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A New Approach to Nonstandard Analysis

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Abstract. In this paper, we propose a new approach to nonstandard analysis without using the ultrafilters. This method is very simple in practice. Moreover, we construct explicitly the total order relation in the new field of the infinitesimal numbers. To illustrate the importance of this work, we suggest comparing a few applications of this approach with the former methods.

1. Introduction

In 1961 Abraham Robinson [14] showed how infinitely large and infinitesimal numbers can be rigorously defined and used to develop the field of non-standard analysis. To better understand his theory, nonconstructively, it is necessary to use the essential proprieties deduced from the model theory and mathematical logic.

After the birth of this theory, more mathematicians have discovered the importance of its applications [7, 1] in physics [3, 2, 9], numerical analysis and variational methods.

In 1977 a new axiomatic representation of hyperreals put forward by Edward Nelson [13], in an attempt to simplify Robinson's method. He proposed to add three axioms on the set theory and obtained a new theory called Internal Set Theory [13, 8].

Another axiomatic method, Alpha-Theory [4], was published in 2003. This theory is more simple compared to that of Nelson. However, it raises a few questions concerning its effectiveness in practice as an axiomatic approach.

According to Robinson's construction, we can see every hyperreal as an element of $\mathbb{R}^{\mathbb{N}}$ modulo a maximal ideal M. The ideal M is defined with

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a non-principal ultrafilter $\mathcal{U}$, whose existence is proved by the axiom of choice. By using the ultrafilter $\mathcal{U}$, we define the order relation in the field of hyperreals. Unfortunately, we cannot determine exactly this order relation because the ultrafilter is unknown.

Our aim, in this article, is to give a new field which contains the infinite and infinitesimal numbers without using the properties of the model theory as well as the ultrafilters, and without adding the new axioms to ZFC (Zermelo-Frankel+Axiom of choice). To understand this theory, it does not require to be a mathematical logic specialist. Only the classical results of analysis and the properties of the analytic functions are sufficient in construction. The new approach is very simple in the sense that we can determine precisely the order relation defined in the new field.

The suggested outline for the current article, therefore, is the following:
Firstly, we shall provide some definitions of the infinite and infinitesimal numbers. Then, we shall present the preceding approaches (Robinson's approach, Internal Set Theory and Alpha-Theory), all of which shall be discussed in Section 8 through studying some concrete examples of each one of them. The purpose of this study is to prove that the choice of the ring $\mathbb{R}^{\mathbb{N}}$ in construction of the hyperreal numbers is too broad to be effective in practice. For instance, we will try to show that in spite of the fact that a hyperreal can be equal to zero, it is impossible to predicate its value. In the subsequent section, we will study the proposed method through presenting the construction of a proper subset $\Delta(\mathbb{R}^{\mathbb{N}})$ of $\mathbb{R}^{\mathbb{N}}$. This set is a unitary ring of $\mathbb{R}^{\mathbb{N}}$. By using a maximal ideal of a new ring denoted by Δ, we obtain a new field called the field of Omicran-reals which is a totally ordered field and an extension of the set of real numbers $\mathbb{R}$.
To illustrate the importance of the new approach, we suggest the following applications:

- For the logarithmic function: We prove the following equalities for every real $x > 0$:

$$\ln(x) = \lim_{\alpha \to 0} \frac{x^{\alpha} - 1}{\alpha},$$

  while $x \neq 1$, we obtain:

$$\frac{x-1}{\ln(x)} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

- Prime numbers: Let $\mathcal{P}$ be the set of prime numbers.
  As $x \to +\infty$, we get
  $$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1},$$
  where $\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}$.
  In addition, we prove that:
  $$p_n \sim n^2(\sqrt[n]{n} - 1), \text{ while } n \to +\infty,$$
  where (p_n) is the sequence of prime numbers.
- The length of a curve: We define the length of the arc $\widetilde{AB}$ and we determine the conditions of rectifiability from the new approach. We calculate easily the length, and we obtain:
  $$l(\widetilde{AB}) = \int_a^b \sqrt{1 + f'^2(x)}dx,$$
  where $l(\widetilde{AB})$ is the length of the arc defined by the curve of the function f between $A(a, f(a))$ and $B(b, f(b))$.
- We calculate the limit by using a new notion called the exact limit.
- We show that it is possible to obtain the finite sum by using the exact limit of a series.
- To calculate the exact limit of a series, we define a new matrix called the black magic matrix, this beautiful matrix admits twelve magical properties and we can determine the Bernoulli numbers by using it.
- We can obtain the standard Euler-Maclaurin formula applied to the zeta function $\zeta(s)$ by using the coefficients of the above matrix.

Finally, we determine in the last section of this paper the relationship between the hyperreal numbers and the Omicran-reals, and we prove that any property which is true for every hyperreal number is also true for every Omicran.

2. Preliminary Results

In this section, we find a few definitions and results that are applied in this work.

(i) The binomial coefficient is defined as:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(ii) The Bernoulli numbers are given below:

$$\begin{cases} B_0 = 1, \\ B_0 + 2B_1 = 0, \\ B_0 + 3B_1 + 3B_2 = 0, \\ B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \\ \vdots \\ B_0 + \binom{n}{1}B_1 + \cdots + \binom{n}{n-1}B_{n-1} = 0. \end{cases}$$

We can verify that $B_{2k+1} = 0$, for every natural $k \geq 1$.

(iii) Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(iv) An important result of the Stirling's formula is given by:

$$\mid B_{2n} \mid\sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}.$$

(v) The standard Euler-Maclaurin formula [6] applied to $x \to x^{-s}$ is given by:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{2N^s} + \frac{N^{1-s}}{s-1} + \sum_{k=1}^{M} T_{k,N}(s) + E(M,N,s),$$

where

$$T_{k,N}(s) = \frac{B_{2k}}{(2k)!}N^{1-s-2k}\prod_{j=0}^{2k-2}(s+j),$$

ζ is the Riemann zeta function defined as

$$\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s},$$

and $s \in \mathbb{C}$.

If $\sigma = \Re(s) > -2M-1$, the error is bounded [6] as:

$$|E(M,N,s)| \leq \left|\frac{s+2M+1}{\sigma+2M+1}T_{M+1,N}(s)\right|.$$

(vi) Let $H(D(0,\varepsilon))$ be the set of the holomorphic functions on the disk $D(0,\varepsilon)$.

We can prove the following theorems [15]:

Theorem 2.1. *If h is a holomorphic function on the disk $D(0,\varepsilon)$, and $h(0) = 0$ then: $h(z) = z^k g(z)$ on a neighborhood of 0, where k is a non-zero integer, and $g \in H(D(0,\varepsilon))$ and $g(0) \neq 0$.*

Theorem 2.2. *The zeros of a nonconstant analytic function are isolated.*

3. The Infinite and Infinitesimal Numbers

Definition 3.1. We define the following assertions:

(i) A totally ordered set $(E, \preceq)$ is called an ordered $\mathbb{R}-$extension if

$$\begin{cases} \mathbb{R} \subseteq E; \\ x \preceq y \quad \Leftrightarrow \quad x \leq y \quad \forall (x,y) \in \mathbb{R}^2. \end{cases}$$

(ii) In addition if $(E, +)$ is a commutative group, we define

$$\begin{aligned} |\alpha| &= \max(\alpha, -\alpha) \\ &= \begin{cases} \alpha, & \text{when } -\alpha \preceq \alpha, \\ -\alpha, & \text{when } \alpha \preceq -\alpha. \end{cases} \end{aligned}$$

(iii) We write $x \prec y$ while $x \preceq y$ *and* $x \neq y$.

(iv) Let I_E be the set defined as follows:

$$I_E = \{\alpha \in E \;/\; 0 \prec \mid \alpha \mid \prec \varepsilon \; \forall \varepsilon \in \mathbb{R}^{+*}\}.$$

I_E is a set of infinitesimal numbers.

Remark 3.2. If it has not the ambiguity, we replace the symbol $\preceq$ by $\leq$, and $\prec$ by $<$.

To construct the new extension of $\mathbb{R}$ which contains the infinite and infinitesimal numbers, it is sufficient to prove the following theorem:

Theorem 3.3. *There exists an extension field* $(E, +, .)$ *of* $(\mathbb{R}, +, .)$, *and partial order* $\leq$ *such that:* $(E, \leq)$ *is an order* $\mathbb{R}-$*extension and* $I_E \neq \emptyset$.

Remark 3.4. An element δ of $I_E \neq \emptyset$ is called infinitesimal.

Notation 1. $\mathbb{N} = \{1, 2, 3, \ldots\}$.

4. Previous Methods

4.1. **Robinson's Approach.** From the works of Abraham Robinson, we know that the heuristic idea of infinite and infinitesimal numbers has obtained a formal rigor. He proved that the field of real numbers $\mathbb{R}$ can be considered as a proper subset of a new field, $^*\mathbb{R}$, which is called the field of hyperreal [14] numbers and contains the infinite and infinitesimal numbers. From the approach of Robinson, we can represent every hyperreal by a sequence of $\mathbb{R}^{\mathbb{N}}$ modulo a maximal ideal $\mathcal{I}$. This ideal is defined by using an ultrafilter $\mathcal{U}$. Unfortunately, the Ultrafilter $\mathcal{U}$ and the order relation defined on $^*\mathbb{R}$ are unknown. Only the existence can be proved by the axiom of choice.

4.2. **Nelson's Approach.** In 1977, Edward Nelson expands the language of set theory by adding a new basic predicate st(x). We obtain a new axiomatic representation of the nonstandard analysis by using the above predicate. To explain the behavior of this unary predicate symbol st(x), Nelson proposes to add three axioms [13]:

(a) Idealization. (I)
(b) Standardization. (S)
(c) Transfer principle. (T)

4.3. **Alpha-Theory.** This axiomatic approach published in 2003 is based on the existence of a new element namely α. In this method, we need five axioms to justify the behavior of this new mathematical object.

In the following section, we begin with the construction of the hyperreals by Robinson. After, we pass to the study of the axiomatic approaches.

5. Construction of the Hyperreal Numbers

Let I be a nonempty set, and $\mathcal{P}(I)$ the power set of I.

Definition 5.1. An ultrafilter $\mathcal{U}$ is a proper subset of $\mathcal{P}(I)$, such that:

(i) Intersections: if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
(ii) Supersets: if $A \subseteq B \subseteq I$, then $B \in \mathcal{U}$.
(iii) For any $A \subseteq I$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

Example 5.2. (i) $\mathcal{F}^i = \{A \subseteq I : i \in A\}$ is an ultrafilter, called the principal ultrafilter generated by i.
(ii) $\mathcal{F}^{co} = \{A \subseteq I : I - A \text{ is finite}\}$ is the cofinite (or Frechet), filter on I. $\mathcal{F}^{co}$ is not an ultrafilter.

To construct the field of hyperreal numbers, we use the unitary ring $\mathbb{R}^{\mathbb{N}}$ as follow:

(a) $\mathbb{R} \subseteq \mathbb{R}^{\mathbb{N}}$: We can identify every sequence $u = (l, l, \ldots, l, \ldots)$ by the real number l.
(b) We define in $\mathbb{R}^{\mathbb{N}}$ the total order relation $\leq$ by:

$$u = (u_1, u_2, \ldots, u_n, \ldots) \leq v = (v_1, v_2, \ldots, v_n, \ldots) \quad \Leftrightarrow \quad \{i : u_i \leq v_i\} \in \mathcal{U},$$

where $\mathcal{U}$ is a nonprincipal ultrafilter of $\mathbb{N}$.
To show the existence of the above ultrafilter, we use the axiom of choice.
(c) $(\mathbb{R}^{\mathbb{N}}, +, .)$ is a commutative ring with unity $(1, 1, \ldots, 1, \ldots)$, but it is not a field, since

$$(1, 0, 1, 0, \ldots)(0, 1, 0, 1, \ldots) = 0_{\mathbb{R}^{\mathbb{N}}}.$$

We construct the field of hyperreal numbers by using the following maximal ideal [14, 11] of $\mathbb{R}^{\mathbb{N}}$:

$$I = \left\{ u \in \mathbb{R}^{\mathbb{N}} : \{i : u_i = 0\} \in \mathcal{U} \right\}.$$

Finally, we deduce that the new field of the hyperreal numbers is given by: ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/I$.

Remark 5.3. For every hyperreal u defined by the sequence (u_i), we set

$$u = \langle u_1, u_2, \ldots, u_n, \ldots \rangle, \text{ or } u = \langle u_i \rangle.$$

(d) We can verify that the hyperreal $\delta = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$ is an infinitesimal number.

6. Internal Set Theory

Edward Nelson developed a new theory, Internal Set Theory, which is different from that of Robinson. According to Nelson's view, we can find both the infinite and infinitesimal numbers in the set of real numbers denoted by ${}^*\mathbb{R}$. In addition, the classical families of real numbers $\mathbb{R} = \{st(x),\ x \in {}^*\mathbb{R}\}$ and natural numbers $\mathbb{N} = \{st(x),\ x \in {}^*\mathbb{N}\}$ are not seen as sets in IST. To clarify this point, we propose to study the properties of a set A by using the axioms added by Nelson. We start by the following abbreviations:

(i) $\forall^{st} x \phi(x)$ to mean $\forall x(x \text{ standard} \quad \Rightarrow \quad \phi(x))$.
(ii) $\exists^{st} x \phi(x)$ to mean $\exists x(x \text{ standard} \wedge \phi(x))$.

We call a formula of IST internal in case it does not involve the new predicate "standard", otherwise we call it external.

A set x is finite if there is no bijection of x with a proper subset of itself.

In IST, the three axioms of Nelson are defined as:

(i) **Transfer:** If $\phi(x, u_1, \ldots, u_n)$ is an internal formula with no other free variables than those indicated, then:

$$\forall^{st} u_1, \ldots, \forall^{st} u_n \left(\forall^{st} x \phi(x, u_1, \ldots, u_n) \rightarrow \forall x \phi(x, u_1, \ldots, u_n) \right).$$

(ii) **Idealization:** For any internal formula B whose free variables include x and y

$$\forall^{st} z \left(z \text{ is finite } \rightarrow \exists y \forall x \in z B(x, y) \right) \leftrightarrow \exists y \forall^{st} x B(x, y).$$

(iii) **Standardization:** For every standard formula $F(z)$ (internal or external), we have:

$$\forall^{st} x \exists^{st} y \forall^{st} z [z \in y \leftrightarrow z \in x \ \wedge \ F(z)].$$

Suppose that there exists a unique x such that $A(x)$ is true, where $A(x)$ is an internal formula whose only its free variable is x. Then that x must be standard, since by transfer $\exists x A(x) \Rightarrow \exists^{st} x A(x)$. For example, the set ${}^*\mathbb{N}$ of all natural numbers, the set ${}^*\mathbb{R}$ of all real numbers, the empty set $\emptyset$, and the real number 0, 1, $\sqrt{\pi}, \ldots$ are all standard sets.

