that \(|| \cdot ||\) satisfies the (*) condition if there exist positive constants \(k_1\) and \(k_2\) such that
\[
\max\{||a||, ||b||\} \leq k_1 ||J(a) + iJ(b)|| \leq k_2 \max\{||a||, ||b||\},
\]
for all \(a, b \in E\).

Note that the (*) condition implies that \((E, \| \cdot \|)\) is a Banach space (Banach algebra, respectively) if and only if \((E_C, || \cdot ||)\) is Banach space (Banach algebra, respectively). Moreover, the existence of a norm (an algebra norm, respectively) \(|| \cdot ||\) on \(E_C\) satisfying the (*) condition guarantees by [3, Proposition I.1.13].

It is shown [2] that if \((A, \| \cdot \|)\) is a real Banach algebra and if \(|| \cdot ||\) is an algebra norm on complex algebra \(A \times A\) satisfying
\[
\max\{||a||, ||b||\} \leq k_1 ||(a, b)|| \leq k_2 \max\{||a||, ||b||\}
\]
for some positive constants \(k_1\) and \(k_2\) and for all \(a, b \in A\), then
\begin{enumerate}
\item[(i)] \(A\) is amenable if and only if \(A \times A\) is amenable [2, Theorem 2.4].
\item[(ii)] \(A\) is weakly amenable if and only if \(A \times A\) is weakly amenable [2, Theorem 2.5].
\end{enumerate}

In Section 2 we assume that \((A, \| \cdot \|)\) is a real Banach algebra, a complex algebra \(A_C\) is the complexification of \(A\) with respect to an injective real algebra homomorphism \(J : A \rightarrow A_C\), \(|| \cdot ||\) is an algebra norm on \(A_C\) satisfying the (*) condition and \((A_C)^*\) is the dual space of \((A_C, || \cdot ||)\). We first show that \(A\) is Arens regular if and only if \(A_C\) is Arens regular. Next we prove that \(A^*\) is a real Banach \(A^{**}\)-module if and only if \((A_C)^*\) is a complex Banach \((A_C)^{**}\)-module. Moreover, we prove that if \(A\) is a real Banach algebra such that \(A^*\) is a real Banach \(A^{**}\)-module, then \(A^{**}\) is \((-1)\)-weakly amenable if and only if \((A_C)^{**}\) is \((-1)\)-weakly amenable. Finally, we give some examples of real Banach algebras which their second duals of some them are and of others are not \((-1)\)-weakly amenable.

2. Main Results and Applications

We first give some lemmas which they will use in the sequel to prove of the main results.

Lemma 2.1. Let \((X, \| \cdot \|)\) be a real Banach space, let \(X_C\) be a complexification of \(X\) with respect to an injective real linear map \(J : X \rightarrow X_C\),
let \( \| \cdot \| \) be a norm on \( X_C \) satisfying the (*) condition with respect to positive constants \( k_1 \) and \( k_2 \) and let \((X_C)^*\) be the dual space of the complex Banach space \((X_C, \| \cdot \|)\).

(i) Let \( \varphi \in X^* \) and define the map \( \varphi_C : X_C \rightarrow \mathbb{C} \) by
\[
\varphi_C(J(x) + iJ(y)) = \varphi(x) + i\varphi(y) \quad (x, y \in X).
\]
Then \( \varphi_C(J(x)) = \varphi(x) \) for all \( x \in X \), \( \varphi_C \in (X_C)^* \), \( \| \varphi_C \|_{op} \leq 2k_1\|\varphi\|_{op} \) and \( \| \varphi \|_{op} \leq \frac{k_2}{k_1}\| \varphi_C \|_{op} \).

(ii) Let \( \lambda \in (X_C)^* \) and define the map \( \lambda_R : X \rightarrow \mathbb{R} \) by
\[
\lambda_R(x) = \text{Re} \lambda(J(x)) \quad (x \in X).
\]
Then \( \lambda_R \in X^* \) and \( \| \lambda_R \|_{op} \leq \frac{k_2}{k_1}\| \lambda \|_{op} \).

(iii) Let \( \lambda \in (X_C)^* \) and define the map \( \lambda_I : X \rightarrow \mathbb{R} \) by
\[
\lambda_I(x) = \text{Im} \lambda(J(x)) \quad (x \in X).
\]
Then \( \lambda_I \in X^* \) and \( \| \lambda_I \|_{op} \leq \frac{k_2}{k_1}\| \lambda \|_{op} \).

Proof. Let \( x \in X \). Then
\[
\varphi_C(J(x)) = \varphi_C(J(x) + iJ(0))
\]
\[
= \varphi(x) + i\varphi(0)
\]
\[
= \varphi(x) + i0
\]
\[
= \varphi(x).
\]
It is easy to see that \( \varphi_C \) is a complex linear functional on \( X_C \). Since
\[
|\varphi_C(J(x) + iJ(y))| = |\varphi(x) + i\varphi(y)|
\]
\[
\leq |\varphi(x)| + |\varphi(y)|
\]
\[
\leq 2\|\varphi\|_{op} \max\{|x|, |y|\}
\]
\[
\leq 2k_1\|\varphi\|_{op}\|J(x) + iJ(y)\|
\]
for all \( x, y \in X \), we deduce that \( \varphi_C \in (X_C)^* \) and \( \| \varphi_C \|_{op} \leq 2k_1\| \varphi \|_{op} \).

On the other hand, we have
\[
|\varphi(x)| = |\varphi_C(J(x))|
\]
\[
\leq \| \varphi_C \|_{op}\|J(x)\|
\]
\[
\leq \| \varphi_C \|_{op}\frac{k_2}{k_1}\|x\|,
\]
for all \( x \in X \). Hence, \( \| \varphi \|_{op} \leq \frac{k_2}{k_1}\| \varphi_C \|_{op} \). Therefore, (i) holds.

Clearly, \( \lambda_R \) is a real linear functional on \( X \). Since
\[
|\lambda_R(x)| = |\text{Re} \lambda(J(x))|
\]
\[
\leq |\lambda(J(x))|
\]
\[
\leq \| \lambda \|_{op}\|J(x)\|
\]
for all $x \in X$, we deduce that $\lambda_R \in X^*$ and $\|\lambda_R\|_{op} \leq \frac{k_2}{k_1} \|\lambda\|$. Hence, (ii) holds.

It is easy to see that $\lambda_I$ is a real linear functional on $X$. Moreover, for each $x \in X$ we have

$$|\lambda_I(x)| = |\text{Im} \lambda(J(x))|$$

$$\leq |\lambda(J(x))|$$

$$\leq \|\lambda\|_{op} \|J(x)\|$$

$$\leq \|\lambda\|_{op} \frac{k_2}{k_1} \|x\|.$$

Hence, $\lambda_I \in X^*$ and $\|\lambda_I\|_{op} \leq \frac{k_2}{k_1} \|\lambda\|_{op}$. Therefore, (iii) holds.

\[\square\]

**Lemma 2.2.** Let $(X, \| \cdot \|)$ be a real Banach space, let $X_C$ be a complexification of $X$ with respect to an injective real linear map $J : X \to X_C$, let $\| \cdot \|$ be a norm on $X_C$ satisfying (**) condition with respect to positive constants $k_1$ and $k_2$ and let $(X_C)^*$ be the dual space of the complex Banach space $(X_C, \| \cdot \|)$. Define the map $J_1 : X^* \to (X_C)^*$ by

$$(2.1) \quad J_1(\varphi) = \varphi_C, \quad (\varphi \in X^*).$$

Then:

(i) $J_1(\varphi)(J(x) + iJ(y)) = \varphi(x) + i\varphi(y)$ for all $\varphi \in X^*$ and $x, y \in X$.

(ii) $J_1$ is a real linear map from $X^*$ into $(X_C)^*$.

(iii) If $\lambda \in (X_C)^*$, then $\lambda = J_1(\lambda_R) + iJ_1(\lambda_I)$.

(iv) $J_1$ is injective and $(X_C)^* = J_1(X^*) \oplus iJ_1(X^*)$.

(v) $(X_C)^*$ is a complexification of $X^*$ with respect to the map $J_1 : X^* \to (X_C)^*$ defined by (2.1) and

$$\max\{ \|\varphi\|_{op}, \|\psi\|_{op} \} \leq \frac{k_2}{k_1} \|J_1(\varphi) + iJ_1(\psi)\|_{op}$$

$$\leq 4k_2 \max\{ \|\varphi\|_{op}, \|\psi\|_{op} \},$$

for all $\varphi, \psi \in X^*$.

**Proof.** By part (i) of Lemma 2.1, $J_1$ is well-defined. Let $\varphi \in X^*$ and $x, y \in X$. Then, by part (i) of Lemma 2.1, we have

$$J_1(\varphi)(J(x) + iJ(y)) = \varphi_C(J(x) + iJ(y))$$

$$= \varphi_C(J(x)) + i\varphi_C(J(y))$$

$$= \varphi(x) + i\varphi(y).$$

Hence, (i) holds.
It is easy to see that \((\varphi + \psi)_C = \varphi_C + \psi_C\) for all \(\varphi, \psi \in \mathcal{X}^*\) and \((\alpha \varphi)_C = \alpha \varphi_C\) for all \(\alpha \in \mathbb{R}\) and \(\varphi \in \mathcal{X}^*\). Hence, (ii) holds.

Let \(\lambda \in (\mathcal{X}_C)^*\). By parts (ii) and (iii) of Lemma 2.1, \(\lambda_R, \lambda_I \in \mathcal{X}^*\). Since

\[
\lambda(J(x) + iJ(y)) = \lambda(J(x)) + \lambda(iJ(y)) \\
= (\text{Re} \lambda(J(x)) + i\text{Im} \lambda(J(x))) \\
+ i(\text{Re} \lambda(J(y)) + i\text{Im} \lambda(J(y))) \\
= (\lambda_R(x) + i\lambda_I(x)) + i(\lambda_R(y) + i\lambda_I(y)) \\
= (\lambda_R(x) + i\lambda_R(y)) + i(\lambda_I(x) + i\lambda_I(y)) \\
= (\lambda_R)(J(x) + iJ(y)) + i(\lambda_I)(J(x) + iJ(y)) \\
= ((\lambda_R) + i((\lambda_I))(J(x) + iJ(y)) \\
= (J_1(\lambda_R) + iJ_1(\lambda_I))(J(x) + iJ(y)),
\]

for all \(x, y \in \mathcal{X}\), we have \(\lambda = J_1(\lambda_R) + iJ_1(\lambda_I)\). Hence, (iii) holds.

Let \(\varphi \in \mathcal{X}^*\) and \(J_1(\varphi) = 0\). Then \(\varphi_C = 0\) and so \(\varphi_C(J(x)) = 0\) for all \(x \in \mathcal{X}\). This implies that \(\varphi(x) = 0\) for all \(x \in \mathcal{X}\) by part (ii) of Lemma 2.1. Hence, \(\varphi = 0\) and so \(J_1\) is injective.

By the definition of the map \(J_1 : \mathcal{X}^* \rightarrow (\mathcal{X}_C)^*\) and (iii), we conclude that

\[\tag{2.2} (\mathcal{X}_C)^* = J_1(\mathcal{X}^*) + iJ_1(\mathcal{X}^*).\]

Let \(\lambda \in J_1(\mathcal{X}^*) \cap iJ_1(\mathcal{X}^*)\). Then there exist \(\varphi, \psi \in \mathcal{X}^*\) such that \(\lambda = J_1(\varphi) = iJ_1(\psi)\). This implies that \(\varphi(x) = i\psi(x)\) for all \(x \in \mathcal{X}\) and so \(\varphi(x) = 0\) for all \(x \in \mathcal{X}\) since \(\varphi\) and \(\psi\) are real-valued functions on \(\mathcal{X}\). Hence, \(\varphi = 0\) and so \(\lambda = J_1(\varphi) = 0\). Thus

\[\tag{2.3} J_1(\mathcal{X}^*) \cap iJ_1(\mathcal{X}^*) = \{0\}.\]

From (2.2) and (2.3) we have \((\mathcal{X}_C)^* = J_1(\mathcal{X}^*) \oplus iJ_1(\mathcal{X}^*)\). Therefore, (iv) holds.