Theorem 6.1. *Let X be a set. Then every element of X is standard if and only if X is a standard finite set.*

Proof. We can apply the idealization principle for $B(x, y) = [y \in X \ \wedge\ x \neq y]$ (see [13, 8] for more details). □

Corollary 6.2. *Every infinite set has a nonstandard element.*

Remark 6.3. From the Corollary 6.2, we deduce that there exists a nonstandard natural number ω.

Theorem 6.4. *There is a finite set F such that for any standard x we have $x \in F$.*

Proof. Just apply (I) to the formula $[(x \in y) \ \wedge\ (y \text{ is finite})]$ (see [13, 8]). □

Theorem 6.5. *Let X be a nonempty set. If X is a standard set, then it admits a standard element.*

Proof. Another version of the transfer principle is giving by:

$$\exists x \phi(x) \rightarrow \exists^{st} x \phi(x),$$

where ϕ is an internal formula. We apply this version for $x \in X$. □

Definition 6.6.
(i) Elements of the ultrapower [10] of $\mathcal{P}(\mathbb{R})$ are the equivalence classes of sequences $(A_i) \in \mathcal{P}(\mathbb{R})^{\mathbb{N}}$, where the sequences (A_i) and (B_i) are defined to be equivalent if and only if we have $\{i \in \mathbb{N} : \ A_i = B_i\} \in \mathcal{U}$.

(ii) We denote by $\langle A_i \rangle$ the equivalence class of (A_i). We define the relation ${}^* \in$ between $x = \langle x_i \rangle \in^* \mathbb{R}$ and $\langle A_i \rangle$ by:

$$x^* \in \langle A_i \rangle \quad \Leftrightarrow \quad \{i : \ x_i \in A_i\} \in \mathcal{U}.$$

(iii) With each equivalence class $\langle A_i \rangle$ in the ultrapower of $\mathcal{P}(\mathbb{R})$ we associate a subset A of ${}^*\mathbb{R}$ as follows:

$$x \in A \quad \Leftrightarrow \quad x^* \in \langle A_i \rangle\,.$$

(iv) The subset A of ${}^*\mathbb{R}$ associated with the equivalence class $\langle A_i \rangle$ is called an internal set.

(v) The collection of all internal subsets of ${}^*\mathbb{R}$ is denoted by ${}^*\mathcal{P}(\mathbb{R})$.

We denote by A the internal set defined by the equivalence class $\langle A_i \rangle$.

Remark 6.7. A standard set ${}^*B$ is given by the equivalence class

$$\langle B, B, \ldots, B, \ldots \rangle ,$$

where $B \in \mathcal{P}(\mathbb{R})$.

Example 6.8. (i) ${}^*[0,1] = \langle [0,1], \ldots, [0,1], \ldots \rangle$, ${}^*\mathbb{R} = \langle \mathbb{R}, \mathbb{R}, \ldots \rangle$ and ${}^*\mathbb{N}$ are all standard sets, and then are internal sets.

(ii) Let ω be the infinite number defined as $\omega = \langle 1, 2, 3, \ldots \rangle$. The set $\{\omega\} = \langle \{i\} \rangle$ is internal but it is not standard.

(iii) For every integer $i \geq 1$ we put $X_i = \left[\frac{1}{i+1}, \frac{1}{i} \right[$ and $X = \langle X_i \rangle$.

The above set is internal and infinite, but we cannot find any standard element in X (because there does not exist a real number x such that $\{i : \ x = x_i\} \in \mathcal{U}$ for $x_i \in X_i$). From the Corollary 6.2, we deduce that X is a nonstandard element.

On the other hand, the set X is bounded from above by 1, we can check that X has a supremum in ${}^*\mathbb{R}$, and we have $\sup X = \left\langle \frac{1}{i} \right\rangle$.

Remark 6.9.

- In the collection of the internal sets [13, 8], we find the standard and the nonstandard sets.
- Every nonempty internal set of hyperreals bounded from above has a supremum in ${}^*\mathbb{R}$. In fact, since the internal set $A = \langle A_i \rangle$ is bounded from above, then there exists $M \in \mathbb{R}$ such that $J = \{i : \ A_i \text{ is bounded from above by M}\} \in \mathcal{U}$.
  We define $s = \langle s_i \rangle$ such that $s_i = \sup(A_i)$ for $i \in J$ and $s_i = 1$ else. We can check easily that $s = \sup(A)$.
- We can prove the above result for every element of ${}^*\mathcal{P}(\mathbb{R})$ by using the transfer principle, but this property is not true for every family of hyperreals (for example, the set $\mathbb{R}$ is bounded from above by every positive infinitely large number L, but it does not have a least upper bound), then we deduce that the set ${}^*\mathcal{P}(\mathbb{R})$ is a proper subset of $\mathcal{P}({}^*\mathbb{R})$. The elements of $\mathcal{P}({}^*\mathbb{R}) \backslash {}^*\mathcal{P}(\mathbb{R})$ are called the external sets. For example, the sets $\mathbb{R}$, $\mathbb{N}$, the infinite numbers and the infinitesimal numbers are all external sets.

7. Alpha-Theory

This approach is based on the existence of a new mathematical object, namely α. Intuitively, this new element, added to $\mathbb{N}$, is considered as a "very large" natural number.

The use of α is governed by the following five axioms [4].

- α_1. **Extension Axiom.** For every sequence, φ, there exists a unique element $\varphi[\alpha]$, called the "ideal value of φ" or the "value of φ at infinity".
- α_2. **Composition Axiom.** If φ and ψ be two sequences and if f is any function such that compositions $f \circ \varphi$ and $f \circ \psi$ make sense, then

$$\varphi[\alpha] = \psi[\alpha] \quad \Rightarrow \quad (f \circ \varphi)[\alpha] = (f \circ \psi)[\alpha].$$

- α_3. **Number Axiom.** Let $c_r : n \to r$ be the constant sequence with value $r \in \mathbb{R}$, then $c_r[\alpha] = r$. If $1_{\mathbb{N}} : n \to n$ is the identity sequence on $\mathbb{N}$, then $1_{\mathbb{N}}[\alpha] = \alpha \notin \mathbb{N}$.
- α_4. **Pair Axiom.** For all sequences φ, ψ and ϑ:

$$\vartheta(n) = \{\varphi(n), \psi(n)\} \text{ for all } n \quad \Rightarrow \quad \vartheta[\alpha] = \{\varphi[\alpha], \psi[\alpha]\}.$$

- α_5. **Internal Set Axiom.** Let ψ be a sequence of atoms, and $c_\emptyset$ be the sequence defined as $c_\emptyset : n \to \emptyset$, then $\psi[\alpha]$ is an atom, and $c_\emptyset[\alpha] = \emptyset$. If ψ is a sequence of nonempty sets, then

$$\psi[\alpha] = \{\varphi[\alpha] \;/\; \varphi[n] \in \psi[n] \text{ for all } n\}.$$

Proposition 7.1. *(i) If $\varphi(n) = \psi(n)$ eventually (i.e. for all but finitely many n), then $\varphi[\alpha] = \psi[\alpha]$.*
(ii) If $\varphi(n) \neq \psi(n)$ eventually, then $\varphi[\alpha] \neq \psi[\alpha]$.

Definition 7.2. Let A be a nonempty set. The star-transform of A is giving by:

$$A^* = \{\varphi[\alpha] \;/\; \varphi : \mathbb{N} \longrightarrow A\}.$$

In the following proposition, we verify that the star-operator preserves all basic operations of sets (except the power set).

Proposition 7.3. *For all A, B, we have* [4]

(i) $A = B \Leftrightarrow A^* = B^*$;
(ii) $A \in B \Leftrightarrow A^* \in B^*$;
(iii) $A \subseteq B \Leftrightarrow A^* \subseteq B^*$;
(iv) $\{A, B\}^* = \{A^*, B^*\}$;
(v) $(A \cup B)^* = (A^* \cup B^*)$;
(vi) $(A \cap B)^* = (A^* \cap B^*)$;
(vii) $(A \setminus B)^* = (A^* \setminus B^*)$;
(viii) $(A \times B)^* = (A^* \times B^*)$.

Definition 7.4. (i) The set of hyperreal numbers is the star-transform $\mathbb{R}^*$ of the set of real numbers:

$$\mathbb{R}^* = \{\varphi[\alpha] \;/\; \varphi : \mathbb{N} \longrightarrow \mathbb{R}\}.$$

(ii) The set of hypernatural numbers is the star-trasform of the set of natural numbers:

$$\mathbb{N}^* = \{\varphi[\alpha] \,/\, \varphi : \mathbb{N} \longrightarrow \mathbb{N}\}.$$

(iii) We define in $\mathbb{R}^*$ the following binary relation:

$$\xi < \zeta \Leftrightarrow (\xi, \zeta) \in \{(x, y) \in \mathbb{R} \times \mathbb{R} \,/\, x < y\}^*.$$

Theorem 7.5. *The hyperreal number system* $(\mathbb{R}^*, +, ., 0, 1, <)$ *is an ordered field.*

Remark 7.6. • An example of an infinitesimal is given by $\frac{1}{\alpha}$, the ideal value of the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$. Other examples of infinitesimals are the following:

$$-\sin\left(\frac{1}{\alpha}\right),\ \frac{\alpha}{3+\alpha^2},\ \log\left(1-\frac{1}{\alpha}\right).$$

- For the infinite numbers, we propose the following examples:

$$\alpha^2 + 1,\ 3 + \sqrt{\alpha},\ \log(7\alpha - 3).$$

8. A Few Remarks About the Previous Approaches

In this section, we shall study some examples to see clearly the difficulties that can be encountered in practice while using the classical approaches of the non-standard analysis. Firstly, we begin with the study of Robinson's approach, afterwards, we proceed to the study of axiomatic approaches. Finally, we conclude with a small discussion as an introduction to the new approach.

To explain our point of view about Robinson's approach, we propose some examples in the following subsection.

8.1. Robinson's Approach.

(1) For the infinitesimal number $\delta = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$, we can not imagine intuitively its nature, because it is defined by the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$ modulo the unknown ideal I.

(2) Let u be a hyperreal number defined as $u = \langle -1, 1, -1, 1, \ldots \rangle$. Despite that the field $({}^*\mathbb{R}, \leq)$ is a totally ordered set, but we cannot determine the sign of u. On the other hand, we have two cases:

   (i) If $u \geq 0$, then there exists an element $F \in \mathcal{U}$ such that: $F = \{i : u_i \geq 0\} = \{i : u_i = 1\}$.
   In this case, we deduce that $F \subseteq 2\mathbb{N} \in \mathcal{U}$, and we find $u = 1$.

   (ii) If $u \leq 0$, we find $2\mathbb{N} + 1 \in \mathcal{U}$, and $u = -1$.

Now, to complicate this problem, we put the following question: where is the sign of the hyperreal number defined as

$$\zeta = \langle \sin(1), \sin(2), \sin(3), \ldots \rangle ?$$

Since the total ordering in the hyperreal numbers is not explicitly defined, then we deduce that the Robinson's approach is very complicated to give us a simple property as the sign of an element of ${}^*\mathbb{R}$. Moreover, the sign of the hyperreal u, which is defined by the sequence (u_i), is not sufficient to know the sign of this sequence at infinity which can be invariant (is not stable from a certain rank).

(3) Let v be the hyperreal defined as:

$$v = \left\langle 1, 10^1, \frac{1}{2}, 10^{10^2}, \frac{1}{3}, 10^{10^3}, \ldots, \frac{1}{i}, 10^{10^i}, \ldots \right\rangle .$$

Where is the nature of this number? Is it infinite or infinitesimal?

If $2\mathbb{N} \in \mathcal{U}$, then v is infinite and otherwise it is infinitesimal. The determination of the nature of an hyperreal is not easy and evident in cases which are general, and we can find other cases which are very complicated than the above example. In addition, if we put (v_i) the sequence which defines the hyperreal v, and w the hyperreal defined by the sequence (v_{i+1}), then:

$$w = \left\langle 10^1, \frac{1}{2}, 10^{10^2}, \frac{1}{3}, 10^{10^3}, \ldots, \frac{1}{i}, 10^{10^i}, \ldots \right\rangle .$$

If $2\mathbb{N} \in \mathcal{U}$ then v is infinite, and w is infinitesimal. Thus, we can find two hyperreals do not have the same nature; the first is defined by a sequence (v_i), the second by its subsequence $(v_{\phi(i)})$. In the above example, the translation of the indices of the sequence (u_i) which defines the infinitesimal number $\langle u_i \rangle$, is sufficient to transform it to an infinite number. This is not well to be effective in practice, for example, if $\langle u_i \rangle = 1$, (in general) we can not know anything about the value of the hyperreal $v = \langle u_{3i+1} \rangle$, v can be zero, infinite, infinitesimal number, etc.. Next, we propose an example of an hyperreal number $\langle u_i \rangle$ which can be zero or an integer $1 \leq i \leq 9$, but it is impossible to determine its value.

(4) For every real number x, let (x_i) be the sequence defined by the decimal representation of x as:

$$x = x_1, x_2 x_3 x_4 x_5 x_6 \ldots .$$

Let $\widetilde{x}$ be the hyperreal defined as $\widetilde{x} = \langle x_i \rangle$. For the number π, we get:

$$\pi = 3.14159265358979323 85 \ldots .$$

Then, $\widetilde{\pi} = \langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, \ldots \rangle$. We attempt to determine the value of this hyperreal, for that, we propose to prove the following lemma.

Lemma 8.1. *Let A be a finite subset of $\mathbb{R}$. For every element $u = (u_i)$ of $A^{\mathbb{N}}$, the hyperreal number $\langle u_i \rangle$ is an element of A.*

Proof. We put $A = \{a_1, a_2, \ldots, a_n\}$, and $F_x = \{i : u_i = x\}$ for every $x \in A$. Let $\mathcal{U}$ be the ultrafilter defined in Robinson's approach. If there exists $1 \leq i_0 \leq n-1$ such that $F_{a_{i_0}} \in \mathcal{U}$, then, $\langle u_i \rangle = a_{i_0}$, and otherwise $F^c_{a_1}, F^c_{a_2}, \ldots, F^c_{a_{n-1}} \in \mathcal{U}$. Then $F^c_{a_1} \cap F^c_{a_2} \cap \ldots \cap F^c_{a_{n-1}} \in \mathcal{U}$, and we deduce that $(F_{a_1} \cup F_{a_2} \cup \ldots \cup F_{a_{n-1}})^c = F_{a_n} \in \mathcal{U}$, which implies that $\langle u_i \rangle = a_n$. □

From the Lemma 8.1, we deduce that $\widetilde{\pi} \in \{0, 1, 2, \ldots, 9\}$ (then can be non invertible). Unfortunately, we do not have any way to determine its value. Let x be the natural number in $\{0, 1, 2, \ldots, 9\}$ such that $\widetilde{\pi} = x$. Consider the hyperreal $\widetilde{\alpha} = \langle \alpha_i \rangle$ defined as:

$$\alpha_i = \begin{cases} \frac{1}{i}, & \text{when } x = \pi_i, \\ 10^{10^i}, & \text{otherwise.} \end{cases}$$

The nature of the number $\widetilde{\alpha}$ is not compatible with the behavior of the sequence used to define it. In fact, the values taken by the sequence (α_i) are very "large" in an infinity of indices. In addition, we can predict the following plausible conjecture:

"The cardinal of the set $\left\{i : \alpha_i = \frac{1}{i}\right\} \cap \{1, 2, \ldots, n\}$ is very small compared to the cardinal of $\left\{i : \alpha_i = 10^{10^i}\right\} \cap \{1, 2, \ldots, n\}$, from a certain rank n_0."

However, this number $\widetilde{\alpha}$ is infinitesimal. Then, we have an incompatibility between the nature of the hyperreal $\langle \alpha_i \rangle$ as an infinitesimal number and the value taken by the sequence (α_i). In addition, we do not have any rule to determine in general the set of indices i that gives us the nature of an hyperreal $\langle u_i \rangle$ defined by the sequence (u_i).

8.2. **Nelson's Approach and Alpha-Theory.** The axiomatic methods allow us to give and explain rigorously the behavior of any new defined notion. Yet, they are not effective enough in practice, especially if the notions of the proposed theory are not explicitly defined. For instance, according to Alpha-Theory, we should define a new mathematical object α. By using the Extension Axiom we justify the existence of the new object. In the same way, the ideal value of every sequence φ denoted by $\varphi[\alpha]$ is defined by the above axiom. Intuitively, $\varphi[\alpha]$ represents the value of φ at infinity. Like Robinson's approach, this ideal value is not explicitly determined in general. Vieri Benci and Mauro Di Nasso confirmed (see [4], p.359):

"Suppose φ is a two-valued sequence, say $\varphi : \mathbb{N} \to \{-1, 1\}$. Then its ideal value makes no surprise, i.e. either $\varphi[\alpha] = -1$ or $\varphi[\alpha] = 1$ (but in general it cannot be decided which is the case)."

In order to find an acceptable solution to this problem, the authors proposed to take $(-1)^\alpha = 1$. They justified this choice as the following (see [4], p.367):

"...For instance, we could consistently postulate that the infinite hypernatural α is even. In this case, the alternating sequence $((-1)^n)_{n\geq 1}$ takes the value $(-1)^\alpha = 1$ at infinity".

Unfortunately, this choice is not convincing and is not sufficient to determine the ideal value in general. If we choose another sequence ψ defined as 1,2,3,1,2,3,1,2,3,1,2,3,.... from the solution proposed by the authors, we need a new postulate for taking $\psi[\alpha] = 2$. On the other hand, the same problem could be raised by Internal Set Theory. To determine the value of a sequence φ at infinity, we are obliged to find an explicit approach to nonstandard analysis. Historically, Internal Set Theory was introduced by Edward Nelson in order to simplify Robinson's approach. However, this approach is not accessible for those mathematicians who lack enough knowledge in logic. For example, if we study the transfer principle in general, it can not be easily and correctly applied without checking some particular conditions. To clarify this point, we propose the following example.

Recall the following property of $\mathbb{N}$: "Every nonempty subset of $\mathbb{N}$ has a least element".

By applying the transfer principle to this formulation, we would get that "Every nonempty subset of $^*\mathbb{N}$ has a least element". But this is clearly false (the collection $^*\mathbb{N} \setminus \mathbb{N}$ has no least element). Here, the transfer principle can not be applied, because the above sentence is not elementary (see [5]). For that, we think that the Nelson's approach is inaccessible for the non-specialist in mathematical logic.

In this paper, we propose a constructive approach to nonstandard analysis without adding any axiom. Only the properties of the classical analysis are sufficient to construct the new field. In addition, we define an explicit total order relation in the new set called the field of Omicran-reals.

8.3. **Discussion.** Abraham Robinson succeeded to show the existence of a total order relation on $\mathbb{R}^{\mathbb{N}}$, but the explicit determination of this relation is very difficult. The judgment of the scientific work of Robinson begins with the study of the choice of $\mathbb{R}^{\mathbb{N}}$. To find or not to find an explicit total order is another question that can be asked after the determination of the initial set in construction. Now, the question we might ask is the following: why we need the ring $\mathbb{R}^{\mathbb{N}}$ to define the field of nonstandard analysis? According to the incompatibility between its nature and the behavior of the sequence which defined it, the hyperreals like $\widetilde{\alpha}$, $\widetilde{\pi}$, $\langle 1, -1, 1, \ldots \rangle$, or $\langle \sin(1), \sin(2), \ldots \rangle$ do not matter in practice. Then, the choice of the ring $\mathbb{R}^{\mathbb{N}}$ is too broad to be effective. In the following section, we attempt to give the answer of the following question: can we construct the field of the infinitesimal numbers by using a proper subset of $\mathbb{R}^{\mathbb{N}}$ with an explicit total order?

9. The Proposed Method

9.1. **The Metalic Map.** Let $D(0,1)$ (resp. $D'(0,1)$) be the open (resp. closed) disk of radius 1 and center 0.

Definition 9.1. Let u be a map from $]0,1]$ to $\mathbb{R}$, such that:

(i) There exists a map $\widetilde{u}$ defined on $D'(0,1)$, and holomorphic in a neighborhood of 0.

(ii) There exists $\varepsilon > 0$, such that $\forall x \in]0, \varepsilon[$ we have $\widetilde{u}(x) = u(x)$.

The map u is called a metalic map, and $\widetilde{u}$ is a metalic extension of u.

Example 9.2. If f is defined in the interval $]0,1]$ as:

$$f(x) = \begin{cases} 2x+1, & \text{when } x \in]\varepsilon, 1], \\ 1 - 3x^2, & \text{when } x \in]0, \varepsilon], \end{cases}$$

then a metalic extension $\widetilde{f}$ is given by: $\widetilde{f}(z) = 1 - 3z^2$ in the disk $D'(0,1)$.

Remark 9.3. If u is metalic, then the two metalic extensions $\widetilde{u}$ and $\widehat{u}$ of u are identic in a disk $D(0,\varepsilon)$ by Theorem 2.2.

Definition 9.4. We set $\Delta_1 = \{\, u, u \text{ is a metalic map}\}$, and we have the following definitions:

(1). $\Delta_1(\mathbb{R}^{\mathbb{N}}) = \{\, \left(u\left(\frac{1}{n}\right)\right)_{n \geq 1}, \text{u is a metalic map}\}$.

(2). $H_0 =$ the set of maps $\widetilde{u}$ defined on the disk $D'(0,1)$ and holomorphic in a neighborhood of 0.

(3). Let $(\mathcal{O}_0, +)$ be a subgroup of $(H_0, +)$ containing the maps defined on the disk $D'(0,1)$ which vanish in a neighborhood of 0.

(4). Let θ_0 be a map defined as

$$\begin{array}{rccc} \theta_0 : & \Delta_1(\mathbb{R}^{\mathbb{N}}) & \longrightarrow & H_0/\mathcal{O}_0, \\ & \left(u\left(\frac{1}{n}\right)\right)_{n \geq 1} & \longmapsto & C(\widetilde{u}), \end{array}$$

which $C(\widetilde{u})$ is the equivalence class of $\widetilde{u}$ modulo $\mathcal{O}_0$. The map θ_0 is well-defined from the unicity of $C(\widetilde{u})$.

(5). We consider the surjective map θ_1 defined as:

$$\begin{array}{rccc}\theta_1: & \Delta_1(\mathbb{R}^{\mathbb{N}}) & \longrightarrow & \theta_0(\Delta_1(\mathbb{R}^{\mathbb{N}})),\\ & \left(u\left(\frac{1}{n}\right)\right)_{n\geq 1} & \longmapsto & C(\widetilde{u}),\end{array}$$

and the set $\overline{\Delta_1(\mathbb{R}^{\mathbb{N}})} = \left\{\theta_1^{-1}(C(\widetilde{u})), C(\widetilde{u}) \in \theta_0(\Delta_1(\mathbb{R}^{\mathbb{N}}))\right\}$.

(6). We define on the set $\Delta_1(\mathbb{R}^{\mathbb{N}})$ the following equivalence relation $\sim$:

$$\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1} \sim \left(v\left(\frac{1}{n}\right)\right)_{n\geq 1} \quad \Leftrightarrow \quad \exists n_0, \forall n \geq n_0, u\left(\frac{1}{n}\right) = v\left(\frac{1}{n}\right).$$

(7). $\overline{\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}}$ is the equivalence class of $\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}$ modulo $\sim$.

Remark 9.5. (a) We can check the equality $\overline{\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}} = \theta_1^{-1}\left(C\left(\widetilde{u}\right)\right)$. Then :

$$\overline{\Delta_1(\mathbb{R}^{\mathbb{N}})} = \left\{\overline{\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}},\ u \in \Delta_1\right\}.$$

(b) The sets Δ_1 and $\overline{\Delta_1(\mathbb{R}^{\mathbb{N}})}$ are commutative groups.

(c) The map defined as:

$$\begin{array}{rccc}\overline{\theta_1}: & \overline{\Delta_1(\mathbb{R}^{\mathbb{N}})} & \longrightarrow & E_1 = \theta_0\left(\Delta_1\left(\mathbb{R}^{\mathbb{N}}\right)\right),\\ & \overline{\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}} & \longmapsto & C(\widetilde{u}),\end{array}$$

is an isomorphism between two groups.

Definition 9.6. Consider the following definitions:

- $\mathcal{A}_2 = \left\{ \frac{1}{u}, u \in \Delta_1 \ \ \forall x \in]0,1] \ \ u(x) \neq 0 \text{ and } \lim\limits_{n\to+\infty} u\left(\frac{1}{n}\right) = 0 \right\}$.
- $\Delta_2 = \left\{v :]0,1] \longrightarrow \mathbb{R} \mid v_{/]0,\varepsilon]} = \left(\frac{1}{u}\right)_{/]0,\varepsilon]} \text{ for } \frac{1}{u} \in \mathcal{A}_2 \text{ and } \varepsilon > 0\right\}$.
- $\Delta_2(\mathbb{R}^{\mathbb{N}}) = \left\{ \left(v\left(\frac{1}{n}\right)\right)_{n\geq 1},\ v \in \Delta_2 \right\}$.

9.2. **Construction of a Unitary Ring.**

Lemma 9.7. *Let $\Delta = \Delta_1 \cup \Delta_2$. Then $(\Delta, +, .)$ is a unitary ring.*

Proof. • The stability of the sum: the set Δ is a non-empty set, because $\Delta_1 \neq \emptyset$ and $\mathbb{R} \subseteq \Delta_1$ (we identify the constant functions by the real numbers). We show that for all $g \in \Delta$, and $h \in \Delta$, we have $g + h \in \Delta$.

First case: If $(g, h) = (u, v) \in \Delta_1^2$, we verify easily that the

function $s = f + g$ is a matalic map, in addition, we have $\widetilde{s} = \widetilde{u} + \widetilde{v}$.
Second case: If $(g, h) \in \Delta_2^2$, there exists a strictly positive real number ε and $(u, v) \in \Delta_1^2$ such that $u(x)v(x) \neq 0$ for every $x \in]0, \varepsilon]$, $\lim_{n \to +\infty} u\left(\frac{1}{n}\right) = \lim_{n \to +\infty} v\left(\frac{1}{n}\right) = 0$, and we have $g_{/]0,\varepsilon]} = \frac{1}{u}_{/]0,\varepsilon]}$ and $h_{/]0,\varepsilon]} = \frac{1}{v}_{/]0,\varepsilon]}$. Since $\widetilde{u}$ and $\widetilde{v}$ are holomorphic functions on a neighborhood of 0, then there exists $(m, n, l) \in \mathbb{N}^3$ such that $\widetilde{u}(z) = z^n b_1(z)$, $\widetilde{v}(z) = z^m b_2(z)$ and $\widetilde{u}(z) + \widetilde{v}(z) = z^l b_3(z)$, where b_1, b_2, b_3 are three holomorphic functions on a neighborhood of 0 and $b_1(0)b_2(0)b_3(0) \neq 0$.
Let

$$\psi(x) = \frac{u(x)v(x)}{u(x) + v(x)}.$$

The map $g + h$ is defined in the interval $]0, 1]$. We can choose the small enough real ε such that $b_1(z)b_2(z)b_3(z) \neq 0$ in the disk $D(0, \varepsilon)$. Then we have

$$\begin{aligned} g(x) + h(x) &= \frac{1}{u(x)} + \frac{1}{v(x)} \\ &= \frac{1}{\psi(x)} \\ &= \frac{x^{l-m-n} b_3(x)}{b_1(x)b_2(x)}, \end{aligned}$$

for every $x \in]0, \varepsilon]$.

✓If $l - m - n \geq 0$, the map $\widetilde{\phi}$ defined as

$$\widetilde{\phi}(z) = \begin{cases} z^{l-m-n} \frac{b_3(z)}{b_1(z)b_2(z)}, & \text{in } D(0, \varepsilon), \\ 1, & \text{if not,} \end{cases}$$

is a metalic extension of $g + h$, then $g + h$ is an element of Δ_1.
✓If $l - m - n < 0$, the map defined as

$$\widetilde{\psi}(z) = \begin{cases} z^{m+n-l} \frac{b_1(z)b_2(z)}{b_3(z)}, & \text{in } D(0, \varepsilon), \\ 1, & \text{if not,} \end{cases}$$

is a metalic extension of ψ, in addition $\lim_{n \to +\infty} \psi\left(\frac{1}{n}\right) = 0$, we deduce that $g + h$ is an element of Δ_2.
Third case: If $(g, h) \in \Delta_1 \times \Delta_2$, there exists $(u, v) \in \Delta_1^2$ such

that $g = u$, $h_{/]0,\varepsilon]} = \frac{1}{v}_{/]0,\varepsilon]}$ and $\lim_{n\to+\infty} v\left(\frac{1}{n}\right) = 0$. Let

$$k(z) = \frac{\widetilde{v}(z)}{\widetilde{u}(z)\widetilde{v}(z) + 1}.$$

Since $\widetilde{v}(0) = 0$, we have $k(0) = 0$, and k is a holomorphic function in a disk $D(0, \varepsilon)$, for a small enough $\varepsilon > 0$. Since the map k is nonzero, then we can choose the ε so that $k(x) \neq 0$ for every $x \in]0, \varepsilon]$. Let ϕ be the function defined on $]0, 1]$ as:

$$\phi(x) = \begin{cases} k(x), & \text{if } x \in]0, \varepsilon], \\ 1, & \text{if not.} \end{cases}$$

We can verify that $g + h_{/]0,\varepsilon]} = (\frac{1}{\phi})_{/]0,\varepsilon]}$, $\phi \in \Delta_1$ then $g + h \in \Delta_2 \subset \Delta$.

Finally, we deduce that $(\Delta, +)$ is a commutative group.

- Now, we can show the stability of the law (.) in Δ, for that, we distinguish three cases:

  (i) We can easily verify that the product of two metalic functions is a metalic function (if $g \in \Delta_1$ and $h \in \Delta_1$ then $gh \in \Delta_1 \subseteq \Delta$).