Applying (ii) and (iv), we deduce that \((\mathcal{X}_C)^*\) is a complexification of \(\mathcal{X}^*\) with respect to the injective real linear map \(J_1 : \mathcal{X}^* \rightarrow (\mathcal{X}_C)^*\) which is defined by (2.1).

Let \(\varphi, \psi \in \mathcal{X}^*\). Since

\[
|\varphi(x)| \leq |\varphi(x) + i\psi(x)| \\
= |(J_1(\varphi)(J(x)) + iJ_1(\psi)(J(x))| \\
= |(J_1(\varphi) + iJ_1(\psi))(J(x))| \\
\leq \|J_1(\varphi) + iJ_1(\psi)\|_\text{op} \|J(x)\| \\
\leq \|J_1(\varphi) + iJ_1(\psi)\|_\text{op} \frac{k_2}{k_1} \|x\|,
\]

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for all $x \in \mathfrak{X}$, we deduce that $\|\varphi\|_{op} \leq \frac{k_2}{k_1} \|J_1(\varphi) + iJ_1(\psi)\|_{op}$. Similarly, we have $\|\psi\|_{op} \leq \frac{k_2}{k_1} \|J_1(\varphi) + iJ_1(\psi)\|_{op}$. Hence,

\begin{equation}
2.4 \quad \max\{\|\varphi\|_{op}, \|\psi\|_{op}\} \leq \frac{k_2}{k_1} \|J_1(\varphi) + iJ_1(\psi)\|_{op}.
\end{equation}

Since

\[
|\langle J_1(\varphi) + iJ_1(\psi), J(x) + iJ(y) \rangle| \\
= |J_1(\varphi)J(x) + iJ(y) + iJ(\psi)(J(x) + iJ(y))| \\
= |(\varphi(x) + i\varphi(y)) + i(\psi(x) + i\psi(y))| \\
\leq |\varphi(x)| + |\varphi(y)| + |\psi(x)| + |\psi(y)| \\
\leq \|\varphi\|_{op}\|x\| + \|\varphi\|_{op}\|y\| + \|\psi\|_{op}\|x\| + \|\psi\|_{op}\|y\| \\
\leq 2\|\varphi\|_{op}\max\{\|x\|, \|y\|\} + 2\|\psi\|_{op}\max\{\|x\|, \|y\|\}
\leq 4k_1\max\{\|\varphi\|_{op}, \|\psi\|_{op}\}\]

for all $x, y \in \mathfrak{X}$, we deduce that

\begin{equation}
2.5 \quad \|J_1(\varphi) + iJ_1(\psi)\|_{op} \leq 4k_1\max\{\|\varphi\|_{op}, \|\psi\|_{op}\}.
\end{equation}

From (2.4) and (2.5) we have

\[
\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} \leq \frac{k_2}{k_1} \|J_1(\varphi) + iJ_1(\psi)\|_{op} \\
\leq 4k_2\max\{\|\varphi\|_{op}, \|\psi\|_{op}\}.
\]

Hence, (v) holds. \hfill \Box

**Lemma 2.3.** Let $(\mathfrak{X}, \| \cdot \|)$ be a real Banach space, let $\mathfrak{X}_{\mathbb{C}}$ be a complexification of $\mathfrak{X}$ with respect to an injective real linear map $J : \mathfrak{X} \to \mathfrak{X}_{\mathbb{C}}$, let $\| \cdot \|$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ satisfying (∗) condition with positive constants $k_1$ and $k_2$ and let $(\mathfrak{X}_{\mathbb{C}})^*$ be the dual space of $(\mathfrak{X}_{\mathbb{C}}, \| \cdot \|)$. Define the map $J_2 : \mathfrak{X}^* \to (\mathfrak{X}_{\mathbb{C}})^*$ by

\begin{equation}
2.6 \quad J_2(\Phi) = \Phi_{\mathbb{C}} \quad (\Phi \in \mathfrak{X}^*).
\end{equation}

Then:

(i) $J_2(\Phi)(J_1(\varphi) + iJ_1(\psi)) = \Phi(\varphi) + i\Phi(\psi)$ for all $\Phi \in \mathfrak{X}^*$ and $\varphi, \psi \in \mathfrak{X}^*$.

(ii) $J_2$ is a real linear map from $\mathfrak{X}^*$ into $(\mathfrak{X}_{\mathbb{C}})^*$.

(iii) If $\Lambda \in (\mathfrak{X}_{\mathbb{C}})^*$, then the maps $\Lambda R, \Lambda I : \mathfrak{X}^* \to \mathbb{R}$ defined by

$\Lambda R(\varphi) = \Re \Lambda(J_1(\varphi)) \quad (\varphi \in \mathfrak{X}^*)$,

$\Lambda I(\varphi) = \Im \Lambda(J_1(\varphi)) \quad (\varphi \in \mathfrak{X}^*)$,

belong to $\mathfrak{X}^*$ and

$\Lambda = J_2(\Lambda R) + iJ_2(\Lambda I)$. 

(iv) $J_2$ is injective and $(\mathfrak{X}_C)^{**} = J_2(\mathfrak{X}^{**}) \oplus iJ_2(\mathfrak{X}^{**})$.

(vi) $(\mathfrak{X}_C)^{**}$ is a complexification of $\mathfrak{X}^{**}$ with respect to the map $J_2 : \mathfrak{X}^{**} \to (\mathfrak{X}_C)^{**}$ defined by (2.10) and
\[
\max\{\|\Phi\|_{op}, \|\Psi\|_{op}\} \leq 4k_1\|J_2(\Phi) + iJ_2(\Psi)\|_{op}
\]
\[
\leq 16k_2\max\{\|\Phi\|_{op}, \|\Psi\|_{op}\},
\]
for all $\Phi, \Psi \in \mathfrak{X}^{**}$.

(vi) $J_2 \circ \pi_X = \pi_{\mathfrak{X}_C} \circ J$, whenever $\pi_Y : Y \to Y^{**}$ is the natural embedding $Y$ in $Y^{**}$ defined by
\[
\pi_Y(y)(\lambda) = \lambda(y) \quad (y \in Y, \lambda \in Y^*)
\]

(vii) $\mathfrak{X}$ is reflexive if and only if $\mathfrak{X}_C$ is reflexive.

**Proof.** By Lemma 2.2, we deduce that the map $J_1 : \mathfrak{X}^* \to (\mathfrak{X}_C)^*$ defined by (2.10) is an injective real linear map, the complex linear space $(\mathfrak{X}_C)^*$ is a complexification of $\mathfrak{X}^*$ with respect to $J_1$,
\[
\lambda = J_1(\lambda_R) + iJ_1(\lambda_I) \quad (\lambda \in (\mathfrak{X}_C)^*),
\]
\[
\max\{\|\varphi\|_{op}, \|\psi\|_{op}\} \leq \frac{k_2}{k_1}\|J_1(\varphi) + iJ_1(\psi)\|_{op}
\]
\[
\leq 4k_2\max\{\|\varphi\|_{op}, \|\psi\|_{op}\},
\]
for all $\varphi, \psi \in \mathfrak{X}^*$, and
\[
J_1(\varphi)(J(x) + iJ(y)) = \varphi(x) + i\varphi(y)
\]
for all $\varphi \in \mathfrak{X}^*$ and $x, y \in \mathfrak{X}$. Hence, by the definition of $J_2$, we deduce that (i), (ii), (iii), (iv) and (v) hold.

To prove (vi), suppose that $x \in \mathfrak{X}$. Then for each $\lambda \in (\mathfrak{X}_C)^*$ we have
\[
((\pi_{\mathfrak{X}_C} \circ J)(x))(\lambda) = (\pi_{\mathfrak{X}_C}(J(x)))(\lambda)
\]
\[
= (J_1(\lambda_R) + iJ_1(\lambda_I))(J(x))
\]
\[
= (J_1(\lambda_R))(J(x)) + i(J_1(\lambda_I))(J(x))
\]
\[
= \lambda_R(x) + i\lambda_I(x)
\]
\[
= \pi_X(x)(\lambda_R) + i\pi_X(x)(\lambda_I)
\]
\[
= J_2(\pi_X(x)(J_1(\lambda_R)) + iJ_2(\pi_X(x))(J_1(\lambda_I))
\]
\[
= J_2(\pi_X(x)(J_1(\lambda_R) + iJ_1(\lambda_I))
\]
\[
= (J_2 \circ \pi_X)(x)(\lambda).
\]

This implies that
\[
(2.7) \quad (\pi_{\mathfrak{X}_C} \circ J)(x) = (J_2 \circ \pi_X)(x).
\]
Since (2.7) holds for all \( x \in \mathcal{X} \), we deduce that \( \pi_{\mathcal{X}^c} \circ J = J_2 \circ \pi_{\mathcal{X}} \). Hence (vi) holds.

To prove (vii) we first assume that \( \mathcal{X} \) is reflexive. Then \( \pi_{\mathcal{X}}(\mathcal{X}) = \mathcal{X}^{**} \).

Let \( \Lambda \in (\mathcal{X}_C)^{**} \). By part (iii) we have

\[
\Lambda = J_2(\Lambda_R) + i J_2(\Lambda_I).
\]

Since \( \Lambda_R, \Lambda_I \in \mathcal{X}^{**} \), there exist \( x, y \in \mathcal{X} \) such that \( \pi_{\mathcal{X}}(x) = \Lambda_R \) and \( \pi_{\mathcal{X}}(y) = \Lambda_I \). Hence, by part (vi) we have

\[
\Lambda = J_2(\pi_{\mathcal{X}}(x)) + i J_2(\pi_{\mathcal{X}}(y)) = (J_2 \circ \pi_{\mathcal{X}})(x) + i(J_2 \circ \pi_{\mathcal{X}})(y) = (\pi_{\mathcal{X}^c} \circ J)(x) + i(\pi_{\mathcal{X}^c} \circ J)(y) = \pi_{\mathcal{X}^c}(J(x) + iJ(y)),
\]

and so \( \Lambda \in \pi_{\mathcal{X}^c}(\mathcal{X}_C) \). Therefore, \( \pi_{\mathcal{X}^c} \) is surjective and so \( \mathcal{X}_C \) is reflexive.

We now assume that \( \mathcal{X} \) is reflexive. Then \( \pi_{\mathcal{X}^c}(\mathcal{X}_C) = (\mathcal{X}_C)^{**} \). Let \( \Phi \in \mathcal{X}^{**} \). Then \( J_2(\Phi) \in (\mathcal{X}_C)^{**} \) and so there exist \( x, y \in \mathcal{X} \) such that

\[
J_2(\Phi) = \pi_{\mathcal{X}^c}(J(x) + iJ(y)).
\]

Hence, by part (vi) we have

\[
J_2(\Phi) + i J_2(0) = J_2(\Phi) = (\pi_{\mathcal{X}^c} \circ J)(x) + i(\pi_{\mathcal{X}^c} \circ J)(y) = (J_2 \circ \pi_{\mathcal{X}})(x) + i(J_2 \circ \pi_{\mathcal{X}})(y) = J_2(\pi_{\mathcal{X}}(x)) + i J_2(\pi_{\mathcal{X}}(y)).
\]

This implies that \( J_2(\Phi) = J_2(\pi_{\mathcal{X}}(x)) \) since \( (\mathcal{X}_C)^{**} = J_2(\mathcal{X}^{**}) \oplus i J_2(\mathcal{X}^{**}) \).