  (ii) In this case, we assume that $g \in \Delta_1$ and $h \in \Delta_2$, we can show that $gh \in \Delta$, in fact, there exists $(u, v) \in \Delta_1^2$, such that $g = u$, $h_{/]0,\varepsilon]} = \frac{1}{v}_{/]0,\varepsilon]}$, $\lim_{n\to+\infty} v\left(\frac{1}{n}\right) = 0$.

  (a) If $\lim_{n\to+\infty} u\left(\frac{1}{n}\right) \neq 0$, then $\frac{\widetilde{v}}{\widetilde{u}}$ is holomorphic in the disk $D(0, \varepsilon)$, which implies that $\dfrac{u}{v} \in \Delta_2 \subseteq \Delta$.

  (b) If $\lim_{n\to+\infty} u\left(\frac{1}{n}\right) = 0$, then $\lim_{n\to+\infty} \widetilde{u}\left(\frac{1}{n}\right) = 0$ and we obtain $\widetilde{u}(0) = 0$. We deduce that $\widetilde{u}(z) = z^k b_1(z)$ in $D(0, \varepsilon)$, and $\widetilde{v}(z) = z^{k'} b_2(z)$, where $b_i(z) \in H(D(0, \varepsilon))$, for $i \in \{0, 1\}$ and $b_i(0) \neq 0$. We get

  $$\frac{\widetilde{u}(z)}{\widetilde{v}(z)} = z^{k-k'} \frac{b_1(z)}{b_2(z)}.$$

  b1. First case: if $k = k'$ then the function $\frac{\widetilde{u}}{\widetilde{v}}$ is holomorphic in $D(0, \varepsilon)$, which implies that $\frac{u}{v} \in \Delta_1$.

b2. Second case: if $k > k'$, then $\lim_{z\to 0} \frac{\widetilde{u}}{\widetilde{v}}(z) = 0$, and $\frac{\widetilde{u}}{\widetilde{v}}$ is a holomorphic function in the disk $D(0,\varepsilon)$. Then $\frac{u}{v} \in \Delta_1$.

b3. Third case: if $k < k'$, then $\lim_{z\to 0} \frac{\widetilde{v}}{\widetilde{u}}(z) = 0$, which implies that $\frac{u}{v} \in \Delta_2$.

(iii) In the case of $g \in \Delta_2$ and $h \in \Delta_2$, we verify easily the stability of the law (.).
Finally, we deduce that $(\Delta, +, .)$ is a commutative and unitary ring, where the constant function $\mathbf{1}_\Delta$ is a multiplicative identity of Δ.

□

9.3. **Construction of the New Field.** Let $\mathcal{I}_0$ be the set defined as:

$$\mathcal{I}_0 = \left\{ u_{/]0,1]} \ / \ u \in \mathcal{O}_0 \ \text{ and } \ u(]0,1]) \subset \mathbb{R} \right\}.$$

Then, it is a set of maps defined on $]0,1]$ which vanish on $]0,\varepsilon]$ (for $0 < \varepsilon \leq 1$).

Now, we can deduce the following proposition.

Proposition 9.8. *$\mathcal{I}_0$ is a maximal ideal of Δ.*

Proof.

- We can prove easily that $\mathcal{I}_0$ is an additive subgroup of Δ.
- $\mathcal{I}_0$ is an ideal of Δ. In fact, if θ is an element of $\mathcal{I}_0$, then $\theta_{/]0,\varepsilon[} = 0$ for some $\varepsilon > 0$. For every $u \in \Delta$, we have $(\theta u)_{/]0,\varepsilon[} = 0$, then $\theta u \in \mathcal{I}_0$.
- Let I be an ideal of Δ such that $\mathcal{I}_0 \subseteq I$. We assume that this inclusion is strict, then there exists $u \in I \setminus \mathcal{I}_0$. Since u is an element of Δ, we can distinguish the following two cases:
  (i) First case: $u \in \Delta_1$. If u admits infinitely many zeros in $]0,\varepsilon[$ for every $\varepsilon > 0$, then $\widetilde{u} = 0$ and we deduce that $u \in \mathcal{I}_0$, which is absurd. Then there exists $\varepsilon > 0$ such that $u(x) \neq 0$ for every $x \in]0,\varepsilon[$. Let v be a function defined in $]0,1]$ by $v(x) = \frac{1}{u(x)}$ in $]0,\varepsilon[$ and $v(x) = 1$ while $\in [\varepsilon, 1]$. We have $u(x)v(x) = 1$ in $]0,\varepsilon[$, then $1 - uv \in \mathcal{I}_0$. Consider $i \in \mathcal{I}_0 \subseteq I$ such that $1 - uv = i$.
  Then $1 = i + uv$ and we deduce that $1 \in I$ which implies that $I = \Delta$.
  (ii) Second case: $u \in \Delta_2$. In this case there exist $\varepsilon > 0$ and $v \in \Delta_1$ such that $u(x) = \frac{1}{v(x)}$ in $]0,\varepsilon[$. Then $1 - uv \in \mathcal{I}_0$ and we deduce that $I = \Delta$.

Finally, we deduce that the ideal $\mathcal{I}_0$ is a maximal ideal of Δ. □

Theorem 9.9. *The ring* $(\Delta/\mathcal{I}_0, +, .)$ *is a field.*

Proof. From the Proposition 9.8, the ideal $\mathcal{I}_0$ is maximal, so we deduce that the ring $(\Delta/\mathcal{I}_0, +, .)$ is a field. □

9.4. **The Field of Omicran-reals.** Consider the set defined as

$$\Delta(\mathbb{R}^{\mathbb{N}}) = \left\{ \left(h\left(\frac{1}{n}\right) \right)_{n\geq 1} : h \in \Delta \right\}.$$

Let $\sim$ be the equivalence relation defined on the set $\Delta(\mathbb{R}^{\mathbb{N}})$ as:

$$\left(g\left(\frac{1}{n}\right) \right)_{n\geq 1} \sim \left(h\left(\frac{1}{n}\right) \right)_{n\geq 1} \quad \Leftrightarrow \quad \exists n_0 \mid \forall n \geq n_0, h\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right).$$

The equivalence class is given by:

$$\overline{\left(g\left(\frac{1}{n}\right) \right)_{n\geq 1}} = \left\{ \left(h\left(\frac{1}{n}\right) \right)_{n\geq 1} : h \in \Delta \text{ and } n_0 \in \mathbb{N} \mid \forall n \geq n_0, h\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \right\}.$$

The map

$$\begin{array}{rccc} \theta : & (\overline{\Delta(\mathbb{R}^{\mathbb{N}})}, +, .) & \longrightarrow & (\Delta/\mathcal{I}_0, +, .), \\ & \overline{(g\left(\frac{1}{n}\right))_{n\geq 1}} & \longmapsto & C(g) = \overline{g}, \end{array}$$

is well-defined, and in addition, we have:

(i) $(\overline{\Delta(\mathbb{R}^{\mathbb{N}})}, +, .)$ is a field.
(ii) θ is an isomorphism.

Lemma 9.10. *Let g be an element of Δ. There exists a positive real number ε such that*

$$\forall x \in]0, \varepsilon[\textit{ we have}: \quad g(x) = \frac{1}{x^m} \sum_{i=0}^{+\infty} a_i x^i,$$

where m is a naturel number, and $s = \sum\limits_{i=0}^{+\infty} a_i z^i$ is a power series with a non zero radius of convergence.

Proof.
- If $g \in \Delta_1$ then $\widetilde{g}$ is holomorphic in a neighborhood of 0, and we have $\widetilde{g}(z) = \sum a_i z^i$ in $D(0, \varepsilon)$.
  Since $\widetilde{g}_{/]0,\varepsilon[} = g_{/]0,\varepsilon[}$, then $g(x) = \sum\limits_{i=0}^{+\infty} a_i x^i$ for every x in $]0, \varepsilon[$.
- If g is an element of Δ_2, then there exist an element u in Δ_1, and a real number $\varepsilon > 0$ such that $g(x) = \frac{1}{u(x)}$ for every x in $]0, \varepsilon[$. We can find $\varepsilon' > 0$ and a holomorphic function b in $D(0, \varepsilon')$ such that $\widetilde{u}(z) = z^m b(z)$ and $b(0) \neq 0$. Since b is holomorphic and $b(0) \neq 0$, then there exists $\varepsilon_1 > 0$ such that $b(z) \neq 0$ in

$D(0,\varepsilon_1)$, we deduce that the map $\frac{1}{b}$ is a holomorphic function in $D(0,\varepsilon_1)$. Then there exists a power series $\sum\limits_{i=0}^{+\infty} a_i z^i$ such that

$$\frac{1}{b(z)} = \sum_{i=0}^{+\infty} a_i z^i \text{ in } D(0,\varepsilon_1).$$

Finally, we deduce that

$$\begin{aligned}\frac{1}{\widetilde{u}(z)} &= \frac{1}{z^m b(z)} \\ &= \frac{1}{z^m}\sum_{i=0}^{+\infty} a_i z^i, \quad \text{in } D(0,\varepsilon_1),\end{aligned}$$

for an small enough ε_1 (we choose $\varepsilon_1 < \varepsilon'$). Now, for an small enough ε we have $g_{/]0,\varepsilon[} = (\frac{1}{u})_{/]0,\varepsilon[}$, and $\widetilde{u}_{/]0,\varepsilon[} = u_{/]0,\varepsilon[}$. We choose $\varepsilon_2 = \min(\varepsilon,\varepsilon_1)$, and we obtain $g(x) = \frac{1}{\widetilde{u}(x)}$ for every $x \in]0,\varepsilon_2[$, which implies that

$$g(x) = \frac{1}{x^m}\sum_{i=0}^{+\infty} a_i x^i,$$

in $]0,\varepsilon_2[$. On the other hand, since $z \to \sum\limits_{i=0}^{+\infty} a_i z^i$ is a holomorphic function on a neighborhood of zero, then the power series $s = \sum\limits_{i=0}^{+\infty} a_i z^i$ has a non zero radius of convergence.

□

Definition 9.11. Consider the following definitions:

d1. The set of formal power series[16] in the indeterminate X with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[[X]]$, and is defined as follows. The elements of $\mathbb{R}[[X]]$ are infinite expressions of the form

$$\sum_{i=0}^{+\infty} a_i X^i = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n + \cdots,$$

where $a_i \in \mathbb{R}$ for every $i \in \mathbb{N}$.

d2. To obtain the structure of a ring, we define in $\mathbb{R}[[X]]$ the addition and the multiplication as follows:

$$\sum_{i=0}^{+\infty} a_i X^i + \sum_{i=0}^{+\infty} b_i X^i = \sum_{i=0}^{+\infty} (a_i + b_i) X^i,$$

$$\left(\sum_{i=0}^{+\infty} a_i X^i\right)\left(\sum_{i=0}^{+\infty} b_i X^i\right) = \sum_{i=0}^{+\infty}\left(\sum_{k=0}^{i} a_k b_{i-k}\right) X^i.$$

d3. The field of fractions of $\mathbb{R}[[X]]$ is denoted by $\mathbb{R}((X))$ and called the field of formal Laurent series[16].

Example 9.12. For the elements of $\mathbb{R}[[X]]$, we propose the following examples:

(i) $f(X) = \sum_{k=0}^{+\infty} k! X^k$.

(ii) $\frac{1}{1-X} = \sum_{k=0}^{+\infty} X^k$.

(iii) $\exp(X) = \sum_{k=0}^{+\infty} \frac{X^k}{k!}$.

Theorem 9.13. *The elements of the field of formal Laurent series $\mathbb{R}((X))$ are infinite expressions of the form*

$$g(X) = \frac{1}{X^m} \sum_{i=0}^{+\infty} a_i X^i,$$

where m is a naturel number, and $a_i \in \mathbb{R}$.

Proof. See [16]. □

Let g be an element of Δ, from the Lemma 9.10 and the Theorem 9.13 there exist a real number $\varepsilon > 0$ and an element g^* of the field of formal Laurent series $\mathbb{R}((X))$ such that

$$g^* = \frac{1}{X^m} \sum_{i=0}^{+\infty} a_i X^i,$$

and we have

$$g(x) = \frac{1}{x^m} \sum_{i=0}^{+\infty} a_i x^i,$$

for every $x \in]0, \varepsilon[$.

Notation 2. Let ε be a strictly positive real number, for every element

$$g^* = \frac{1}{X^m}\sum_{i=0}^{+\infty} a_i X^i$$

of $\mathbb{R}((X))$, we denote by $g^*_{/]0,\varepsilon[}$ the real function defined in $]0,\varepsilon[$ as:

$$g^*_{/]0,\varepsilon[} : x \longrightarrow \frac{1}{x^m}\sum_{i=0}^{+\infty} a_i x^i.$$

We can prove that g^* is unique for every element g of Δ. For that, consider two elements of $\mathbb{R}((X))$ as

$$g^* = \frac{1}{X^m}\sum_{i=0}^{+\infty} a_i X^i, \qquad g^\circ = \frac{1}{X^n}\sum_{i=0}^{+\infty} b_i X^i,$$

such that $g^*_{/]0,\varepsilon[} = g^\circ_{/]0,\varepsilon[} = g_{/]0,\varepsilon[}$. Assume that $m \geq n$. For every $x \in]0,\varepsilon[$ we have

$$\begin{aligned} g^*(x) = g^\circ(x) \quad \Rightarrow \quad \frac{1}{x^m}\sum_{i=0}^{+\infty} a_i x^i &= \frac{1}{x^n}\sum_{i=0}^{+\infty} b_i x^i \\ &= \frac{1}{x^m}\sum_{i=0}^{+\infty} b_i x^{i+m-n}. \end{aligned}$$

Then

$$\sum_{i=0}^{+\infty} a_i x^i = \sum_{i=0}^{+\infty} b_i x^{i+m-n}.$$

From the properties of analytic functions, we deduce that:

$$\begin{cases} a_i = 0, & \text{when } i < m-n, \\ a_i = b_{i-m+n}, & \text{else.} \end{cases}$$

Then

$$\begin{aligned} g^* &= \frac{1}{X^m}\sum_{i=0}^{+\infty} a_i X^i \\ &= \frac{1}{X^m}\sum_{i=m-n}^{+\infty} b_{i-m+n} X^i \\ &= \frac{1}{X^m}\sum_{i=0}^{+\infty} b_i X^{i+m-n} \\ &= \frac{1}{X^n}\sum_{i=0}^{+\infty} b_i X^i \\ &= g^\circ. \end{aligned}$$

We set $\delta = X$ and $g(\delta) = g^*$ and define the following map:

$$\begin{array}{rccc} \vartheta: & (\Delta,+,.) & \longrightarrow & (\mathbb{R}((\delta)),+,.), \\ & g & \longmapsto & g(\delta), \end{array}$$

which satisfies the following properties:

(i) ϑ is a ring homomorphism: let f and g be two elements of Δ. Then we have $\vartheta(f+g) = (f+g)^*$. Since $f^* + g^*$ is an element of $\mathbb{R}((\delta))$, and $(f^*+g^*)_{/]0,\varepsilon[} = (f+g)_{/]0,\varepsilon[}$ then $(f+g)^* = f^* + g^*$ from the uniqueness of $(f+g)^*$ in $\mathbb{R}((\delta))$ which satisfies $(f+g)^*)_{/]0,\varepsilon[} = (f+g)_{/]0,\varepsilon[}$. Then, we deduce that $\vartheta(f+g) = \vartheta(f) + \vartheta(g)$. In the same way, one can prove that $\vartheta(f.g) = \vartheta(f)\vartheta(g)$.

(ii) $\ker(\vartheta) = \mathcal{I}_0$: if $g^* = 0$, then there exists a real number $\varepsilon > 0$ such that $g_{/]0,\varepsilon[} = 0$, we deduce that $g \in \mathcal{I}_0$.

From the properties of ϑ, we deduce that the following ring homomorphism

$$\begin{array}{rccc} \overline{\vartheta}: & (\Delta/\mathcal{I}_0,+,.) & \longrightarrow & (\mathbb{R}((\delta)),+,.), \\ & \overline{g} & \longmapsto & g(\delta), \end{array}$$

is injective.

Theorem 9.14. *There exists a set $\mathcal{O}$ and a total order $\leq$ such that:*

(i) $(\mathcal{O},+,.)$ *is an extension field of* $(\mathbb{R},+,.)$.

(ii) $(\mathcal{O},\leq)$ *is an ordered* $\mathbb{R}-$*extension.*

(iii) $I_{\mathcal{O}} \neq \emptyset$.

Proof. Let $\mathcal{O}$ be the set defined as $\mathcal{O} = \overline{\vartheta}(\Delta/\mathcal{I}_0) = \vartheta(\Delta)$. We denote by $R\left(\sum\limits_{i=0}^{+\infty} a_i z^i\right)$ the radius of convergence of $\sum\limits_{i=0}^{+\infty} a_i z^i$ and we can verify easily that:

$$\begin{aligned}\mathcal{O} &= \{g(\delta) : g \in \Delta\} \\ &= \left\{ \frac{1}{\delta^m} \sum_{i=0}^{+\infty} a_i \delta^i : \text{where } m \in \mathbb{N} \text{ and } R\left(\sum_{i=0}^{+\infty} a_i z^i\right) \neq 0 \right\}.\end{aligned}$$

From what precedes, the map

$$\begin{array}{rrcl} \vartheta^* : & (\Delta/\mathcal{I}_0, +, .) & \longrightarrow & (\mathcal{O}, +, .), \\ & \overline{g} & \longmapsto & g(\delta), \end{array}$$

is a ring isomorphism. Since $(\Delta/\mathcal{I}_0, +, .)$ is a field, then $(\mathcal{O}, +, .)$ is a subfield of $\mathbb{R}((\delta))$.
The map $\varphi = \vartheta^* \circ \theta$ defined as:

$$\begin{array}{rrcl} \varphi : & (\overline{\Delta(\mathbb{R}^{\mathbb{N}})}, +, .) & \longrightarrow & (\mathcal{O}, +, .), \\ & \overline{g((\frac{1}{n}))_{n\geq 1}} & \longmapsto & g(\delta), \end{array}$$

is a ring isomorphism.

c1. Let l be a real number. If g is a constant element of Δ such that $g = l$, then we can identify l by the image of $\bar{l} = \overline{(g\left(\frac{1}{n}\right))_{n\geq 1}}$ by φ, and we find $\varphi(\bar{l}) = l$. Using this identification, we deduce that $\mathbb{R} \subseteq \mathcal{O}$.

c2. We can define on $\mathcal{O}$ the following relation $\leq$:
$g(\delta) \leq h(\delta)$ if and only if there exists a natural number n_0, such that $g\left(\frac{1}{n}\right) \leq h\left(\frac{1}{n}\right)$ for every $n \geq n_0$. It is easy to check that $\leq$ is reflexive, transitive and antisymmetric. Then it is an partial order.

c3. To show that the set $(\mathcal{O}, \leq)$ is an ordered $\mathbb{R}$-extension, we need to show that the relation $\leq$ is total.
Consider $g, h \in \Delta_0$. Assume that these propositions (*not* $g(\delta) \leq h(\delta)$) and (*not* $h(\delta) \leq g(\delta)$) are true. To conclude, we need to find a contradiction. Since the above propositions are true, then:
for every $k \in \mathbb{N}$, there exists $n_k > k$ and $n_k^{'} > k$ such that

$$g\left(\frac{1}{n_k}\right) > h\left(\frac{1}{n_k}\right), \qquad g\left(\frac{1}{n_k^{'}}\right) < h\left(\frac{1}{n_k^{'}}\right).$$

(i) We assume that $g, h \in \Delta_1$. From the intermediate value theorem we deduce that there exists $\beta_k \in \left| \frac{1}{n_k}, \frac{1}{n_k^{'}} \right|$ such that $(g-h)(\beta_k) = 0$ (we can choose β_k so that the sequence (β_k) is strictly decreasing). Then the holomorphic function $\widetilde{g} - \widetilde{h}$ has an infinite number of roots in a neighborhood of

0. From the Theorem 2.2, we deduce that the function $\widetilde{g}-\widetilde{h}$ is the zero function. Then $g=h$, which is absurd.

(ii) Now, suppose that $g,h\in\Delta_2$. Then, there exists $(u,v)\in\Delta_1^2$, such that $u_{/]0,\varepsilon]}=(\frac{1}{g})_{/]0,\varepsilon]}$ and $v_{/]0,\varepsilon]}=(\frac{1}{h})_{/]0,\varepsilon]}$. Since u and v are two elements of Δ_1, then, $u(\delta)\leq v(\delta)$ or $v(\delta)\leq u(\delta)$. Finally, we deduce that $g(\delta)$ and $h(\delta)$ are comparable.

(iii) In the case of $g\in\Delta_1$ and $h\in\Delta_2$, there exists $h_1\in\Delta_1$ and $\varepsilon>0$, such that $h_{/]0,\varepsilon[}=(\frac{1}{h_1})_{/]0,\varepsilon[}$. Since h_1 is a metalic function, then the sequence $(h_1\left(\frac{1}{n}\right))_{n\geq 1}$ admits a constant sign from a certain rank. In fact, if it is not the case, then, for every $k\in\mathbb{N}$ there exist $n_k>k$ and $n_k^{'}>k$ such that $h_1(\frac{1}{n_k})>0$ and $h_1(\frac{1}{n_k^{'}})<0$. From the intermediate value theorem, there exists $\beta_k\in\left|\frac{1}{n_k},\frac{1}{n_k^{'}}\right|$ such that $(h_1)(\beta_k)=0$ (we can choose β_k such that the sequence (β_k) is strictly decreasing). From the Theorem 2.2, h_1 is the zero function on a neighborhood of 0, which is absurd. Then, we deduce that the sequence $(h_1\left(\frac{1}{n}\right))_{n\geq 1}$ admits a constant sign. Since $\lim\limits_{n\to+\infty}h_1\left(\dfrac{1}{n}\right)=0$, then $\lim\limits_{n\to+\infty}\dfrac{1}{h_1\left(\frac{1}{n}\right)}$ exists and we have $\lim\limits_{n\to+\infty}\dfrac{1}{h_1\left(\frac{1}{n}\right)}=\pm\infty$, which implies that

$$\lim_{n\to+\infty}h\left(\frac{1}{n}\right)=\pm\infty.$$

(a) If we have $\lim\limits_{n\to+\infty}h\left(\dfrac{1}{n}\right)=+\infty$, then $g(\delta)\leq h(\delta)$ (because $g\left(\frac{1}{n}\right)\leq h\left(\frac{1}{n}\right)$ from a certain rank).

(b) In other case, we have $\lim\limits_{n\to+\infty}h\left(\dfrac{1}{n}\right)=-\infty$, then we find $h(\delta)\leq g(\delta)$ (because $h\left(\frac{1}{n}\right)\leq g\left(\frac{1}{n}\right)$ from a certain rank).

c4. Now, it remains to show that $I_{\mathcal{O}}\neq\emptyset$. For that, it is necessary to find an element $\delta\in\mathcal{O}$ which is infinitesimal. For $u:x\longrightarrow x$, we have $u\in\Delta$ (more precisely Δ_1) and $\delta=u(\delta)$. In addition, we have $0<\delta<\varepsilon$ for every real strictly positive ε, because there exists $p\in\mathbb{N}$ such that $0<u\left(\frac{1}{n}\right)<\varepsilon$ for every integer $n>p$. Then δ is an infinitesimal number.

□

Conclusions 1. Finally, we deduce that:

(1) $(\mathcal{O},+,.)$ is an extension field of $(\mathbb{R},+,.)$.
(2) $(\mathcal{O},\leq)$ is an ordered $\mathbb{R}-$extension, which contains the infinitesimal element δ.
The field $(\mathcal{O},+,.)$ is called the field of **Omicran-reals** and an element of $\mathcal{O}$ is called an Omicran (or an Omicran-real).

10. Applications of the Field of Omicran-reals

10.1. The Exact Limit.

Proposition 10.1. *The map φ defined as:*

$$\begin{array}{rccc} \varphi: & (\overline{\Delta(\mathbb{R}^{\mathbb{N}})},+,.) & \longrightarrow & (\mathcal{O},+,.), \\ & \overline{(g\left(\frac{1}{n}\right))_{n\geq 1}} & \longmapsto & g(\delta), \end{array}$$

is an isomorphism.

If we want to define a new concept more precise than the limit that allows to give the value taken by the sequence $\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}$ at infinity, then this concept (called exact limit) is dependent to the values taken by $\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}$ from a certain rank n_0. Intuitively, the equivalence class $\overline{\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}}$ is the only concept can give these values independently from n_0. On the other hand, if f is an element of Δ, then we can identify the equivalence class $\overline{\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}}$ by $f(\delta)$ from the Proposition 10.1, so we deduce that we can define the new concept as follows.

Definition 10.2. Let $f\in\Delta$. The Omicran $f(\delta)=\varphi\left(\overline{\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}}\right)$ is called the exact limit of the sequence $\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}$.

We set

$$\lim_{\text{exact}} f\left(\frac{1}{n}\right)=f(\delta).$$

Remark 10.3. We remark that $\lim\limits_{\text{exact}}=\varphi\circ s$, where s is a canonical surjection defined as:

$$\begin{array}{rccc} s: & (\Delta(\mathbb{R}^{\mathbb{N}}),+,.) & \longrightarrow & (\overline{\Delta(\mathbb{R}^{\mathbb{N}})},+,.), \\ & \left(f\left(\frac{1}{n}\right)\right)_{n\geq 1} & \longmapsto & \overline{\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}}. \end{array}$$

Example 10.4. We propose the following examples:

(1) $\lim\limits_{\text{exact}}\dfrac{1}{n}=\delta$.
(2) $\lim\limits_{\text{exact}}\sin\left(\dfrac{1}{n}\right)=\sin(\delta)$.
(3) $\lim\limits_{\text{exact}}\dfrac{1}{n+1}=\dfrac{\delta}{\delta+1}$.

(4) We can verify that there does not exist an element $f \in \Delta$, such that $f\left(\frac{1}{n}\right) = (-1)^n$ from a certain rank. Then we can not define the exact limit $\lim\limits_{\text{exact}} (-1)^n$.

(5) Generally, from the proprieties of the elements of Δ, we can show that if $(x_n)_{n\geq 1}$ does not admit a constant sign from a certain rank, then this sequence does not admit an exact limit, for instance, if $x_n = \frac{(-1)^n}{n}$, then $\lim\limits_{n\to+\infty} x_n = 0$, but we can not define the exact limit of $(x_n)_{n\geq 1}$.

10.2. **The Projection of an Element of $\mathcal{O}$.**

Definition 10.5. Let f be a metalic function, and $x \in \mathcal{O}$ such that $x = f(\delta)$. If we find an element $x^* \in \mathbb{R}$ such that $\mid x - x^* \mid \leq \mid x - y \mid$, $\forall y \in \mathbb{R}$, then, the real x^* is called the projection of x in $\mathbb{R}$.

Remark 10.6. The distance from x to $\mathbb{R}$ is given by

$$d_{\mathbb{R}}(x) = \inf_{y\in\mathbb{R}} \mid x - y \mid = \mid x - x^* \mid .$$

Example 10.7. For example, we have:

- $\delta^* = 0$.
- $\left(\dfrac{1}{\delta^{\,2} + 1}\right)^* = 1$.

Theorem 10.8. *Let f be a metalic function, and $x \in \mathcal{O}$ such that $x = f(\delta)$. The projection x^* of x onto $\mathbb{R}$ exists and it is unique. In addition, we have:*

$$x^* = \lim_{n\to+\infty} f\left(\frac{1}{n}\right).$$

Proof. Let $x_0 = \lim f\left(\frac{1}{n}\right)$. Then: $\forall \varepsilon > 0\ \exists n_0 \mid \forall n \geq n_0, \mid f\left(\frac{1}{n}\right) - x_0 \mid \leq \varepsilon$

$$\begin{aligned}
&\Leftrightarrow \quad \text{for every } n \geq n_0, \text{ we have } -\varepsilon \leq f\left(\tfrac{1}{n}\right) - x_0 \leq \varepsilon \\
&\Leftrightarrow \quad \lim_{\text{exact}} f\left(\frac{1}{n}\right) - x_0 \leq \varepsilon \quad \text{and} \quad -\varepsilon \leq \lim_{\text{exact}} f\left(\frac{1}{n}\right) - x_0 \\
&\Leftrightarrow \quad f(\delta) - x_0 \leq \varepsilon \ \text{ and } -\varepsilon \leq f(\delta) - x_0 \\
&\Leftrightarrow \quad \mid f(\delta) - x_0 \mid \leq \varepsilon.
\end{aligned}$$

Next, we can show that $\mid f(\delta) - x_0 \mid \leq \mid f(\delta) - y \mid$ for any $y \in \mathbb{R}$. Assume that there exists $y \in \mathbb{R}$ such that $\mid f(\delta) - y \mid \leq \mid f(\delta) - x_0 \mid \leq \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$. This $\mid y - x_0 \mid \leq\ 2\varepsilon$ for any $\varepsilon \in \mathbb{R}^+$, which implies that $y = x_0 = x^*$. Finally, we deduce the existence and uniqueness of $x^* \in \mathbb{R}$ such that

$$\mid f(\delta) - x^* \mid \leq \mid f(\delta) - y \mid, \quad \forall y \in \mathbb{R}.$$

In addition, we have $x^* = \lim\limits_{n\to+\infty} f\left(\dfrac{1}{n}\right)$. □

Theorem 10.9. *Let f be a metalic map, and $x = f(\delta)$. Then we have:*

$$| x - x^* | \leq \varepsilon, \quad \forall \varepsilon > 0,$$

where x^* is the unique element of $\mathbb{R}$ which verifies this property.