Therefore, \( \Phi = \pi_{\mathcal{X}}(x) \) since \( J_2 \) is injective. Hence, \( \pi_{\mathcal{X}} \) is surjective and so \( \mathcal{X} \) is reflexive. Thus, (vii) holds.

\[\Box\]

**Lemma 2.4.** Let \( (A, \| \cdot \|) \) be a real Banach algebra, let \( A_C \) be a complexification of \( A \) with respect to an injective real algebra homomorphism \( J : A \rightarrow A_C \), let \( \| \cdot \| \) be an algebra norm on \( A_C \) satisfying the (\( * \)) condition and let \( (A_C)^* \) be the dual space of \( (A_C, \| \cdot \|) \).

(i) If \( a \in A \) and \( \varphi \in A^* \), then

\[
J_1(a \cdot \varphi) = J(a) \cdot J_1(\varphi), \quad J_1(\varphi \cdot a) = J_1(\varphi) \cdot J(a).
\]

(ii) If \( \varphi \in A^* \) and \( \Phi \in A^{**} \), then

\[
J_1(\varphi \cdot \Phi) = J_1(\varphi) \cdot J_2(\Phi), \quad J_1(\Phi \cdot \varphi) = J_2(\Phi) \cdot J_1(\varphi).
\]

(iii) If \( \Phi, \Psi \in A^{**} \), then

\[
J_2(\Phi \square \Psi) = J_2(\Phi) \square J_2(\Psi), \quad J_2(\Phi \triangle \Psi) = J_2(\Phi) \triangle J_2(\Psi).
\]
(iv) If $\Lambda \in (A_C)^{**}$ and $\lambda \in (A_C)^*$, then

$$\Lambda \cdot \lambda = J_1(\Lambda_R \cdot \lambda_R - \Lambda_I \cdot \lambda_I) + i J_1(\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R),$$

$$\lambda \cdot \Lambda = J_1(\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) + i J_1(\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R).$$

**Proof.** Let $a \in A$ and $\varphi \in A^*$. Then, by Lemma 2.3, we have

$$J_1(a \cdot \varphi)(J(b)) = (a \cdot \varphi)(b)$$

$$= \varphi(ba)$$

$$= J_1(\varphi)(ba)$$

$$= J_1(\varphi)(J(b)J(a))$$

$$= (J(a) \cdot J_1(\varphi))(J(b)),$$

for all $b \in A$. This implies that

$$J_1(a \cdot \varphi)(J(b) + iJ(c)) = J_1(a \cdot \varphi)(J(b)) + i J_1(a \cdot \varphi)(J(c))$$

$$= (J(a) \cdot J_1(\varphi))(J(b)) + i (J(a) \cdot J_1(\varphi))(J(c))$$

$$= (J(a) \cdot J_1(\varphi))(J(b) + iJ(c)),$$

for all $b,c \in A$. Hence,

$$J_1(a \cdot \varphi) = J(a) \cdot J_1(\varphi).$$

Similarly, we can show that

$$J_1(\varphi \cdot a) = J_1(\varphi) \cdot J(a).$$

Therefore, (i) holds.

Let $\varphi \in A^*$ and $\Phi \in A^{**}$. Then, by (i), we have

$$J_1(\varphi \cdot \Phi)(J(a)) = J_1(\varphi \cdot \Phi)(a)$$

$$= \Phi(a \cdot \varphi)$$

$$= J_2(\Phi)(J_1(a \cdot \varphi))$$

$$= J_2(\Phi)(J(a) \cdot J_1(\varphi))$$

$$= (J_1(\varphi \cdot \Phi))(J(a))$$

for all $a \in A$. This implies that

$$J_1(\varphi \cdot \Phi)(J(a) + iJ(b)) = J_1(\varphi \cdot \Phi)(J(a)) + i J_1(\varphi \cdot \Phi)(J(b))$$

$$= (J_1(\varphi \cdot \Phi)(J(a))$$

$$+ i(J_1(\varphi \cdot \Phi))(J(b))$$

$$= (J_1(\varphi \cdot \Phi))(J(a) + iJ(b)),$$

for all $a,b \in A$. Hence,

$$J_1(\varphi \cdot \Phi) = J_1(\varphi) \cdot J_2(\Phi).$$
Similarly, we can show that
\[ J_1(\Phi \cdot \varphi) = J_2(\Phi) \cdot J_1(\varphi). \]
Therefore, (ii) holds.

Let \( \Phi, \Psi \in A^{**} \). Then, by (ii), we have
\[ J_2(\Phi \square \Psi)(J_1(\varphi)) = (\Phi \square \Psi)(\varphi) = \Phi(\Psi \cdot \varphi) = J_2(\Phi)(J_1(\Psi \cdot \varphi)) = J_2(\Phi)(J_2(\Psi) \cdot J_1(\varphi)) = (J_2(\Phi) \square J_2(\Psi))(J_1(\varphi)), \]
for all \( \varphi \in A^* \). This implies that
\[ J_2(\Phi \square \Psi)(J_1(\varphi) + iJ_1(\psi)) = J_2(\Phi \square \Psi)(J_1(\varphi)) + iJ_2(\Phi \square \Psi)(J_1(\psi)) = (J_2(\Phi) \square J_2(\Psi))(J_1(\varphi)) + iJ_2(\Phi) \square J_2(\Psi)(J_1(\psi)) = (J_2(\Phi) \square J_2(\Psi))(J_1(\varphi) + iJ_1(\psi)), \]
for all \( \varphi, \psi \in A^* \). Hence,
\[ J_2(\Phi \square \Psi) = J_2(\Phi) \square J_2(\Psi). \]
Similarly, we can show that
\[ J_2(\Phi \triangle \Psi) = J_2(\Phi) \triangle J_2(\Psi). \]
Therefore, (iii) holds.

Let \( \Lambda \in (A_C)^{**} \) and \( \lambda \in (A_C)^* \). Then by part (iii) of Lemma 2.3 and part (iii) of Lemma 2.2, we have
\[ (2.8) \quad \Lambda = J_2(\Lambda_R) + iJ_2(\Lambda_I), \quad \lambda = J_1(\lambda_R) + iJ_1(\lambda_I). \]
Applying (2.8) and (ii), we get
\[
\begin{align*}
\Lambda \cdot \lambda &= (J_2(\Lambda_R) + iJ_2(\Lambda_I)) \cdot (J_1(\lambda_R) + iJ_1(\lambda_I)) \\
&= (J_2(\Lambda_R) \cdot J_1(\lambda_R) - J_2(\Lambda_I) \cdot J_1(\lambda_I)) + i(J_2(\Lambda_R) \cdot J_1(\lambda_I) + J_2(\Lambda_I) \cdot J_1(\lambda_R)) \\
&= (J_1(\Lambda_R \cdot \lambda_R) - J_1(\Lambda_I \cdot \lambda_I)) + i(J_1(\Lambda_R \cdot \lambda_I) + J_1(\Lambda_I \cdot \lambda_R)) \\
&= J_1(\Lambda_R \cdot \lambda_R - \Lambda_I \lambda_I) + iJ_1(\Lambda_R \cdot \lambda_I + \Lambda_I \cdot \lambda_R).
\end{align*}
\]
Similarly, we can show that
\[ \lambda \cdot \Lambda = J_1(\lambda_R \cdot \Lambda_R - \lambda_I \cdot \Lambda_I) + iJ_1(\lambda_R \cdot \Lambda_I + \lambda_I \cdot \Lambda_R). \]
Hence, (iv) holds. \( \square \)
Theorem 2.5. Let \( (A, \| \cdot \|) \) be a real Banach algebra, let \( A_C \) be a complexification of \( A \) with respect to an injective real algebra homomorphism \( J : A \rightarrow A_C \), let \( \| \cdot \| \) be an algebra norm on \( A_C \) satisfying the \((*)\) condition and let \( (A_C)^* \) be the dual space of \( (A_C, \| \cdot \|) \). Then \( A \) is Arens regular if and only if \( A_C \) is Arens regular.

Proof. We first assume that \( A \) is Arens regular. Then

\[
\Phi \square \Psi = \Phi \triangle \Psi,
\]

for all \( \Phi, \Psi \in A^{**} \). Let \( \Lambda, \Gamma \in (A_C)^{**} \). Then, by part (iii) of Lemma 2.9, we have \( \Lambda_R, \Lambda_I, \Gamma_R, \Gamma_I \in A^{**} \) and

\[
\Lambda = J_2(\Lambda_R) + iJ_2(\lambda_I), \quad \Gamma = J_2(\Gamma_R) + iJ_2(\Gamma_I).
\]

Since (2.10) holds for all \( \Phi, \Psi \in A^{**} \), we have

\[
\Lambda_R \square \Gamma_R = \Lambda_R \triangle \Gamma_R, \quad \Lambda_R \square \Gamma_I = \Lambda_R \triangle \Gamma_I,
\]

(2.11)

\[
\Lambda_I \square \Gamma_R = \Lambda_I \triangle \Gamma_R, \quad \Lambda_I \square \Gamma_I = \Lambda_I \triangle \Gamma_I.
\]

By Lemma 2.3 and according to (2.10) and (2.11), we get

\[
\Lambda \square \Gamma = (J_2(\Lambda_R) + iJ_2(\lambda_I)) \square (J_2(\Gamma_R) + iJ_2(\Gamma_I))
\]

\[
= (J_2(\Lambda_R) \square J_2(\Gamma_R) - J_2(\Lambda_I) \square J_2(\Gamma_I))
+ i ((J_2(\Lambda_R) \square J_2(\Gamma_I)) + (J_2(\Lambda_I) \square J_2(\Gamma_R)))
\]

\[
= (J_2(\Lambda_R \square \Gamma_R) - J_2(\Lambda_I \square \Gamma_I))
+ i (J_2(\Lambda_R \square \Gamma_I) + J_2(\Lambda_I \square \Gamma_R))
\]

\[
= (J_2(\Lambda_R \triangle \Gamma_R) - J_2(\Lambda_I \triangle \Gamma_I))
+ i (J_2(\Lambda_R \triangle \Gamma_I) + J_2(\Lambda_I \triangle \Gamma_R))
\]

\[
= (J_2(\Lambda_R \triangle \Gamma_R) - J_2(\Lambda_I \triangle \Gamma_I))
+ i (J_2(\Lambda_R \triangle \Gamma_I) + J_2(\Lambda_I \triangle \Gamma_R))
\]

\[
= (J_2(\Lambda_R \triangle \Gamma_R) - J_2(\Lambda_I \triangle \Gamma_I))
+ i (J_2(\Lambda_R \triangle \Gamma_I) + J_2(\Lambda_I \triangle \Gamma_R))
\]

\[
\triangle \Lambda \Gamma.
\]

Therefore, \( (A_C)^{**} \) is Arens regular.

We now assume that \( A_C \) is Arens regular. Then

\[
\Lambda \square \Gamma = \Lambda \triangle \Gamma,
\]

for all \( \Lambda, \Gamma \in (A_C)^{**} \). Let \( \Phi, \Psi \in A^{**} \). Then, by Lemma 2.3, we have \( J_2(\Phi), J_2(\Psi) \in (A_C)^{**} \) and so by (2.12) we have

\[
J_2(\Phi) \square J_2(\Psi) = J_2(\Phi) \triangle J_2(\Psi).
\]

Moreover,

\[
J_2(\Phi \square \Psi) = J_2(\Phi) \square J_2(\Psi), \quad J_2(\Phi) \triangle J_2(\Psi) = J_2(\Phi \triangle \Psi),
\]
We prove the result for the first Arens product \((2.16)\) and \((2.17)\) we get \(J_2(\Phi \square \Psi) = J_2(\Phi \triangle \Psi)\). This implies that \(\Phi \square \Psi = \Phi \triangle \Psi\), since \(J_2\) is injective. Therefore, \(A\) is Arens regular. \(\square\)

**Theorem 2.6.** Let \((A, \| \cdot \|)\) be a real Banach algebra, let \(A_C\) be a complexification of \(A\) with respect to an injective real algebra homomorphism \(J: A \to A_C\), let \(\| \cdot \|\) be an algebra norm on \(A_C\) satisfying the \((*)\) condition and let \((A_C)^*\) be the dual space of \((A_C, \| \cdot \|)\). Then \(A^*\) is a real Banach \(A^{**}\)-module if and only if \((A_C)^*\) is a complex Banach \((A_C)^{**}\)-module.