Proof. There exists n_0 such that

$$| f\left(\frac{1}{n}\right) - x^* | \leq \varepsilon, \quad \forall n \geq n_0.$$

Then

$$x^* - \varepsilon \leq f\left(\frac{1}{n}\right) \leq x^* + \varepsilon, \quad \forall n \geq n_0.$$

Then, we deduce that

$$x^* - \varepsilon \leq f(\delta) \leq x^* + \varepsilon,$$

$$| x - x^* | \leq \varepsilon.$$

To show the uniqueness of x^*, we assume that there exists another element $y \in \mathbb{R}$ such that $| x - y | \leq \varepsilon$. Then: $| x^* - y | \leq 2\varepsilon$, finally we get $y = x^*$. □

Theorem 10.10. *If $f \in \Delta_1$, then the real $\lim_{n \to +\infty} f\left(\frac{1}{n}\right)$ is the projection of $\lim_{exact} f\left(\frac{1}{n}\right)$ onto $\mathbb{R}$, so we get:*

$$\left(\lim_{exact} f\left(\frac{1}{n}\right)\right)^* = \lim_{n \to +\infty} f\left(\frac{1}{n}\right).$$

10.3. **Necessary Conditions for the Existence of the Exact Limit.**

Definition 10.11. Let $(x_n)_{n \geq 1}$ be a real sequence. We say that $(x_n)_{n \geq 1}$ has an exact limit, if there exists $f \in \Delta$ such that $x_n = f\left(\frac{1}{n}\right)$ from a certain rank $n_0 \in \mathbb{N}$. In this case, we have

$$\lim_{\text{exact}} x_n = f(\delta).$$

In this subsection, we propose the following remarks concerning the existence of the exact limit of a real sequence $(x_n)_{n \geq 1}$:

(1) Let $(x_n)_{n \geq 1}$ be a sequence of real numbers. Assume that the exact limit of $(x_n)_{n \geq 1}$ exists. Then, there exists a function $f \in \Delta$ such that $\lim_{\text{exact}} x_n = f(\delta)$.

If $f \in \Delta_1$, then f is a metalic function. Let $\widetilde{f}$ be a metalic extension of f, then we have $f\left(\frac{1}{n}\right) = \widetilde{f}\left(\frac{1}{n}\right) = x_n$ from a certain rank. Since $\widetilde{f}$ is holomorphic at 0, then the limit of $(x_n)_{n \geq 1}$ exists and we have $\lim_{n \to +\infty} x_n = \widetilde{f}(0)$. Finally, we conclude that the existence of the exact limit implies the existence of the limit. In addition, we have $\lim\limits_{n\to+\infty} x_n = \widetilde{f}(0)$. Generally, we get

$$\lim_{\text{exact}} x_n = f(\delta) \quad \Rightarrow \quad \lim_{n\to+\infty} x_n = \begin{cases} \widetilde{f}(0), & \text{while } f \in \Delta_1; \\ \pm\infty, & \text{while } f \in \Delta_2. \end{cases}$$

(2) The reciprocal of the above implication is not true. We can find a convergent sequence which does not have an exact limit (for example, $x_n = \frac{(-1)^n}{n}$).

(3) If a sequence $(x_n)_{n\geq 1}$ has the exact limit, then $(x_n)_{n\geq 1}$ admits a constant sign from a certain rank.
In addition, if $x_n > 0$ from a certain rank, then, we have $\lim\limits_{\text{exact}} x_n > 0$.

(4) If the sequence $(x_n)_{n\geq 1}$ has the exact limit, from the properties of the elements of Δ, we can show that the sequence $(x_{n+1} - x_n)_{n\geq 1}$ admits a constant sign from a certain rank.

Theorem 10.12. *Let $(a_n)_{n\geq 1}$ be a real sequence, and f be a holomorphic function on $D(0,\varepsilon) \setminus \{0\}$ such that*

(i) $f(]0,\varepsilon[) \subseteq \mathbb{R}$,
(ii) $f\left(\frac{1}{n}\right) = a_n$ from a certain rank,
(iii) f is bounded on $D(0,\varepsilon) \setminus \{0\}$.

Then the sequence $(a_n)_{n\geq 1}$ has an exact limit, and we have $\lim\limits_{\text{exact}} a_n = f(\delta)$.

Proof. 0 is an artificial singularity of f. □

10.4. **The Exact Derivative.**

Definition 10.13. Let f be a real function that is differentiable at a point $x_0 \in \mathbb{R}$. If the function $h \longrightarrow \frac{f(x_0+h)-f(x_0)}{h}$ is metalic, then the exact limit of $\left(\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{\frac{1}{n}}\right)_{n\geq 1}$ exists. We put

$$\begin{aligned} \widehat{f}(x_0) &= \lim_{\text{exact}} \frac{f(x_0+\frac{1}{n}) - f(x_0)}{\frac{1}{n}} \\ &= \frac{f(x_0+\delta) - f(x_0)}{\delta}. \end{aligned}$$

The Omicran $\widehat{f}(x_0)$ is called the exact derivative of the function f at x_0.

Example 10.14. Consider the function f defined as $f : x \longrightarrow x^2$. The exact derivative of f at x_0 is given by $\widehat{f}(x_0) = 2x_0 + \delta$.

Theorem 10.15. *Let f be a real function that is differentiable at a point $x_0 \in \mathbb{R}$. If the function $h \longrightarrow \frac{f(x_0+h)-f(x_0)}{h}$ is metalic, then*

$$(\widehat{f}(x_0))^* = f'(x_0)$$

Proof. We can apply the Theorem 10.10. □

Example 10.16. For $f : x \longrightarrow x^2$, the exact derivative of f at x_0 is $\widehat{f}(x_0) = 2x_0 + \delta$, and the derivative at x_0 is $f'(x_0) = 2x_0$. We can verify easily that $(2x_0 + \delta)^* = 2x_0$.

Lemma 10.17. *Let f be a metalic function such that for every integer $k \in \mathbb{N}$, the function $t \longrightarrow f(x_0 + kt)$ is metalic. Then*

$$f(x_0+N\delta) = f(x_0)+\delta(\widehat{f}(x_0)+\widehat{f}(x_0+\delta)+\widehat{f}(x_0+2\delta)+\cdots+\widehat{f}(x_0+(N-1)\delta).$$

Proof. From the definition of $\widehat{f}$, we have

$$\begin{aligned} f(x_0 + \delta) &= f(x_0) + \delta\widehat{f}(x_0), \\ f(x_0 + 2\delta) &= f(x_0 + \delta) + \delta\widehat{f}(x_0 + \delta), \\ &\vdots \\ f(x_0 + N\delta) &= f(x_0 + (N-1)\delta) + \delta\widehat{f}(x_0 + (N-1)\delta). \end{aligned}$$

By summing these equalities, we find the desired result. □

Application 1. (Calculation of the sum Σk^n)

(1) For $n = 1$, if $f(x) = x^2$, then $\widehat{f}(x) = 2x + \delta$.
From the Lemma 10.17 in the case of $x_0 = 0$, we find

$$N^2\delta^2 = (\delta(\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \cdots + \widehat{f}((N-1)\delta)),$$

which implies that $N^2\delta^2 = \delta.(\sum_{k=0}^{N-1} 2k\delta + \delta)$. Then

$$\begin{aligned} N^2 &= \sum_{k=0}^{N-1}(2k+1) \\ &= 2\sum_{k=0}^{N-1} k + N, \end{aligned}$$

and we deduce that

$$\frac{N^2 - N}{2} = \sum_{k=0}^{N-1} k.$$

(2) In the case of $n = 2$, we choose $f(x) = x^3$ and we obtain $\widehat{f}(x) = 3x^2 + 3x\delta + \delta^2$. By using the Lemma 10.17 for $x_0 = 0$, we find here

$$\begin{aligned} N^3\delta^3 &= \delta.(\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \cdots + \widehat{f}((N-1)\delta)), \\ &= \delta.\sum_{k=0}^{N-1}(3k^2\delta^2 + 3k\delta.\delta + \delta^2), \\ &= \delta^3.(\sum_{k=0}^{N-1} 3k^2 + 3k + 1). \end{aligned}$$

Then $N^3 = 3\displaystyle\sum_{k=0}^{N-1} k^2 + 3\sum_{k=0}^{N-1} k + N$,

and we deduce that $\displaystyle\sum_{k=0}^{N-1} k^2 = \frac{N^3 - N - 3\sum_{k=0}^{N-1} k}{3}$.

Finally, we get: $\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6}$.

Similarly, we can calculate $\sum_{k=0}^{N-1} k^3$, $\sum_{k=0}^{N-1} k^4, \ldots$.

Application 2. (The Riemann sum)

Let f and g be two metalic functions such that: $f(\delta) = g(\delta)$. Then $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ from a certain rank.

Consider the function defined as follows:

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

From the Lemma 10.17, we deduce that there exists a natural number n_0 such that we have:

$$\begin{aligned} f\left(x_0 + \frac{N}{n}\right) = f(x_0) + \frac{1}{n}\left(f_n(x_0) + f_n\left(x_0 + \frac{1}{n}\right) + f_n\left(x_0 + \frac{2}{n}\right)\right. \\ \left. + \cdots + f_n\left(x_0 + \frac{N-1}{n}\right)\right), \quad \forall n \geq n_0. \end{aligned}$$

Assume that f is twice differentiable on $\mathbb{R}$, and f'' is continuous on $\mathbb{R}$. By using the Taylor's formula with Lagrangian Remainder, we obtain:

$$f_n\left(a + \frac{k}{n}\right) = f'\left(a + \frac{k}{n}\right) + \frac{1}{2n}f''(\xi_{k,n}),$$

where $\xi_{k,n} \in]a, a+\frac{N}{n}[$. Then

$$f\left(a+\frac{N}{n}\right) = f(a) + \frac{1}{n}.\sum_{k=0}^{N-1} f_n\left(a+\frac{k}{n}\right)$$
$$= f(a) + \frac{1}{n}.\sum_{k=0}^{N-1}\left(f'\left(a+\frac{k}{n}\right) + \frac{1}{2n}f''(\xi_{k,n})\right).$$

Assume that $b > a$. We can choose $N = \lfloor (b-a)n \rfloor$, and then we get

$$f\left(a+\frac{N}{n}\right) - f(a) = \frac{1}{n}.\sum_{k=0}^{N-1} f'\left(a+\frac{k}{n}\right) + \frac{1}{2n^2}\sum_{k=0}^{N-1} f''(\xi_{k,n}).$$

Since $N = \lfloor (b-a)n \rfloor$, then $b-a-\frac{1}{n} < \frac{N}{n} \leq b-a$.
Let $M = \sup_{[a,b]} \left|f''(x)\right| < +\infty$. We have

$$\left|\frac{1}{2n^2}\sum_{k=0}^{N-1} f''(\xi_{k,n})\right| \leq \frac{MN}{2n^2}$$
$$\leq \frac{M(b-a)}{2n} \longrightarrow 0 \quad (n \to +\infty).$$

In addition, we have $\lim_{n\to+\infty} f\left(a+\frac{N}{n}\right) = f(b)$. We pass to the limit and we find

$$f(b) - f(a) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{\lfloor (b-a)n \rfloor - 1} f'\left(a+\frac{k}{n}\right).$$

For $b = 1$ and $a = 0$, we get

$$f(1) - f(0) = \int_0^1 f'(t)dt = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f'\left(\frac{k}{n}\right).$$

10.5. **The Logarithmic Function.** We know that

$$\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n = e^x, \forall x \in \mathbb{R}.$$

Let x be a real number. The function

$$f : z \longrightarrow (1+xz)^{\frac{1}{z}} = e^{\frac{1}{z}\ln(1+zx)},$$

is a holomorphic function on $D(0,\varepsilon) \setminus \{0\}$. In addition, we have

$$\ln(1+zx) = zx - \frac{z^2x^2}{2} + o(z^2x^2), \quad (\text{for } \mid z \mid << 1).$$

So we deduce that $\lim_{z\to 0} \frac{\ln(1+zx)}{z} = x$, and $\lim_{z\to 0} f(z) = e^x$. Then, the function f can be extended to a holomorphic function on a neighborhood of 0, which implies that $\lim_{\text{exact}} \left(1+\frac{x}{n}\right)^n$ exists and we have $\lim_{\text{exact}} \left(1+\frac{x}{n}\right)^n = (1+x\delta)^{\frac{1}{\delta}}$ and $((1+x\delta)^{\frac{1}{\delta}})^* = e^x$. Then, the real number e^x is an infinitesimal approximation of $(1+\delta x)^{\frac{1}{\delta}}$.

We put $\xi_\alpha(x) = (1+\alpha x)^{\frac{1}{\alpha}}$, for every $\alpha > 0$. If the function $x \longrightarrow \xi_\alpha(x)$ has an inverse, then, we have $\xi_\alpha^{-1}(x) = \frac{x^\alpha - 1}{\alpha}$.

We attempt to prove that the omicran $\frac{x^\delta - 1}{\delta}$ exists for every $x > 0$, and it represents an infinitesimal approximation of the real number $\ln(x)$.

Let x be a real number in $\mathbb{R}^{*+}$. The map defined as

$$g : z \longrightarrow \frac{x^z - 1}{z} = \frac{e^{z\ln(x)} - 1}{z},$$

is a holomorphic function on $D(0,\varepsilon) \setminus \{0\}$, and $\lim_{z\to 0} g(z) = \ln(x)$. Then 0 is an artificial singularity of g, and we deduce that the exact limit of the sequence $\left(n\left(x^{\frac{1}{n}} - 1\right)\right)_{n\geq 1}$ exists and we have $\lim_{\text{exact}} n\left(x^{\frac{1}{n}} - 1\right) = \frac{x^\delta - 1}{\delta}$. We define the original logarithm by

$$\ln_o : x \longrightarrow \frac{x^\delta - 1}{\delta}.$$

The function ξ is called the function of original exponential. We set

$$\xi(x) = \exp_o(x) = (1+\delta x)^{\frac{1}{\delta}},$$

and we deduce that

$$(\ln_o(x))^* = \ln(x).$$

Then

$$\ln(x) = \lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha}.$$

Application 3. From the above results, we can show the following equality:

$$\ln(x) = \lim_{\alpha\to 0} \frac{x^\alpha - x^{-\alpha}}{2\alpha}.$$

Remark 10.18. We have

$$\begin{aligned} \frac{\ln(x)}{\ln(y)} &= \lim_{\alpha\to 0} \frac{x^\alpha - 1}{y^\alpha - 1} \\ &= \lim_{\alpha\to 0} \frac{x^\alpha - x^{-\alpha}}{y^\alpha - y^{-\alpha}}. \end{aligned}$$

Application 4. By using the above results, we can show the following theorem.

Theorem 10.19. *For all $x > 0$ and $x \neq 1$, we have*

$$\frac{x-1}{\ln(x)} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

Proof. We have

$$\left(x^{\frac{1}{n}} - 1\right)\left(\sum_{k=0}^{n-1} x^{\frac{k}{n}}\right) = x - 1.$$

Then,

$$n\left(x^{\frac{1}{n}} - 1\right)\left(\frac{1}{n}\sum_{k=0}^{n-1} x^{\frac{k}{n}}\right) = x - 1.$$

Since $\lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{n\to\infty} n(x^{\frac{1}{n}} - 1) = \ln(x)$, then

$$\frac{x-1}{\ln(x)} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

□

Application 5. Consider $\mathcal{P}$ as the set of prime numbers. We define the prime-counting function [12] at real values of x by

$$\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}.$$

Theorem 10.20. *(Hadamard and de la Valle Poussin)*
As $x \to +\infty$, we have

$$\pi(x) \sim \frac{x}{\ln(x)}.$$

Proof. See [12]. □

Theorem 10.21. *As $x \to +\infty$, we have*

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

Proof. We can verify that $\frac{x}{\ln(x)} \sim \frac{1}{x^{\frac{1}{x}}-1}$. In fact, as $x \to +\infty$, we have

$$\frac{1}{x^{\frac{1}{x}} - 1} \sim \frac{1}{e^{\frac{\ln(x)}{x}} - 1} = \frac{1}{\frac{e^{\frac{\ln(x)}{x}}-1}{\frac{\ln(x)}{x}}} \frac{x}{\ln(x)}.$$

Since, $\lim_{x\to+\infty} \frac{e^{\frac{\ln(x)}{x}} - 1}{\frac{\ln(x)}{x}} = 1$, then $\frac{1}{x^{\frac{1}{x}}-1} \sim \frac{x}{\ln(x)}$.
Finally, we obtain

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

□

Application 6. Let (p_n) be the sequence of prime numbers. Then we have the following theorems.

Theorem 10.22. *We have*

$$p_n \sim n\ln(n), \ \textit{while} \quad n \to +\infty.$$

Proof. See [12].

□

Theorem 10.23. *We have*

$$p_n \sim n^2(\sqrt[n]{n}-1), \ \textit{while} \quad n \to +\infty.$$

Proof. We can verify that $\lim\limits_{n\to\infty} \dfrac{n(\sqrt[n]{n}-1)}{\ln(n)} = 1$. Let n be a natural number greater than 2. We have

$$\sqrt[n]{n} - 1 = e^{\frac{1}{n}\ln(n)} - 1.$$

Then

$$\sqrt[n]{n} - 1 = \left(\frac{e^{\frac{1}{n}\ln(n)} - 1}{\frac{\ln(n)}{n}}\right)\left(\frac{\ln(n)}{n}\right).$$

On the other hand, we have $\lim\limits_{n\to\infty} \dfrac{\ln(n)}{n} = 0$, then $\lim\limits_{n\to\infty} \dfrac{e^{\frac{1}{n}\ln(n)} - 1}{\frac{\ln(n)}{n}} = 1$.

So, we obtain

$$\sqrt[n]{n} - 1 \sim \frac{\ln(n)}{n}, \ \text{while } n \to +\infty.$$

Then

$$n^2(\sqrt[n]{n}-1) \sim n\ln(n), \ \text{while } n \to +\infty.$$

From the Theorem 10.22, we deduce that

$$n^2(\sqrt[n]{n}-1) \sim p_n, \ \text{while } n \to +\infty.$$

Finally, we obtain the desired result. □

10.6. **The Omicran-reals in Geometry.**

10.6.1. *The Geometric Point.* Let f be a metalic function, and $\widetilde{f}(\delta)$ be an infinitesimal number. The sequence $\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}$ admits a constant sign from a certain rank. Assume that the above sequence is positive from a certain rank n_0. Since $\varphi(\widetilde{f}(\delta)) = \overline{\left(f\left(\frac{1}{n}\right)\right)_{n\geq 1}}$, then we can represent $\widetilde{f}(\delta)$ by the family of segments $(I_n)_{n\geq 1}$, where $I_n =]0, f\left(\frac{1}{n}\right)]$.

Definition 10.24. Let x_A be an Omicran of $\mathcal{O}$. An elementary geometric point of $\mathcal{O}$ is a segment of the type $[x_A, x_A+\delta[$, where $[x, y[= \{z \in \mathcal{O},\ x \leq z < y\}$.

10.6.2. *The Length of a Curve* $\mathcal{C}_f$. We define the length of an elementary geometric point by

$$l([x_A, x_A+\delta[) = \delta,$$

where $x_A = g(\delta)$, and g is a metalic function. The real x_A^* represents the projection of x_A onto $\mathbb{R}$.

Let $\widetilde{f}$ be a holomorphic function on an open set U such that $D'(0,1) \subset U$. Assume that $\widetilde{f}([0,1]) \subset \mathbb{R}$. We set $\widetilde{f}_{/[0,1]} = f$. The map f is a metalic function and $\widetilde{f}$ is a metalic extension of f. Assume that x_A^* is an element of $[0,1]$. The map $x \longrightarrow f(x_A^* + x)$ is metalic. Its metalic extension is given by $x \longrightarrow \widetilde{f}(x_A^* + x)$. Consider $A(x_A, f(x_A))$ and $A'(x_A+\delta, f(x_A+\delta))$ which are two ordered pairs of $\mathcal{O}^2$. Let ϕ be the function defined as

$$\phi : z \longrightarrow z\sqrt{1 + \left(\frac{\widetilde{f}(\widetilde{g}(z)+z) - \widetilde{f}(\widetilde{g}(z))}{z}\right)^2}.$$

The map $\theta : z \to \frac{\widetilde{f}(\widetilde{g}(z)+z)-\widetilde{f}(\widetilde{g}(z))}{z}$ is holomorphic on $D(0,\varepsilon) \setminus \{0\}$, and we have

$$\theta(z) = \frac{\widetilde{f}(\widetilde{g}(z)+z) - \widetilde{f}(x_A^*)}{z} - \frac{\widetilde{f}(\widetilde{g}(z)) - \widetilde{f}(x_A^*)}{z},$$

where $\lim\limits_{z\to 0} \widetilde{g}(z) = \widetilde{g}(0) = x_A^*$ (the projection of x_A onto $\mathbb{R}$). Then $\lim\limits_{z\to 0} \theta(z)$ exists, and we have

$$\lim_{z\to 0} \theta(z) = (\widetilde{g}'(0)+1)f'(x_A^*) - \widetilde{g}'(0)f'(x_A^*) = f'(x_A^*) \in \mathbb{R}.$$

So, we deduce that $\lim\limits_{z\to 0} \phi(z) = 0$, and ϕ is continuously extendable at 0. Then the function ϕ is holomorphically extendable at 0, which justifies the existence of the exact limit of the sequence $(\phi\left(\frac{1}{n}\right))_{n\geq 1}$, and we have

$$\begin{aligned}\lim_{\text{exact}} (\phi\left(\frac{1}{n}\right)) &= \phi(\delta) \\ &= \delta\sqrt{1 + \left(\frac{f(x_A+\delta) - f(x_A)}{\delta}\right)^2} \in \mathcal{O}.\end{aligned}$$

We define the length of the segment $[A, A'[$ by

$$l([A, A'[) = \delta\sqrt{1 + \left(\frac{f(x_A+\delta) - f(x_A)}{\delta}\right)^2}.$$

We put

$$\psi(x_A) = \delta\sqrt{1 + \left(\frac{f(x_A+\delta) - f(x_A)}{\delta}\right)^2}$$
$$= \delta\sqrt{1 + \widehat{f}(x_A)^2}.$$

Let f be a metalic function defined on $[0,1]$, and let $A(0, f(0))$ and $B(1, f(1))$ be two points of the plane which define with f the arc $\widetilde{AB}$. If the exact limit of the series $\displaystyle\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1+\widehat{f}\left(\frac{k}{n}\right)^2}$ exists, then we define the exact length of the arc $\widetilde{AB}$ by

$$l(\widetilde{AB}) = \lim_{\text{exact}} \frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1+\widehat{f}\left(\frac{k}{n}\right)^2}.$$

The length of the arc $\widetilde{AB}$ is the real denoted by $l^*(\widetilde{AB})$ and is defined by

$$l^*(\widetilde{AB}) = \left(\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1+\widehat{f}\left(\frac{k}{n}\right)^2}\right)^*$$
$$= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1+f_n\left(\frac{k}{n}\right)^2},$$

where $f_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$. Since f is a metalic function, then it can be extended to a function which is twice differentiable at 0.

Assume that the function f is twice differentiable on $]0,1[$. Then

$$f_n\left(\frac{k}{n}\right) = \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{1}{n}}$$
$$= f'\left(\frac{k}{n}\right) + \frac{1}{2n}f''(\xi_k),$$

where $1 \le k \le n-1$, and $\xi_k \in]\frac{k}{n}, \frac{k+1}{n}[$.

Consider $M_1 = \sup_{]0,1[}(\mid f'(x) \mid)$ and $M_2 = \sup_{]0,1[}(\mid f''(x) \mid)$. We have

$$f_n^2\left(\frac{k}{n}\right) = \left(f'\left(\frac{k}{n}\right) + \frac{1}{2n}f''(\xi_k)\right)^2$$
$$= f'^2\left(\frac{k}{n}\right) + \varepsilon_{n,k},$$

where

$$\varepsilon_{n,k} = f'\left(\frac{k}{n}\right)\frac{1}{n}f''(\xi_k) + \frac{1}{4n^2}f''^2(\xi_k).$$

If $M_1 < +\infty$ and $M_2 < +\infty$, we obtain

$$\mid \varepsilon_{n,k} \mid \leq \frac{M_1M_2 + M_2^2}{n}.$$

Then, $\lim\limits_{n\to+\infty} \sup\limits_{k} \mid \varepsilon_{n,k} \mid = 0$, and we have

$$\begin{aligned}\sqrt{1 + f_n^2\left(\frac{k}{n}\right)} &= \sqrt{1 + f'^2\left(\frac{k}{n}\right) + \varepsilon_{n,k}} \\ &= \sqrt{1 + f'^2\left(\frac{k}{n}\right)} + \frac{\varepsilon_{n,k}}{2\sqrt{\beta_{n,k}}},\end{aligned}$$

where $\beta_{n,k} \in \left|1 + f'^2(\frac{k}{n}), 1 + f'^2(\frac{k}{n}) + \varepsilon_{n,k}\right|$. Then

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1 + f_n^2\left(\frac{k}{n}\right)} = \frac{1}{n}\sqrt{1 + f_n^2(0)} + \frac{1}{n}\sum_{k=1}^{n-1}\sqrt{1 + f'^2\left(\frac{k}{n}\right)} \\ + \frac{1}{2n}\sum_{k=1}^{n-1}\frac{\varepsilon_{n,k}}{\sqrt{\beta_{n,k}}}.\end{aligned}$$

Since $\lim\limits_{n\to+\infty} \sup\limits_{k} \mid \varepsilon_{n,k} \mid = 0$, then we can verify that $\beta_{n,k} > \frac{1}{2}$ from a certain rank n_0, and we have

$$\begin{aligned}\left|\frac{1}{2n}\sum_{k=1}^{n-1}\frac{\varepsilon_{n,k}}{\sqrt{\beta_{n,k}}}\right| &\leq \frac{\sqrt{2}}{2n}\sum_{k=1}^{n-1}|\varepsilon_{n,k}| \\ &\leq \frac{\sqrt{2}(M_1M_2 + M_2^2)}{2n}.\end{aligned}$$

Then:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1 + f_n^2\left(\frac{k}{n}\right)} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sqrt{1 + f'^2\left(\frac{k}{n}\right)}.$$

By using the Riemann sum, we deduce that the length of the arc $\widetilde{AB}$ is

$$l^*(\widetilde{AB}) = \int_0^1 \sqrt{1 + f'^2(x)}dx.$$

10.7. **The Exact Limit of the Series.** Let $s_n = \sum_{k=1}^{n} a_k$ be a convergent series, where $(a_k)_{k\geq 1}$ is a sequence of real numbers. Assume that the series $(s_n)_{n\geq 1}$ has the exact limit. Then, there exists a holomorphic function $\widetilde{f}$ on a neighborhood of zero such that $\lim\limits_{\text{exact}} s_n = \widetilde{f}(\delta)$. Then:

$$\widetilde{f}(\delta) = \lim_{\text{exact}} \sum_{k=1}^{n} a_k,$$

which implies that $\sum\limits_{k=1}^{n} a_k = \widetilde{f}\left(\frac{1}{n}\right)$, from a certain rank $n_0 \in \mathbb{N}$.
Since $a_n = s_n - s_{n-1}$, we deduce that

$$a_n = \widetilde{f}\left(\frac{1}{n}\right) - \widetilde{f}\left(\frac{1}{n-1}\right), \text{ from a certain rank.}$$

If $(a_n)_{n\geq 1}$ has the exact limit $\lim\limits_{\text{exact}} a_n$, then, we can find a holomorphic function g on a neighborhood of 0 such that $a_n = g\left(\frac{1}{n}\right)$ from a certain rank. In this case we have $\lim\limits_{\text{exact}} a_n = g(\delta)$. Since $\lim\limits_{n\to\infty} a_n = 0$, then $g(0) = 0$ and there exists p such that

$$g\left(\frac{1}{n}\right) = \widetilde{f}\left(\frac{1}{n}\right) - \widetilde{f}\left(\frac{1}{n-1}\right), \quad \forall n \geq p.$$

Since f and g are holomorphic functions on a neighborhood of 0, then there exists $\varepsilon > 0$ such that

$$g(z) = \widetilde{f}(z) - \widetilde{f}\left(\frac{z}{1-z}\right), \quad \forall z \in D(0,\varepsilon).$$

On the other hand, we have $\widetilde{f}(0) = \lim\limits_{n\to+\infty} \widetilde{f}\left(\frac{1}{n}\right) = \sum\limits_{k=1}^{+\infty} a_k$. Then, we deduce the following theorem.

Theorem 10.25. *Let g be a metalic function, and $(s_n)_{n\geq 1}$ be the convergent series defined as $s_n = \sum\limits_{k=1}^{n} g\left(\frac{1}{k}\right)$. If the exact limit of $(s_n)_{n\geq 1}$ exists, then there exists a function $\widetilde{f}$ which is holomorphic at 0 such that $\widetilde{f}(\delta) = \lim\limits_{exact} \sum\limits_{k=1}^{n} g\left(\frac{1}{k}\right)$. This function is given by*

$$\begin{cases} \widetilde{f}(0) = \sum\limits_{k=1}^{+\infty} g\left(\frac{1}{k}\right), \\ g(z) = \widetilde{f}(z) - \widetilde{f}\left(\frac{z}{1-z}\right), \quad \textit{in a neighborhood of 0.} \end{cases}$$

Remark 10.26. (Calculating of a finite sum)

If $\widetilde{f}(\delta) = \lim\limits_{\text{exact}} \sum\limits_{k=1}^{n} a_k$, then $\widetilde{f}\left(\dfrac{1}{n}\right) = \sum\limits_{k=1}^{n} a_k$ from a certain rank n_0.