**Proof.** We prove the result for the first Arens product \(\square\) on \(A^{**}\) and \((A_C)^{**}\). Similarly, one can show that the result hold for the second Arens product \(\triangle\) on \(A^{**}\) and \((A_C)^{**}\).

We first assume that \(A^*\) is a real Banach \(A^{**}\)-module. Then

\[
(\Phi \cdot \varphi) \cdot \Psi = \Phi \cdot (\varphi \cdot \Psi),
\]

and

\[
\varphi \cdot (\Phi \square \Psi) = (\varphi \cdot \Phi) \cdot \Psi,
\]

\[
\Phi \square (\varphi \Psi) = \Phi \cdot (\varphi \cdot \Psi),
\]

for all \((\varphi, \Phi, \Psi) \in A^* \times A^{**} \times A^{**}\). Let \((\Lambda, \lambda, \Gamma) \in (A_C)^{**} \times (A_C)^* \times (A_C)^{**}\). Then \(\Lambda R_1, \Lambda I, \Gamma R, \Gamma I \in A^{**}\). Applying part (iv) of Lemma 2.4 and (2.15), we get

\[
(\Lambda \cdot \lambda) \cdot \Gamma
\]

\[
= (J_1(\Lambda R \cdot \lambda R - \Lambda I \cdot \lambda I) + iJ_1(\Lambda R \cdot \lambda I + \Lambda I \cdot \lambda R)) \cdot \Gamma
\]

\[
= J_1((\Lambda R \cdot \lambda R - \Lambda I \cdot \lambda I) \cdot \Gamma R - (\Lambda R \cdot \lambda I + \Lambda I \cdot \lambda R) \cdot \Gamma I)
\]

\[
+ iJ_1((\Lambda R \cdot \lambda R - \Lambda I \cdot \lambda I) \cdot \Gamma I + (\Lambda R \cdot \lambda I + \Lambda I \cdot \lambda R) \cdot \Gamma R)
\]

\[
= J_1(\Lambda R \cdot (\lambda R \cdot \Gamma R) - \Lambda I \cdot (\lambda I \cdot \Gamma R) - \Lambda R \cdot (\lambda I \cdot \Gamma I) - \Lambda I \cdot (\lambda R \cdot \Gamma I))
\]

\[+ iJ_1(\Lambda R \cdot (\lambda R \cdot \Gamma I) - \Lambda I \cdot (\lambda I \cdot \Gamma I) + \Lambda R \cdot (\lambda I \cdot \Gamma R) + \Lambda I \cdot (\lambda R \cdot \Gamma R))
\]

\[
= \Lambda \cdot (J_1(\lambda R \cdot \Gamma R - \lambda I \cdot \lambda I) + iJ_1(\lambda R \cdot \Gamma I + \lambda I \cdot \Gamma R))
\]

\[
= \Lambda \cdot (\lambda \cdot \Gamma)\]

Applying part (ii) of Lemma 2.4 and (2.16), we get

\[
\lambda \cdot (\lambda \cdot \Gamma)
\]

\[
= (J_1(\lambda R) + iJ_1(\lambda I))\]
\[ J(\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) + iJ(\Lambda_{I} \Gamma R - \Lambda_{R} \Gamma I) \]
\[ = J_{2}(\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) - \lambda_{I} \Lambda \cdot (\Lambda_{I} \Gamma R - \Lambda_{R} \Gamma I) \]
\[ + iJ(\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) \]
\[ = J_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) - \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{R} \Gamma I)) \]
\[ + iJ_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) + \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{R} \Gamma I)) \]
\[ + iJ_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) - \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{R} \Gamma I)) \]
\[ = J_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) + \lambda_{R} \cdot (\Lambda_{I} \Gamma R - \Lambda_{I} \Gamma I) + \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{I} \Gamma I)) \]
\[ + iJ_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) - \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{I} \Gamma I)) \]
\[ = J_{1}(\lambda_{R} \cdot (\Lambda_{R} \Gamma R - \Lambda_{I} \Gamma I) + \lambda_{R} \cdot (\Lambda_{I} \Gamma R - \Lambda_{I} \Gamma I) + \lambda_{I} \cdot (\Lambda_{I} \Gamma R - \Lambda_{I} \Gamma I)) \]
\[ \cdot (J_{2}(\Gamma R) + iJ_{2}(\Gamma I)) \]
\[ = (\lambda \cdot \Lambda) \cdot \Gamma. \]

Similarly, applying part (ii) of Lemma 2.17 and (2.19) we get
\[ (\Lambda \square \Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda). \]

Therefore, \((A_{C})^{\ast}\) is a complex Banach \((\Lambda_{C})^{\ast\ast}\)-module.

We now assume that \((A_{C})^{\ast}\) is a complex Banach \((\Lambda_{C})^{\ast\ast}\)-module. Then
\[ (2.18) \quad (\Lambda \cdot \lambda) \cdot \Gamma = \Lambda \cdot (\lambda \cdot \Gamma), \]
\[ (2.19) \quad \lambda \cdot (\Lambda \square \Gamma) = (\lambda \cdot \Lambda) \cdot \Gamma, \]
\[ (2.20) \quad (\Lambda \square \Gamma) \cdot \lambda = \Lambda \cdot (\Gamma \cdot \lambda), \]
for all \((\Lambda, \lambda, \Gamma) \in (A_{C})^{\ast\ast} \times (A_{C})^{\ast} \times (A_{C})^{\ast}\). Let \((\Phi, \varphi, \Psi) \in A^{\ast\ast} \times A^{\ast} \times A^{\ast}\). Then \((J_{2}(\Phi), J_{1}(\varphi), J_{2}(\Psi)) \in (A_{C})^{\ast\ast} \times (A_{C})^{\ast} \times (A_{C})^{\ast}\). By (2.18), we have
\[ (2.21) \quad (J_{2}(\Phi) \cdot J_{1}(\varphi)) \cdot J_{2}(\Psi) = J_{2}(\Phi) \cdot (J_{1}(\varphi) \cdot J_{2}(\Psi)). \]

Applying part (ii) of Lemma 2.17 and (2.21), we get
\[ J_{1}((\Phi \cdot \varphi) \cdot \Psi) = J_{1}(\Phi \cdot \varphi) \cdot J_{2}(\Psi) \]
\[ = J_{2}(\Phi) \cdot J_{1}(\varphi) \cdot J_{2}(\Psi) \]
\[ = J_{2}(\Phi) \cdot J_{1}(\varphi) \cdot J_{2}(\Psi) \]
\[ = J_{2}(\Phi) \cdot J_{1}(\varphi) \cdot J_{2}(\Psi) \]
\[ = J_{2}(\Phi) \cdot J_{1}(\varphi) \cdot J_{2}(\Psi). \]

This implies that \((\Phi \cdot \varphi) \cdot \Psi = \Phi \cdot (\varphi \cdot \Psi)\), since \(J_{1}\) is injective.

By (2.19), we have
\[ (2.22) \quad J_{1}(\varphi) \cdot (J_{2}(\Phi) \cdot J_{2}(\Psi)) = (J_{1}(\varphi) \cdot J_{2}(\Phi)) \cdot J_{2}(\Psi). \]
Applying part (iii) of Lemma 2.6 and (2.7), we get
\[
J_1(\varphi \cdot (\Phi \square \Psi)) = J_1(\varphi) \cdot J_2(\Phi \square \Psi)
\]
\[
= J_1(\varphi) \cdot (J_2(\Phi) \square J_2(\Psi))
\]
\[
= (J_1(\varphi) \cdot J_2(\Phi)) \cdot J_2(\Psi)
\]
\[
= J_1(\varphi \cdot \Phi) \cdot J_2(\Psi).
\]
This implies that \(\varphi \cdot (\Phi \square \Psi) = (\varphi \cdot \Phi) \cdot \Psi\), since \(J_1\) is injective. Similarly, we can show that
\[
(\Phi \square \Psi) \cdot \varphi = \Phi \cdot (\Psi \cdot \varphi).
\]
Therefore, \(A^*\) is a real Banach \(A^{**}\)-module. \(\square\)

Applying Theorem 2.5 and [11, Example 2], we give an example of a real Banach algebra \(A\) for which \(A^\ast\) is not a real Banach \(A^{**}\)-module.

**Example 2.7.** Let \(Z\) be the set of all integer numbers and \(l^1(Z)\) denote the complex Banach algebra consisting of all sequence \(\{a_n\}_{n=-\infty}^{\infty}\) in \(\mathbb{C}\) for which \(\sum_{n=-\infty}^{\infty} |a_n| < \infty\) with convolution product \(*\) defined by
\[
a \ast b = \{c_n\}_{n=-\infty}^{\infty}, \quad a = \{a_n\}_{n=-\infty}^{\infty}, b = \{b_n\}_{n=-\infty}^{\infty} \in l^1(Z),
\]
where \(c_n = \sum_{j=-\infty}^{\infty} a_{n-j}b_j\) for all \(n \in Z\) and with the \(l^1\)-norm \(\| \cdot \|_1\) defined by
\[
\|a\|_1 = \sum_{n=-\infty}^{\infty} |a_n|, \quad a = \{a_n\}_{n=-\infty}^{\infty} \in l^1(Z).
\]
It is shown [11, Example 2] that \((l^1(Z))^\ast\) is not a complex Banach \((l^1(Z))^{**}\)-module.

Let \(\tau : Z \rightarrow Z\) be a bijection additive map. Define
\[
l^1(Z, \tau) = \{ \{a_n\}_{n=-\infty}^{\infty} \in l^1(Z) : a_{\tau(n)} = \overline{a_n} \ (n \in Z) \}.
\]
It is easy to see that \(l^1(Z, \tau)\) is a real closed subalgebra of \(l^1(Z)\) and
\[
l^1(Z) = l^1(Z, \tau) \oplus il^1(Z, \tau).
\]
Hence, \(l^1(Z, \tau)\) is a real Banach algebra with the algebra norm \(\| \cdot \|_1\) and \(l^1(Z)\) is the complexification of \(l^1(Z, \tau)\) with respect to the injective real algebra homomorphism \(J : l^1(Z, \tau) \rightarrow l^1(Z)\) defined by
\[
J(a) = a, \quad a = \{a_n\}_{n=-\infty}^{\infty} \in l^1(Z, \tau).
\]
Since \(\|a - ib\|_1 = \|a + ib\|_1\) for all \(a = \{a_n\}_{n=-\infty}^{\infty}, b = \{b_n\}_{n=-\infty}^{\infty} \in l^1(Z, \tau)\), we deduce that
\[
\max\{\|a\|_1, \|b\|_1\} \leq \|a + ib\|_1 \leq 2 \max\{\|a\|_1, \|b\|_1\}
\]
for all \( a = \{a_n\}_{n=-\infty}^{\infty}, b = \{b_n\}_{n=-\infty}^{\infty} \in l^1(\mathbb{Z}, \tau) \). Therefore, \((l^1(\mathbb{Z}, \tau))^*\) is not a real Banach \((l^1(\mathbb{Z}, \tau))^*)\)-module by Theorem 2.10.

Note that the map \( \tau: \mathbb{Z} \rightarrow \mathbb{Z} \) is a bijection additive map if and only if either \( \tau(n) = n \) for all \( n \in \mathbb{Z} \) or \( \tau(n) = -n \) for all \( n \in \mathbb{Z} \).

We now discuss the relationship between the \((-1)\)-weak amenability of \( A^{**} \) and \((-1)\)-weak amenability of \((A_C)^{**}\). For this purpose we need the following lemma.

**Lemma 2.8.** Let \((A, \| \cdot \|)\) be a real Banach algebra, let \( A_C \) be a complexification of \( A \) with respect to an injective real algebra homomorphism \( J: A \rightarrow A_C \), let \( \| \cdot \| \) be an algebra norm on \( A_C \) satisfying (*) condition and let \((A_C)^{**}\) be the second dual of \((A_C, \| \cdot \|)\). Suppose that \( A^* \) is a real Banach \( A^{**}\)-module. Then:

(i) If \( d \in Z^1_{\mathbb{R}}(A^{**}, A^*) \) and \( \Phi \in A^{**} \), then \( J_1(d(\Phi)) \in (A_C)^* \).

(ii) If \( d \in Z^1_{\mathbb{R}}(A^{**}, A^*) \) then \( \Delta_d \in Z^1_{\mathbb{R}}((A_C)^{**}, (A_C)^*) \), where the map \( \Delta_d: (A_C)^{**} \rightarrow (A_C)^* \) is defined by

\[
\Delta_d(J_2(\Phi) + iJ_2(\Psi)) = J_1(d(\Phi)) + iJ_1(d(\Psi)), \quad \Phi, \Psi \in A^{**}.
\]

(iii) The map \( J_Z: Z^1_{\mathbb{R}}(A^{**}, A^*) \rightarrow Z^1_{\mathbb{R}}((A_C)^{**}, (A_C)^*) \) defined by

\[
J_Z(d) = \Delta_d, \quad d \in Z^1_{\mathbb{R}}(A^{**}, A^*)
\]

is an injective real linear map.

(iv) The complex linear space \( Z^1_{\mathbb{R}}((A_C)^{**}, (A_C)^*) \) is a complexification of the real linear space \( Z^1_{\mathbb{R}}(A^{**}, A^*) \) with respect to the injective linear map \( J_Z \).

(v) If \( \varphi \in A^* \), then \( J_Z(\delta_{\varphi}) = \delta_{J_1(\varphi)} \).

(vi) If \( \lambda \in (A_C)^* \), then \( \delta_{\lambda} = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}) \).

(vii) \( H^1_{\mathbb{R}}(A^{**}, A^*) = \{0\} \) if and only if \( H^1_{\mathbb{R}}((A_C)^{**}, (A_C)^*) = \{0\} \).

**Proof.** Let \( d \in Z^1_{\mathbb{R}}(A^{**}, A^*) \) and \( \Phi \in A^{**} \). Then \( d(\Phi) \in A^* \) and so \( J_1(d(\Phi)) \in (A_C)^* \) by Lemma 2.2. Hence, (i) holds.

Let \( d \in Z^1_{\mathbb{R}}(A^{**}, A^*) \) and define \( \Delta_d: (A_C)^{**} \rightarrow (A_C)^* \) by (2.23). Then \( \Delta_d \) is well-defined by (i). It is easy to see that \( \Delta_d \) is a complex linear map from \((A_C)^{**}\) to \((A_C)^*\). Since \( \| \cdot \| \) be an algebra norm on \( A_C \) satisfying (*) condition, there exist positive constants \( k_1 \) and \( k_2 \) such that

\[
\max\{|a|, |b|\} \leq k_1 \|J(a) + iJ(b)\| \leq k_2 \max\{|a|, |b|\}
\]

for all \( a, b \in A \). Applying part (v) of Lemma 2.2 and part (v) of Lemma 2.3, we get

\[
\|\Delta_d(J_2(\Phi) + iJ_2(\Psi))\|_{op} = \|J_1(d(\Phi) + iJ_1(d(\Psi)))\|_{op} \\
\leq 4k_1 \max\{\|d(\Phi)\|_{op}, \|d(\Psi)\|_{op}\} \\
\leq 4k_1 \|d\|_{op} \max\{\|\Phi\|_{op}, \|\Psi\|_{op}\}
\]
for all $\Phi, \Psi \in A^{**}$. Therefore, $\Delta_d$ is a bounded complex linear operator and

$$
\|\Delta_d\|_{op} \leq 16k_1^2\|d\|_{op}.
$$

By Theorem 2.1, $(A_C)^*$ is complex Banach $(A_C)^{**}$-module. Since $d \in Z_{\mathbb{R}}^1(A^{**}, A^*)$, by Lemma 2.3, for all $\Phi, \Psi \in A^{**}$ we have

$$
\Delta_d(J_2(\Phi)\Box J_2(\Psi)) = \Delta_d(J_2(\Phi \Box \Psi))
$$

This implies that for all $\Phi, \Psi, \Phi', \Psi' \in A^{**}$ we have

$$
\Delta_d(J_2(\Phi) + iJ_2(\Psi)) \Box (J_2(\Phi') + iJ_2(\Psi'))
$$

$$
= \Delta_d(J_2(\Phi)\Box J_2(\Psi')) - (J_2(\Phi)\Box J_2(\Psi'))
$$

$$
+ i((J_2(\Phi)\Box J_2(\Psi')) + (J_2(\Psi)\Box J_2(\Phi')))
$$

$$
= \Delta_d(J_2(\Phi)\Box J_2(\Psi')) - i\Delta_d(J_2(\Psi)\Box J_2(\Phi'))
$$

$$
+ i\Delta_d(J_2(\Phi)\Box J_2(\Psi')) + i\Delta_d(J_2(\Psi)\Box J_2(\Phi'))
$$

$$
= \Delta_d(J_2(\Phi)) \cdot J_2(\Phi') + J_2(\Phi) \cdot \Delta_d(J_2(\Phi'))
$$

$$
- \Delta_d(J_2(\Psi)) \cdot J_2(\Psi') - J_2(\Psi) \cdot \Delta_d(J_2(\Psi'))
$$

$$
+ i(\Delta_d(J_2(\Phi)) \cdot J_2(\Psi') + J_2(\Phi) \cdot \Delta_d(J_2(\Psi')))
$$

$$
+ i(\Delta_d(J_2(\Psi)) \cdot J_2(\Phi') + J_2(\Psi) \cdot \Delta_d(J_2(\Phi')))
$$

$$
= \Delta_d(J_2(\Phi)) + i\Delta_d(J_2(\Psi)) \cdot (J_2(\Phi') + iJ_2(\Psi'))
$$

$$
+ (J_2(\Phi) + iJ_2(\Psi)) \cdot (\Delta_d(J_2(\Phi')) + i\Delta_d(J_2(\Psi')))
$$

$$
= \Delta_d(J_2(\Phi) + iJ_2(\Psi)) \cdot (J_2(\Phi') + iJ_2(\Psi'))
$$

Therefore, $\Delta_d \in Z_{\mathbb{R}}^1((A_C)^{**}, (A_C)^*)$. Hence, (ii) holds.

It is clear that the map $J_Z : Z_{\mathbb{R}}^1(A^{**}, A^*) \rightarrow Z_{\mathbb{R}}^1((A_C)^{**}, (A_C)^*)$, defined by (2.3), is a real linear map. Let $d \in Z_{\mathbb{R}}^1(A^{**}, A^*)$ and $J_Z(d) = 0$. Then $\Delta_d = 0$ and so for each $\Phi \in A^{**}$ we have

$$
0 = \Delta_d(J_2(\Phi)) = J_1(d(\Phi)).
$$
This implies that $d(\Phi) = 0$ for all $\Phi \in A^{**}$, since $J_1$ is injective. Hence, $d = 0$ and so $J_Z$ is injective.

Assume that $D \in Z^1_{(A_C)^*}(A_C)^*$. Define the maps $D_R, D_I : A^{**} \rightarrow A^*$ by

\begin{align*}
D_R(\Phi) &= (D(J_2(\Phi)))_R, \quad (\Phi \in A^{**}), \\
D_I(\Phi) &= (D(J_2(\Phi)))_I, \quad (\Phi \in A^{**}).
\end{align*}

By Lemma 2.1, $D_R$ is well-defined. It is easy to see that $D_R$ is a real linear map from $A^{**}$ to $A^*$. Applying part (iii) of Lemma 2.1 and part (v) of Lemma 2.3, we have

$$\|D_R(\Phi)\|_\circ \leq \frac{k_2}{k_1} \|D(J_2(\Phi))\|_\circ \leq \frac{k_2}{k_1} \|D\|_\circ \|J_2(\Phi)\|_\circ \leq \frac{k_2}{k_1} \|D\|_\circ \frac{4k_2}{k_1} \|\Phi\|_\circ = \frac{4k_2^2}{k_1} \|D\|_\circ \|\Phi\|_\circ$$

for all $\Phi \in A^{**}$. Hence, $D_R$ is a bounded real linear operator and

$$\|D_R\|_\circ \leq \frac{4k_2^2}{k_1} \|D\|_\circ.$$

On the other hand, for all $\Phi, \Psi \in A^{**}$ we have

\begin{align*}
D_R(\Phi \square \Psi) &= (D(J_2(\Phi \square \Psi)))_R \\
&= (D(J_2(\Phi) \square J_2(\Psi)))_R \\
&= (D(J_2(\Phi)) \cdot J_2(\Psi) + J_2(\Phi) \cdot D(J_2(\Psi)))_R \\
&= (D(J_2(\Phi)) \cdot J_2(\Psi))_R + (J_2(\Phi) \cdot D(J_2(\Psi)))_R \\
&= (D(J_2(\Phi)))_R \cdot \Phi + \Phi \cdot D(J_2(\Psi))_R \\
&= D_R(\Phi) \cdot \Psi + \Phi \cdot D_R(\Psi).
\end{align*}

Therefore, $D_R$ is a real $A^*$-derivation on $A^{**}$ and so $D_R \in Z^1_{(A^*)}(A^{**}, A^*)$.

Similarly, we can show that $D_I \in Z^1_{(A^*)}(A^{**}, A^*)$.

Now we show that

\begin{equation}
D = J_Z(D_R) + iJ_Z(D_I).
\end{equation}

Let $\Phi \in A^{**}$. For each $a \in A$ we have

\begin{align*}
D(J_2(\Phi))(J(a)) &= \text{Re} D(J_2(\Phi))(J(a)) + i\text{Im} D(J_2(\Phi))(J(a)) \\
&= D_R(\Phi)(a) + iD_I(\Phi)(a)
\end{align*}
\[ J_1(D_R(\Phi))(J(a)) + iJ_1(D_I(\Phi))(J(a)) \]
\[ = (J_1(D_R(\Phi)) + iJ_1(D_I(\Phi)))(J(a)) \]
\[ = (J_Z(D_R)(J_2(\Phi)) + iJ_Z(D_I)(J_2(\Phi)))(J(a)) \]
\[ = ((J_Z(D_R) + iJ_Z(D_I))(J_2(\Phi)))(J(a)). \]

This implies that

\[ (2.28) \quad D(J_2(\Phi)) = (J_Z(D_R) + iJ_Z(D_I))(J_2(\Phi)), \]

since \( D(J_2(\Phi)) \) and \((J_Z(D_R) + iJ_Z(D_I))(J_2(\Phi))\) are complex linear mappings from \( A_C \) to \( \mathbb{C} \). Since \( D \) and \( J_Z(D_R) + iJ_Z(D_I) \) are complex linear mappings from \((A_C)^*\) to \((A_C)^*\) and \((2.28)\) holds for each \( \Phi \in A^{**} \), we deduce that

\[ D(J_2(\Phi) + iJ_2(\Psi)) = (J_Z(D_R) + iJ_Z(D_I))(J_2(\Phi) + iJ_2(\Psi)) \]

for all \( \Phi, \Psi \in A^{**} \). Hence, \((2.27)\) holds. Since \((2.27)\) holds for all \( D \in Z^1_C((A_C)^*, (A_C)^*) \), we have

\[ (2.29) \quad Z^1_C((A_C)^*, (A_C)^*) = J_Z(Z^1_R(A^{**}, A^*)) + iJ_Z(Z^1_R(A^{**}, A^*)). \]

Let \( D \in J_Z(Z^1_R(A^{**}, A^*)) \cap iJ_Z(Z^1_R(A^{**}, A^*)) \). Then there exist two functions \( d_1, d_2 \in Z^1_R(A^{**}, A^*) \) such that \( D = J_Z(d_1) = iJ_Z(d_2) \). Hence, for each \( \Phi \in A^{**} \) we have

\[ J_1(d_1(\Phi)) = (J_Z(d_1))(J_2(\Phi)) \]
\[ = (iJ_Z(d_2))(J_2(\Phi)) \]
\[ = iJ_1(d_2(\Phi)), \]

and so \( J_1(d_1(\Phi)) = 0 \), since \( J_1(A^*) \cap iJ_1(A^*) = \{0\} \). This implies that \( d_1(\Phi) = 0 \) for all \( \Phi \in A^{**} \), since \( J_1 \) is injective. Hence, \( d_1 = 0 \) and so \( D = J_Z(d_1) = 0 \). Therefore,

\[ (2.30) \quad J_Z(Z^1_R(A^{**}, A^*)) \cap iJ_Z(Z^1_R(A^{**}, A^*)) = \{0\}. \]

From \((2.29)\) and \((2.30)\) we obtain

\[ Z^1_C((A_C)^*, (A_C)^*) = J_Z(Z^1_R(A^{**}, A^*)) \oplus iJ_Z(Z^1_R(A^{**}, A^*)). \]

Therefore, \((iv)\) holds.

Let \( \varphi \in A^* \). Since

\[ J_Z(\delta_\varphi)(J_2(\Phi) + iJ_2(\Psi)) = J_1(\delta_\varphi(\Phi)) + iJ_1(\delta_\varphi(\Psi)) \]
\[ = J_1(\Phi \cdot \varphi - \varphi \cdot \Phi) + iJ_1(\Psi \cdot \varphi - \varphi \cdot \Psi) \]
\[ = (J_1(\Phi \cdot \varphi) - J_1(\varphi \cdot \Phi)) \]
\[ + i(J_1(\Psi \cdot \varphi) - J_1(\varphi \cdot \Psi)) \]
\[ = (J_2(\Phi) \cdot J_1(\varphi) - J_1(\varphi) \cdot J_2(\Phi)). \]
for all \( \Phi, \Psi \in A^{**} \), we deduce that \( J_Z(\delta_{\varphi}) = \delta_{J_1(\varphi)} \). Hence (v) holds.

Let \( \lambda \in (A_C)^* \). By parts (ii) and (iii) of Lemma 2.1 and part (iii) of Lemma 2.2, we have

\[
(2.31) \quad \lambda = J_1(\lambda_R) + iJ_1(\lambda_I).
\]

Since \( J_Z(\delta_{\lambda_R}), \delta_{J_1(\lambda_R)} \in Z_C^1((A_C)^**, (A_C)^*) \) and

\[
J_Z(\delta_{\lambda_R})(J_2(\Phi)) = J_1(\delta_{\lambda_R}(\Phi))
\]

\[
= J_1(\Phi \cdot \lambda_R - \lambda_R \cdot \Phi)
\]

\[
= J_1(\Phi \cdot \lambda_R) - J_1(\lambda_R \cdot \Phi)
\]

\[
= J_2(\Phi) \cdot J_1(\lambda_R) - J_1(\lambda_R) \cdot J_2(\Phi)
\]

\[
= \delta_{J_1(\lambda_R)}(J_2(\Phi))
\]

for all \( \Phi \in A^{**} \), we conclude that

\[
J_Z(\delta_{\lambda_R})(J_2(\Phi) + iJ_2(\Psi)) = \delta_{J_1(\lambda_R)}(J_2(\Phi) + iJ_2(\Psi))
\]

for all \( \Phi, \Psi \in A^{**} \). Hence,

\[
(2.32) \quad J_Z(\delta_{\lambda_R}) = \delta_{J_1(\lambda_R)}.
\]

Similar to the argument above we can obtain

\[
(2.33) \quad J_Z(\delta_{\lambda_I}) = \delta_{J_1(\lambda_I)}.
\]

Applying (2.32), (2.33) and (2.31), we get

\[
J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}) = \delta_{J_1(\lambda_R) + iJ_1(\lambda_I)}
\]

\[
= \delta_{J_1(\lambda_R) + iJ_1(\lambda_I)}
\]

\[
= \delta_{\lambda}.
\]

Hence, (vi) holds.

To prove (vii), we first assume that

\[
(2.34) \quad H_{\xi}^1(A^{**}, A^*) = \{0\}.
\]

Let \( D \in Z_C^1((A_C)^**, (A_C)^*) \). By (iv), there exist unique elements \( d, d' \in Z^1_{\mathbb{R}}(A^{**}, A^*) \) such that

\[
(2.35) \quad D = J_Z(d) + iJ_Z(d').
\]

By (2.34), there exist \( \varphi, \varphi' \in A^* \) such that

\[
(2.36) \quad d = \delta_{\varphi}, \quad d' = \delta_{\varphi'}.
\]
Set $\lambda = J_1(\varphi) + iJ_1(\varphi')$. Then $\lambda \in (A_C)^*$ and
\begin{equation}
\varphi = \lambda_R, \quad \varphi' = \lambda_I.
\end{equation}
From (2.38), (2.39) and (2.40) we obtain
\begin{equation}
D = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}).
\end{equation}
Since $\lambda \in (A_C)^*$, we deduce that
\begin{equation}
\delta_{\lambda} = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}),
\end{equation}
by (vi). From (2.38) and (2.39), we have $D = \delta_{\lambda}$ and so
\begin{equation*}
H^1_c((A_C)^*, (A_C)^*) = \{0\}.
\end{equation*}
We now assume that
\begin{equation}
H^1_c((A_C)^*, (A_C)^*) = \{0\}.
\end{equation}
Let $d \in Z^1_c(A^{**}, A^*)$. Then $J_Z(d) \in Z^1_c((A_C)^*, (A_C)^*)$. By (2.40), there exists $\lambda \in (A_C)^*$ such that $J_Z(d) = \delta_{\lambda}$, and so by (vi) we have
\begin{equation}
J_Z(d) + iJ_Z(0) = J_Z(d) = J_Z(\delta_{\lambda_R}) + iJ_Z(\delta_{\lambda_I}).
\end{equation}
Applying (2.41) and (iv), we deduce that $J_Z(d) = J_Z(\delta_{\lambda_R})$ and so $d = \delta_{\lambda_R}$, since $J_Z$ is injective. Therefore, $H^1_R(A^{**}, A^*) = \{0\}$ and so (vii) holds.

**Theorem 2.9.** Let $(A, \| \cdot \|)$ be a real Banach algebra, let $A_C$ be a complexification of $A$ with respect to an injective real algebra homomorphism $J : A \to A_C$, let $\| \cdot \|$ be an algebra norm on $A_C$ satisfying the (*) condition, and let $(A_C)^*$ be the dual space of $(A_C, \| \cdot \|)$. Then $A^{**}$ is $(-1)$-weakly amenable if and only if $(A_C)^*$ is $(-1)$-weakly amenable.

**Proof.** We first assume that $A^{**}$ is $(-1)$-weakly amenable. Then $A^*$ is a real Banach $A^{**}$-module and $H^1_c((A_C)^*, A^*) = \{0\}$. Hence, $(A_C)^*$ is a complex Banach $(A_C)^{**}$-module by Theorem 2.3 and $H^1_c((A_C)^{**}, (A_C)^*) = \{0\}$ by part (vii) of Lemma 2.8. Therefore, $(A_C)^{**}$ is $(-1)$-weakly amenable.

We now assume that $(A_C)^{**}$ is $(-1)$-weakly amenable. Then $(A_C)^*$ is a complex Banach $(A_C)^{**}$-module and $H^1_c((A_C)^{**}, (A_C)^*) = \{0\}$. Hence, $A^*$ is a real Banach $A^{**}$-module by Theorem 2.4 and so we conclude that $H^1_c(A^{**}, A^*) = \{0\}$ by part (vii) of Lemma 2.8. Therefore, $A^{**}$ is $(-1)$-weakly amenable.

Here, as applications of Theorem 2.9, we give some examples of real Banach algebras which their second duals of some them are and of others are not $(-1)$-weakly amenable.
Example 2.10. Let $A = \mathbb{R}$ with the zero multiplication. Then $A$ is a real Banach algebra with the Euclidean norm $| \cdot |$. Set $A_C = \mathbb{C}$ with the zero multiplication. Clearly, $A_C$ is a complex Banach algebra with Euclidean norm $| \cdot |$ and $A_C = A + iA$. Hence, $A_C$ is a complexification of $A$ with respect to the injective real algebra homomorphism $J : A \to A_C$ defined by $J(a) = a + (a \in \mathbb{R})$. Moreover,

$$\max\{|a|, |b|\} \leq |a + ib| \leq 2 \max\{|a|, |b|\},$$

for all $a, b \in A$. It is known [11, Example 2.2] that $(A_C)^{**}$ is not $(-1)$-weakly amenable. Therefore, $A^{**}$ is not $(-1)$-weakly amenable by Theorem 2.3.

Example 2.11. Let $S$ be a discrete semigroup. We denote by $l^1(S)$ the set of all complex-valued functions $f$ on $S$ for which $\sum_{s \in S} |f(s)| < \infty$. Then $l^1(S)$ is a self-adjoint complex Banach algebra with the convolution product $\ast$ defined by

$$(f \ast g)(r) = \sum_{s,t \in S, s \ast t = r} f(s)g(t), \quad f, g \in l^1(S),$$

and with the algebra norm $\| \cdot \|_1$ defined by

$$\|f\|_1 = \sum_{s \in S} |f(s)|, \quad f \in l^1(S).$$

Let $\tau : S \to S$ be a self-map of $S$ satisfying $\tau(st) = \tau(s)\tau(t)$ for all $s, t \in S$ and $\tau(\tau(s)) = s$ for all $s \in S$. It is easy to see $\tilde{f} \circ \tau \in l^1(S)$ for all $f \in l^1(S)$. Define

$$l^1(S, \tau) = \{f \in l^1(S) : \tilde{f} \circ \tau = f\}.$$ 

Then $l^1(S, \tau)$ is a real closed subalgebra of $l^1(S)$ and

$$l^1(S) = l^1(S, \tau) \oplus il^1(S, \tau).$$

Hence, $l^1(S)$ is the complexification of $l^1(S, \tau)$ with respect to the injective real algebra homomorphism $J : l^1(S, \tau) \to l^1(S)$ defined by $J(f) = f$ (if $f \in l^1(S, \tau)$). Since $\|f - ig\|_1 = \|f + ig\|_1$ for all $f, g \in l^1(S, \tau)$, we deduce that

$$\max\{|f|_1, |g|_1\} \leq |f + ig|_1 \leq 2 \max\{|f|_1, |g|_1\},$$

for all $f, g \in l^1(S, \tau)$. It is known [11, Example 2.3] that if $S^2 \neq S$ then $(l^1(S))^\ast\ast$ is not $(-1)$-weakly amenable. Therefore, if $S^2 \neq S$ then $(l^1(S, \tau))^\ast\ast$ is not $(-1)$-weakly amenable by Theorem 2.3.

Example 2.12. Let $N^{<\omega} = \cup_{k \in N} N^k$ and let $P$ be the set of all elements $p = (p_1, \ldots, p_k) \in N^{<\omega}$ such that $k \geq 2$ and $p_j < p_{j+1}$ for all $j \in$
{1, \ldots, k-1}. For a sequence \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) in \( F \) and for \( p = (p_1, \ldots, p_k) \in P \), define \( N(\alpha, p) \) by

\[
2(N(\alpha, p))^2 = \left( \sum_{j=1}^{k-1} |\alpha_{p_{j+1}} - \alpha_{p_j}|^2 \right) + |\alpha_{p_n} - \alpha_{p_1}|^2.
\]

For each sequence \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) in \( F \), we set

\[
N(\alpha) = \sup\{N(\alpha, p) : p \in P\}.
\]

Then \( N(\alpha) \in [0, \infty] \) for all sequence \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) in \( F \). Define

\[
J_F = \{ \alpha = \{\alpha_n\}_{n=1}^{\infty} : \alpha \in F, N(\alpha) < \infty \}.
\]

Then \( J_F \) is a closed subalgebra of Banach algebra \((l_F^\infty(N), \| \cdot \|_\infty)\) over \( F \), where \( l_F^\infty(N) \) is the set of all sequence \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) in \( F \) for which \( \sup\{|\alpha_n| : n \in N\} < \infty \) and \( \| \cdot \|_\infty \) is the algebra norm on \( l_F^\infty(N) \) over \( F \) defined by

\[
\|\alpha\|_\infty = \sup\{|\alpha_n| : n \in N\}, \quad (\alpha = \{\alpha_n\}_{n=1}^{\infty} \in l_F^\infty(N)).
\]

\( J_F \) is called the James algebra over \( F \). It is clear that \( J_F \) is a real subalgebra of \( J_C \) and \( J_C = J_R \oplus iJ_R \). Hence, \( J_C \) is a complexification of \( J_R \) with the injective real algebra homomorphism \( J : J_R \rightarrow J_C \) defined by \( J(\alpha) = \alpha \quad (\alpha \in J_R) \). It is easy to see that

\[
\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|\alpha + i\beta\|_\infty \leq 2\max\{\|\alpha\|_\infty, \|\beta\|_\infty\},
\]

for all \( \alpha = \{\alpha_n\}_{n=1}^{\infty}, \beta = \{\beta_n\}_{n=1}^{\infty} \in J_R \).

By \( R \), Theorem 4.1.45], we have some properties of \( J_C \) as:

\begin{itemize}
  \item[(i)] \( J_C \) is Arens regular,
  \item[(ii)] \( J_C \) is weakly amenable,
  \item[(iii)] \( J_C \) is not amenable.
\end{itemize}

It is shown \( S \), Example 2.2] that \((J_C)^{**}\) is \((-1)\)-weakly amenable. Therefore, we deduce that \( J_R \) is weakly amenable by \( S \), Theorem 2.5], \( J_R \) is not amenable by \( S \), Theorem 2.4], \( J_R \) is Arens regular by Theorem \( 2.3 \) and \((J_R)^{**}\) is \((-1)\)-weakly amenable by Theorem \( 2.3 \).

**Example 2.13.** Let \( 1 < p < \infty \) and let \( l^p(Z) \) denote the set of all sequences \( \alpha = \{\alpha_n\}_{n=-\infty}^{\infty} \) in \( C \) for which \( \sum_{n=-\infty}^{\infty} |\alpha_n|^p < \infty \). Then \( l^p(Z) \) with the pointwise addition and scalar multiplication is a complex Banach space with the norm \( \| \cdot \|_p \) defined by

\[
\|\alpha\|_p = \left( \sum_{n=-\infty}^{\infty} |\alpha_n|^p \right)^{\frac{1}{p}}, \quad (\alpha = \{\alpha_n\}_{n=-\infty}^{\infty} \in l^p(Z)).
\]

Moreover, \( l^p(Z) \) with the pointwise multiplication becomes a complex algebra and \( \| \cdot \|_p \) is a complete algebra norm on \( l^p(Z) \). Hence, \( (l^p(Z), \| \cdot \|_p) \).
\[\|p\|_p\) is a complex Banach algebra. For each \(m \in \mathbb{Z}\) we have \(e_m \in \ell^p(\mathbb{Z})\) and \(e_m e_m = e_m\) whenever \(e_m = \{e_{m,n}\}_{n=-\infty}^{\infty}\) and

\[
e_{m,n} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad (n \in \mathbb{Z}).
\]

Moreover, \(\ell^p(\mathbb{Z})\) generates by \(\{e_m : m \in \mathbb{Z}\}\). Hence, \(\ell^p(\mathbb{Z})\) is weakly amenable by \[\text{Proposition 2.8.72(i)}\]. Therefore, \((\ell^p(\mathbb{Z}))^*\) is \((-1)\)-weakly amenable since \(\ell^p(\mathbb{Z})\) is reflexive.

Let \(\tau : \mathbb{Z} \rightarrow \mathbb{Z}\) be a bijection additive map. Define

\[\ell^p(\mathbb{Z}, \tau) = \{ \alpha = \{\alpha_n\}_{n=-\infty}^{\infty} \in \ell^p(\mathbb{Z}) : \alpha_{\tau(n)} = \overline{\alpha}_n, \quad (n \in \mathbb{Z}) \}.
\]

It is easy to see that \(\ell^p(\mathbb{Z}, \tau)\) is closed real subalgebra of \(\ell^p(\mathbb{Z})\) and \(\ell^p(\mathbb{Z}) = \ell^p(\mathbb{Z}, \tau) \oplus i\ell^p(\mathbb{Z}, \tau)\). Hence, \((\ell^p(\mathbb{Z}, \tau), \| \cdot \|_p)\) is a real Banach algebra and \(\ell^p(\mathbb{Z})\) is a complexification of \(\ell^p(\mathbb{Z}, \tau)\) with respect to the injective real algebra homomorphism \(J : \ell^p(\mathbb{Z}, \tau) \rightarrow \ell^p(\mathbb{Z})\) defined by \(J(\alpha) = \alpha \cdot (\alpha \in \ell^p(\mathbb{Z}, \tau))\). Since \(\|\alpha - i\beta\|_p = \|\alpha + i\beta\|_p\) for all \(\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}, \beta = \{\beta_n\}_{n=-\infty}^{\infty} \in \ell^p(\mathbb{Z}, \tau)\), we deduce that

\[
\max\{\|\alpha\|_p, \|\beta\|_p\} \leq \|\alpha + i\beta\|_p \leq 2 \max\{\|\alpha\|_p, \|\beta\|_p\}
\]

for all \(\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}, \beta = \{\beta_n\}_{n=-\infty}^{\infty} \in \ell^p(\mathbb{Z}, \tau)\). Therefore, \(\ell^p(\mathbb{Z}, \tau)\) is reflexive by the reflexivity of \(\ell^p(\mathbb{Z})\) and part (vii) of Lemma \[\text{2.5}\]. \(\ell^p(\mathbb{Z}, \tau)\) is weakly amenable by \[\text{2.4 Theorem 2.5}\] and \((\ell^p(\mathbb{Z}, \tau))^*\) is \((-1)\)-weakly amenable by Theorem \[\text{2.4}\].

**Example 2.14.** Let \(X\) be a compact Hausdorff space. We denote by \(C_\mathbb{F}(X)\) the algebra of all \(\mathbb{F}\)-valued continuous functions on \(X\) over \(\mathbb{F}\). Then \(C_\mathbb{F}(X)\) is a Banach algebra over \(\mathbb{F}\) with the uniform norm \(\| \cdot \|_X\) defined by

\[
\|f\|_X = \sup\{|f(x)| : x \in X\}, \quad (f \in C(X)).
\]

We write \(C(X)\) instead of \(C_\mathbb{C}(X)\).

A self-map \(\tau : X \rightarrow X\) is called a topological involution on \(X\) if \(\tau\) is continuous and \(\tau(\tau(x)) = x\) for all \(x \in X\). Clearly, \(f \circ \tau \in C(X)\) for all \(f \in C(X)\). Define

\[C(X, \tau) = \{ f \in C(X) : \bar{f} \circ \tau = f \}.
\]

Then \(C(X, \tau)\) is a real closed subalgebra of \(C(X)\), \(1_X \in C(X, \tau)\) and \(i1_X \notin C(X, \tau)\), where \(1_X\) is the constant function on \(X\) with value 1. Moreover, \(C(X) = C(X, \tau) \oplus iC(X, \tau)\). Hence, \(C(X)\) is a complexification of \(C(X, \tau)\) with respect to the injective real algebra homomorphism \(J : C(X, \tau) \rightarrow C(X)\) defined by \(J(f) = f\) \((f \in C(X, \tau))\). Since \(\|f - ig\|_X = \|f + ig\|_X\) for all \(f, g \in C(X, \tau)\), we deduce that

\[
\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2 \max\{\|f\|_X, \|g\|_X\},
\]
for all \( f, g \in C(X, \tau) \). Real Banach algebra \( C(X, \tau) \) was first defined by Kulkarni and Limaye in [1]. For further general facts about \( C(X, \tau) \) and certain real subalgebras we refer to [1].

Clearly, \( C(X) \) is a complex \( C^* \)-algebra with the natural algebra involution \( f \mapsto \bar{f} : C(X) \to C(X) \). Hence, \( C(X) \) is Arens regular and, by [1], Corollary 3.7], \( (C(X))^* \) is \((-1)\)-weakly amenable. Therefore, if \( \tau \) is a topological involution on \( X \) then \( C(X, \tau) \) is Arens regular by Theorem 2.5 and \( (C(X, \tau))^* \) is \((-1)\)-weakly amenable by theorem 2.9.

Example 2.15. Let \((X, d)\) be an infinite compact metric space and let \( \alpha \in (0, 1] \). We denote by \( \text{Lip}_F(X, d^\alpha) \) the set of all \( F \)-valued functions \( f \) on \( X \) for which

\[
p_{(X, d^\alpha)}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X, x \neq y \right\} < \infty.
\]

Clearly, \( \text{Lip}_F(X, d^\alpha) \) is a subalgebra of \( C_F(X) \) and \( 1_X \in \text{Lip}_F(X, d^\alpha) \). Moreover, \( \text{Lip}_F(X, d^\alpha) \) is a Banach algebra over \( F \) with the \( \alpha \)-Lipschitz norm

\[
\|f\|_{\text{Lip}(X, d^\alpha)} = \|f\|_X + p_{(X, d^\alpha)}(f), \quad (f \in \text{Lip}_F(X, d^\alpha)).
\]

\( \text{Lip}_F(X, d^\alpha) \) is called the Lipschitz algebra of order \( \alpha \) on \((X, d)\) over \( F \). This algebra was first introduced by Sherbert in [1]. We write \( \text{Lip}(X, d^\alpha) \) instead of \( \text{Lip}_C(X, d^\alpha) \).

Let \((X, d)\) be a metric space. A Lipschitz mapping on \((X, d)\) is a self-map \( \tau : X \to X \) for which there exist a positive constant \( M \) such that \( d(\tau(x), \tau(y)) \leq Md(x, y) \) for all \( x, y \in X \). For a Lipschitz mapping \( \tau : X \to X \) on \((X, d)\), the constant Lipschitz of \( \tau \) is denoted by \( p(\tau) \) and defined by

\[
p(\tau) = \sup \left\{ \frac{d(\tau(x), \tau(y))}{d(x, y)} : x, y \in X, x \neq y \right\}.
\]

A self-map \( \tau : X \to X \) is called a Lipschitz involution on \((X, d)\) if \( \tau \) is a Lipschitz mapping and \( \tau(\tau(x)) = x \) for all \( x \in X \).

Let \((X, d)\) be a compact metric space, let \( \alpha \in (0, 1] \) and let \( \tau : X \to X \) be a Lipschitz involution on \((X, d)\). It is easy to see that \( f \circ \tau \in \text{Lip}(X, d^\alpha) \) for all \( f \in \text{Lip}(X, d^\alpha) \). Define

\[
\text{Lip}(X, d^\alpha, \tau) = \{ f \in \text{Lip}(X, d^\alpha) : f \circ \tau = f \}.
\]

Then \( \text{Lip}(X, d^\alpha, \tau) \) is a real closed subalgebra of \( \text{Lip}(X, d^\alpha) \), containing \( 1_X \), \( i1_X \notin \text{Lip}(X, d^\alpha, \tau) \) and

\[
\text{Lip}(X, d^\alpha) = \text{Lip}(X, d^\alpha, \tau) \oplus i\text{Lip}(X, d^\alpha, \tau).
\]

Hence, \( (\text{Lip}(X, d^\alpha, \tau), \| \cdot \|_{\text{Lip}(X, d^\alpha)}) \) is a real Banach algebra and the complex algebra \( \text{Lip}(X, d^\alpha) \) is a complexification of \( \text{Lip}(X, d^\alpha, \tau) \) with
respect to the injective real algebra homomorphism $J : \text{Lip}(X, d^\alpha, \tau) \to \text{Lip}(X, d^\alpha)$ by $J(f) = f$ ($f \in \text{Lip}(X, d^\alpha, \tau)$). Moreover,

$$\max \{\|f\|_{\text{Lip}(X, d^\alpha)}, \|g\|_{\text{Lip}(X, d^\alpha)}\} \leq C\|f + ig\|_{\text{Lip}(X, d^\alpha)}$$

$$\leq 2C \max \{\|f\|_{\text{Lip}(X, d^\alpha)}, \|g\|_{\text{Lip}(X, d^\alpha)}\}$$

for all $f, g \in \text{Lip}(X, d^\alpha, \tau)$, where $C = (p(\tau))^{\alpha}$ (see [1]).

By [20, Theorem 9.2], $\text{Lip}(X, d^\alpha)$ has a nonzero continuous point derivation. Hence, $(\text{Lip}(X, d^\alpha))^{**}$ is not $(-1)$-weakly amenable by [11, Theorem 2.6]. Therefore, if $\tau : X \to X$ is a Lipschitz involution on $(X, d)$ then $(\text{Lip}(X, d^\alpha, \tau))^{**}$ is not $(-1)$-weakly amenable by Theorem 2.9.

**Example 2.16.** Let $(X, d)$ be a compact metric space, let $K$ be a nonempty compact subset of $X$ and let $\alpha \in (0, 1]$. We denote by $\text{Lip}(X, K, d^\alpha)$ the set of all $f \in C(X)$ for which $f|_K \in \text{Lip}(K, d^\alpha)$. Then $\text{Lip}(X, K, d^\alpha)$ is a complex subalgebra of $C(X)$ and $\text{Lip}(X, d^\alpha)$ is a complex subalgebra of $\text{Lip}(X, K, d^\alpha)$. Moreover, $\text{Lip}(X, K, d^\alpha) = C(X)$ if $K$ is finite and $\text{Lip}(X, K, d^\alpha) = \text{Lip}(X, d^\alpha)$ if $X \setminus K$ is finite.

Furthermore, $\text{Lip}(X, K, d^\alpha)$ is a complex Banach algebra with the algebra norm $\| \cdot \|_{\text{Lip}(X, K, d^\alpha)}$ defined by

$$\|f\|_{\text{Lip}(X, K, d^\alpha)} = \|f\|_X + p(K, d^\alpha)(f), \quad f \in \text{Lip}(X, K, d^\alpha).$$

$\text{Lip}(X, K, d^\alpha)$ is called extended Lipschitz algebra of order $\alpha$ on $(X, d)$ with respect to $K$. This algebra was first studied in [1].

By [16, Theorem 3.3], $\text{Lip}(X, K, d^\alpha)$ has a nonzero continuous point derivation if $\text{int}(K) \cap K' \neq \emptyset$ where $\text{int}(K)$ is the set of all interior points of $K$ and $K'$ is the set of all limit points of $K$ in $(X, d)$. Therefore, if $\text{int}(K) \cap K' \neq \emptyset$ then $(\text{Lip}(X, K, d^\alpha))^{**}$ is not $(-1)$-weakly amenable by [12, Theorem 2.6].

Let $(X, d)$ be a compact metric space, let $K$ be compact subset of $X$, let $\alpha \in (0, 1]$ and let $\tau$ be a Lipschitz involution on $(X, d)$ such that $\tau(K) = K$. Clearly, $f \circ \tau \in \text{Lip}(X, K, d^\alpha)$ for all $f \in \text{Lip}(X, K, d^\alpha)$. Define

$$\text{Lip}(X, K, d^\alpha, \tau) = \{f \in \text{Lip}(X, K, d^\alpha) : f \circ \tau = f\}.$$ 

It is easy to see that $\text{Lip}(X, K, d^\alpha, \tau)$ is a real closed subalgebra of $\text{Lip}(X, K, d^\alpha)$, $1_X \in \text{Lip}(X, K, d^\alpha, \tau)$ and

$$\text{Lip}(X, K, d^\alpha) = \text{Lip}(X, K, d^\alpha, \tau) \oplus i\text{Lip}(X, K, d^\alpha, \tau).$$

Hence, $(\text{Lip}(X, K, d^\alpha, \tau), \| \cdot \|_{\text{Lip}(X, K, d^\alpha)})$ is a real Banach algebra and $\text{Lip}(X, K, d^\alpha)$ is a complexification of $\text{Lip}(X, K, d^\alpha, \tau)$ with the injective real algebra homomorphism $J : \text{Lip}(X, K, d^\alpha, \tau) \to \text{Lip}(X, K, d^\alpha)$.
defined by \( J(f) = f \quad (f \in \text{Lip}(X, K, d^\alpha, \tau)) \). Moreover,
\[
\max\{\|f\|_B, \|g\|_B\} \leq C\|f + ig\|_B \\
\leq 2C \max\{\|f\|_B, \|g\|_B\},
\]
for all \( f, g \in \text{Lip}(X, K, d^\alpha, \tau) \) where \( B = \text{Lip}(X, K, d^\alpha) \) and \( C = (p(\tau))^{\alpha} \).

Therefore, if \( \text{int}(K) \cap K' \neq \emptyset \) and \( \tau : X \to X \) is a Lipschitz involution on \( (X, d) \) with \( \tau(K) = K \), then \( \text{Lip}(X, K, d^\alpha, \tau) \) is not weakly amenable by [2, Theorem 2.5] and \( (\text{Lip}(X, K, d^\alpha, \tau))^{**} \) is not \((-1)\)-weakly amenable by Theorem [4].

**Example 2.17.** Let \( (X, d) \) be an infinite compact metric space and \( \alpha \in (0, 1) \). We denote by \( \text{lip}_F(X, d^\alpha) \) the set of all \( f \in \text{Lip}_F(X, d^\alpha) \) for which \( \lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} = 0 \), i.e., for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \frac{|f(x) - f(y)|}{d^\alpha(x,y)} < \varepsilon \) whenever \( x, y \in X \) with \( 0 < d(x,y) < \delta \). Then \( \text{lip}_F(X, d^\alpha) \) is a closed subalgebra of \( \text{Lip}_F(X, d^\alpha) \) over \( F \), and \( 1_X \in \text{lip}_F(X, d^\alpha) \). Hence, \( (\text{lip}_F(X, d^\alpha), \|\cdot\|_{\text{Lip}(X, d^\alpha)}) \) is a Banach algebra over \( F \). This algebra is called the little Lipschitz algebra of order \( \alpha \) on \( (X, d) \) over \( F \) and was first introduced by Sherbert in [20]. We write \( \text{lip}(X, d^\alpha) \) instead of \( \text{lip}_C(X, d^\alpha) \).

Let \( (X, d) \) be an infinite compact metric space, let \( \alpha \in (0, 1) \) and let \( B = \text{lip}(X, d^\alpha) \). For each \( x \in X \) the map \( e_{B,x} : B \to \mathbb{C} \) defined by
\[
e_{B,x}(f) = f(x), \quad f \in B,
\]
belongs to \( B^* \). Moreover, \( \|e_{B,x} - e_{B,y}\|_{\text{op}} \leq d^\alpha(x,y) \) for all \( x, y \in X \) and so the map \( E_{B,X} : X \to B^* \) defined by
\[
E_{B,X}(x) = e_{B,x}, \quad x \in X,
\]
is a continuous function from \( (X, d) \) to \( (B^*, \|\cdot\|_{\text{op}}) \). We know [4, Theorem 3.5] that the map \( \eta : B^{**} \to \text{Lip}(X, d^\alpha) \) defined by
\[
\eta(\Lambda) = \Lambda \circ E_{B,X}, \quad \Lambda \in B^{**},
\]
is a complex linear isometry from \( (B^{**}, \|\cdot\|_{\text{op}}) \) onto \( (\text{Lip}(X, d^\alpha), \|\cdot\|_{\text{Lip}(X, d^\alpha)}) \). It is shown [4, Theorem 3.8] that \( B \) is Arens regular and \( \eta \) is an algebra homomorphism. This implies that \( B^* \) is a complex Banach \( B^{**} \)-module.

Let \( \tau : X \to X \) be a Lipschitz involution on \( (X, d) \). It is easy to see that \( \tilde{f} \circ \tau \in B \) for all \( f \in B \). Define
\[
\text{lip}(X, d^\alpha, \tau) = \{ f \in B = \text{lip}(X, d^\alpha) : \tilde{f} \circ \tau = f \}.
\]
Then \( \text{lip}(X, d^\alpha, \tau) \) is a real closed subalgebra of \( B \) and
\[
\text{lip}(X, d^\alpha) = \text{lip}(X, d^\alpha, \tau) \oplus i\text{lip}(X, d^\alpha, \tau).
\]
Therefore, \((\text{lip}(X,d^\alpha), \| \cdot \|_{\text{lip}(X,d^\alpha)})\) is a real Banach algebra and the complex algebra \(\text{lip}(X,d^\alpha)\) is a complexification of \(\text{lip}(X,d^\alpha, \tau)\) with respect to the injective real algebra homomorphism \(J : \text{lip}(X,d^\alpha, \tau) \rightarrow \text{lip}(X,d^\alpha)\) defined by \(J(f) = f \quad (f \in \text{lip}(X,d^\alpha, \tau))\). Moreover,
\[
\max\{\|f\|_{\text{Lip}(X,d^\alpha)}, \|g\|_{\text{Lip}(X,d^\alpha)}\} \leq C\|f + ig\|_{\text{Lip}(X,d^\alpha)} \\
\leq 2C \max\{\|f\|_{\text{Lip}(X,d^\alpha)}, \|g\|_{\text{Lip}(X,d^\alpha)}\},
\]
for all \(f, g \in \text{lip}(X,d^\alpha, \tau)\) where \(C = (p(\tau))^{\alpha}\) (see [1]).

By Theorem 2.5, we deduce that \(\text{lip}(X,d^\alpha, \tau)\) is Arens regular.

Let \(T = \{Z \in \mathbb{C} : |z| = 1\}\), let \(d\) be the Euclidean metric on \(T\) and let \(\alpha \in (\frac{1}{2}, 1)\). By [3, Theorem 2.2], \((\text{lip}(T,d^\alpha))^\ast\) is not \((-1)\)-weakly amenable. Therefore, if \(\tau : T \rightarrow T\) be a Lipschitz involution on \(T\) then \((\text{lip}(T,d^\alpha, \tau))^\ast\) is not \((-1)\)-weakly amenable by Theorem 2.9.

Note that the map \(\tau : T \rightarrow T\) defined by one of the following:
\[
\tau(z) = z \quad (z \in T), \quad \tau(z) = -z \quad (z \in T), \\
\tau(z) = \bar{z} \quad (z \in T), \quad \tau(z) = -\bar{z} \quad (z \in T), \\
\tau(z) = iz \quad (z \in T), \quad \tau(z) = -iz \quad (z \in T),
\]
is a Lipschitz involution on \((T,d)\).

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References