Example 10.27. We have $\lim\limits_{\text{exact}} \sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)} = \dfrac{1}{1+\delta}$. Then, we deduce that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1+\frac{1}{n}}, \quad \forall n \geq 1.$$

10.8. **The Calculation of the Exact Limit of $\sum a_k$.** Let (s_n) be the series defined as $s_n = \sum\limits_{k=1}^{n} g\left(\frac{1}{k}\right)$. Assume that this series is convergent, and g is a metalic function. Then g is holomorphic on a neighborhood of 0. The existence of the exact limit of (s_n) implies that there exists a holomorphic function $\widetilde{f}$ on a neighborhood of 0 and we have

$$g(z) = \widetilde{f}(z) - \widetilde{f}\left(\frac{z}{1-z}\right), \text{ on the disk } D(0,\varepsilon).$$

Let $g(z) = \sum\limits_{n=0}^{+\infty} \beta_n z^n$ and $\widetilde{f}(z) = \sum\limits_{n=0}^{+\infty} \alpha_n z^n$, where $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ are real sequences.
We have

$$\widetilde{f}(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n + o(z^n).$$

Then,

$$\begin{aligned}
\widetilde{f}\left(\frac{z}{1-z}\right) &= \widetilde{f}(z + z^2 + \cdots + z^n + o(z^n)) \\
&= \alpha_0 + \alpha_1(z + \cdots + z^n + o(z^n)) + \cdots \\
&\quad + \alpha_n(z + \cdots + z^n + o(z^n))^n + o(z^n) \\
&= \alpha_0 + \alpha_1 z + (\alpha_1 + \alpha_2)z^2 + (\alpha_1 + 2\alpha_2 + \alpha_3)z^3 \\
&\quad + (\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4)z^4 + (\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + \alpha_5)z^5 \\
&\quad + (\alpha_1 + 5\alpha_2 + 10\alpha_3 + 10\alpha_4 + 5\alpha_5 + \alpha_6)z^6 + \cdots \\
&\quad + \left(\alpha_1 + \binom{n-1}{1}\alpha_2 + \binom{n-1}{2}\alpha_3 + \cdots \right.\\
&\quad \left. + \binom{n-1}{n-2}\alpha_{n-1} + \alpha_n\right) z^n + o(z^n).
\end{aligned}$$

Since $g(z) = \widetilde{f}(z) - \widetilde{f}\left(\frac{z}{1-z}\right)$, we deduce that

$$\begin{cases} \beta_0 = \beta_1 = 0, \\ \beta_k = -\alpha_1 - \binom{k-1}{1}\alpha_2 - \binom{k-1}{2}\alpha_3 - \cdots - \binom{k-1}{k-2}\alpha_{k-1}, \quad \forall 2 \leq k \leq n. \end{cases}$$

Remark 10.28. Since $\beta_0 = \beta_1 = 0$, then $g(z) = z^2 g_1(z)$, where g_1 is a holomorphic function on a neighborhood of 0.

Now, from the above results, we deduce that

$$\begin{cases} \beta_0 = \beta_1 = 0, \\ \beta_2 = -\alpha_1, \\ \beta_3 = -\alpha_1 - 2\alpha_2, \\ \beta_4 = -\alpha_1 - 3\alpha_2 - 3\alpha_3, \\ \quad \vdots \\ \beta_n = -\alpha_1 - (n-1)\alpha_2 - \cdots - \binom{n-1}{k}\alpha_{k+1} - \cdots - \binom{n-1}{n-2}\alpha_{n-1}. \end{cases}$$

Then,

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & \dots & \dots & 0 \\ -1 & -2 & \ddots & & \vdots \\ -1 & -3 & -3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -1 & -(n-1) & \dots & \dots & -(n-1) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

Consider the matrix defined as

$$M_n = \begin{pmatrix} -1 & 0 & \dots & \dots & 0 \\ -1 & -2 & \ddots & & \vdots \\ -1 & -3 & -3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -1 & -n & \dots & \dots & -n \end{pmatrix}.$$

We have $\det(M_n) = (-1)^n n!$, then M_n is invertible and we have:

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix} = M_{n-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

So, the above system admits a unique solution $(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$.
If $\limsup \sqrt[n]{\mid \alpha_n \mid} = \dfrac{1}{R} > 0$, then the function $\widetilde{f}(z) = \sum_{n=0}^{+\infty} \alpha_n z^n$ is holomorphic on the disk $D(0, R)$.

In this case, the exact limit $\lim_{\text{exact}} \sum_{k=1}^{n} g\left(\frac{1}{k}\right)$ exists. In addition, we have

$$\widetilde{f}(\delta) = \lim_{\text{exact}} \sum_{k=1}^{n} g\left(\frac{1}{k}\right) = \lim_{\text{exact}} \sum_{k=1}^{n} a_k,$$

and we get

$$\widetilde{f}(0) = \left(\lim_{\text{exact}} \sum_{k=1}^{n} a_k\right)^* = \sum_{k=1}^{+\infty} a_k.$$

11. The Black Magic Matrix

11.1. The Calculation of the Exact Limit Using the Black Magic Matrix. Let g be a metalic function and $\widetilde{f}$ be a holomorphic function in a neighborhood of 0. Assume that the series $\sum_{k=1}^{n} g(\frac{1}{k})$ admits the exact limit $\widetilde{f}(\delta)$. Let $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ be two real sequences such that

$$\widetilde{f}(z) = \alpha_0 + \sum_{k=1}^{+\infty} \alpha_k z^k, \qquad g(z) = \sum_{k=0}^{+\infty} \beta_k z^k.$$

We have

$$\lim_{\text{exact}} \sum_{k=1}^{n} g\left(\frac{1}{k}\right) = \widetilde{f}(\delta).$$

Then,

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix} = M_{n-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

Definition 11.1. The black magic matrix of order n is defined as $\psi^{(n)} = M_n^{-1}$.

We obtain

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix}.$$

The real α_0 is given by $\alpha_0 = \widetilde{f}(0) = \sum_{k=1}^{+\infty} g\left(\frac{1}{k}\right)$.

Remark 11.2. We can verify that

$$\sum_{k=1}^{m} g\left(\frac{1}{k}\right) = \alpha_0 + \lim_{n\to+\infty} \begin{pmatrix} \frac{1}{m} & \frac{1}{m^2} & \frac{1}{m^3} & \cdots & \frac{1}{m^{n-1}} \end{pmatrix} \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix},$$

from a certain rank m_0.

11.2. **The Magical Properties of $\boldsymbol{\psi^{(n)}}$.**

Property 11.3. The matrix $\psi^{(n)}$ is given by $\psi^{(n)} = M_n^{-1}$,

where $M_n[i,j] = \begin{cases} -\binom{i}{j-1}, & \text{if } 1 \le j-1 \le i \le n, \\ 0, & \text{otherwise.} \end{cases}$

We deduce that the matrix $\psi^{(n)}$ is invertible and it is a lower triangular matrix.

Property 11.4. We have $\psi^{(n)}_{i,i} = \dfrac{-1}{i}$. Then, the determinant of $\psi^{(n)}$ is given by $\det(\psi^{(n)}) = \frac{(-1)^n}{n!}$ and we have $\mathrm{tr}(\psi^{(n)}) = -H(n)$, where $(H(n))_{n\ge 1}$ is the harmonic series which is defined as $H(n) = \displaystyle\sum_{i=1}^{n} \frac{1}{i}$.

Proof. The matrix M_n is lower triangular and we have

$$Sp(M_n) = \{-i, \text{ for } 1 \le i \le n\}.$$

Then, $\psi^{(n)}$ is lower triangular, and we get

$$Sp(\psi^{(n)}) = \left\{\frac{-1}{i}, \text{ for } 1 \le i \le n\right\}.$$

□

Property 11.5. For every $1 \le i \le n-1$, we have

$$\psi^{(n)}_{i+1,i} = \frac{1}{2}.$$

Proof. We have

$$\delta_{i+1,i} = \sum_{k=1}^{n} M_n[i+1,k]\psi^{n}_{k,i}.$$

Then,

$$\sum_{k=i}^{i+1} M_n[i+1,k]\psi^{(n)}_{k,i} = 0.$$

So, we deduce that

$$\psi^{(n)}_{i+1,i} = -\frac{M_n[i+1,i]\psi^{(n)}_{i,i}}{M_n[i+1,i+1]},$$

$$\psi^{(n)}_{i+1,i} = -\frac{\binom{i+1}{i-1}\psi^{(n)}_{i,i}}{\binom{i+1}{i}}.$$

Finally, we obtain

$$\psi^{(n)}_{i+1,i} = \frac{1}{2}.$$

□

Property 11.6. For every $(m,p) \in \mathbb{N}^2$, such that $2 \leq m$, and $2m+p \leq n$, we have

$$\psi^{(n)}_{2m+p,1+p} = 0.$$

In particular, for every $2 \leq m \leq \frac{n}{2}$, we get

$$\psi^{(n)}_{n,n-2m+1} = 0.$$

Proof. We can see the demonstration in the following. □

Property 11.7. For every $1 \leq m \leq n-1$, we have

$$\psi^{(n)}_{m,m}\psi^{(n)}_{m+1,m-1} = \frac{1}{12}.$$

Then,

$$\psi^{(n)}_{m+1,m-1} = \frac{-m}{12}.$$

Proof. We have $\psi^{(n)}M_n = I_n$. Then,

$$\sum_{k=1}^{n} \psi^{(n)}_{i,k} M_n[k,j] = \delta_{ij}.$$

In particular,

$$\sum_{k=1}^{n} \psi^{(n)}_{m+1,k} M_n[k,m-1] = \delta_{m+1,m-1}.$$

Then,

$$\sum_{k=m-1}^{m+1} \psi^{(n)}_{m+1,k} M_n[k,m-1] = 0,$$

which implies that

$$\psi^{(n)}_{m+1,m-1}M_n[m-1,m-1]+\psi^{(n)}_{m+1,m}M_n[m,m-1]+\psi^{(n)}_{m+1,m+1}M_n[m+1,m-1] = 0.$$

Then,

$$-(m-1)\psi^{(n)}_{m+1,m-1} - \frac{m(m-1)}{4} + \frac{m(m-1)}{6} = 0.$$

Finally, we get

$$\psi^{(n)}_{m+1,m-1} = \frac{-m}{12}.$$

□

Property 11.8. For every $(i,j) \in \mathbb{N}^2$ such that $1 \le i,j \le n$, we have

$$\psi^{(n+1)}_{i,j} = \psi^{(n)}_{i,j}.$$

Proof. From the definition of M_n, we have

$$M_{n+1} = \begin{pmatrix} M_n & 0 \\ X_n & -n-1 \end{pmatrix},$$

where $X_n = -\left(\binom{n+1}{0}, \binom{n+1}{1}, \ldots, \binom{n+1}{n-1}\right)$. To prove $\psi^{(n+1)}_{i,j} = \psi^{(n)}_{i,j}$, it is sufficient to show that there exists a row vector Y_n such that

$$\psi^{(n+1)} = \begin{pmatrix} \psi^{(n)} & 0 \\ Y_n & \frac{-1}{n+1} \end{pmatrix}.$$

On the other hand, we have

$$M_{n+1}\psi^{(n+1)} = I_{n+1},$$

then,

$$\begin{pmatrix} M_n & 0 \\ X_n & -n-1 \end{pmatrix} \begin{pmatrix} \psi^{(n)} & 0 \\ Y_n & \frac{-1}{n+1} \end{pmatrix} = I_{n+1},$$

which implies that,

$$\begin{pmatrix} M_n\psi^{(n)} & 0 \\ X_n\psi^{(n)} - (n+1)Y_n & 1 \end{pmatrix} = I_{n+1}.$$

Finally, we deduce that $X_n\psi^{(n)} - (n+1)Y_n = 0$. Then, we can choose Y_n as the form $Y_n = \dfrac{X_n\psi^{(n)}}{n+1}$, and we get

$$\psi^{(n+1)} = \begin{pmatrix} \psi^{(n)} & 0 \\ \dfrac{1}{n+1}X_n\psi_n & -\frac{1}{n+1} \end{pmatrix}.$$

Finally, we deduce that $\psi^{(n+1)}_{i,j} = \psi^{(n)}_{i,j}$, for every $1 \le i,j \le n$.

□

Remark 11.9. From Property 11.8, we deduce that $\psi^{(i)}_{i,j} = \psi^{(n)}_{i,j}$, for every $1 \le i,j \le n$. We set $\psi^{(n)}_{i,j} = \psi_{i,j}$.

Property 11.10. For every $1 < i \leq n$, we have

$$\sum_{k=1}^{n} \psi_{i,k} = 0, \qquad \sum_{k=1}^{n} \psi_{1,k} = -1.$$

Then,

$$\sum_{i=1}^{n} C_i = C_1 + C_2 + \cdots + C_n = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $C_1, C_2, \ldots, C_n$ are the column vectors of the matrix $\psi^{(n)}$.

Proof. We know that $\psi_n M_n = I_n$, then $\sum_{k=1}^{n} \psi_n[i,k] M_n[k,1] = \delta_{i1}$.
So, we deduce that

$$\begin{cases} \sum_{k=1}^{n} \psi_{1,k} M_n[k,1] = 1, \\ \\ \sum_{k=1}^{n} \psi_{i,k} M_n[k,1] = 0, \quad \text{if } i \neq 1. \end{cases}$$

Then,

$$\begin{cases} \sum_{k=1}^{n} \psi_{1,k} = -1, \\ \\ \sum_{k=1}^{n} \psi_{i,k} = 0, \qquad \text{if } i \neq 1. \end{cases}$$

□

Property 11.11. For every $1 \leq i \leq n$, we have

$$\sum_{k=1}^{n} (-1)^k \psi_{i,k} = (-1)^{i+1},$$

which implies,

$$\sum_{i=1}^{n} (-1)^{i-1} C_i = C_1 - C_2 + \cdots + (-1)^{n-1} C_n = \begin{pmatrix} -1 \\ 1 \\ \vdots \\ (-1)^n \end{pmatrix},$$

where $C_1, C_2, \ldots, C_n$ are the column vectors of the matrix $\psi^{(n)}$.

Proof. From Example 10.27, we have

$$\lim_{\text{exact}} \sum_{k=1}^{n} \frac{-1}{k(k+1)} = \frac{-1}{1+\delta}.$$

Then $\lim_{\text{exact}} \sum_{k=1}^{n} g(\frac{1}{k}) = \widetilde{f}(\delta)$ for $g(z) = \frac{-z^2}{1+z} = \sum_{k=2}^{+\infty} (-1)^k z^k$ and we have

$$\widetilde{f}(z) = \frac{-1}{1+z} = \sum_{k=0}^{+\infty} (-1)^{k+1} z^k.$$

By using Property 11.12, we deduce that

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{n+1} \end{pmatrix} = \psi^{(n)} \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^n \end{pmatrix},$$

finally, we deduce that

$$\sum_{k=1}^{n} (-1)^k \psi_{i,k} = (-1)^{i+1}.$$

□

Property 11.12. Let g be a metalic function such that $g(z) = \sum_{k=0}^{+\infty} \beta_k z^k$ on a neighborhood of 0. Assume that the series $\sum_{k=1}^{n} g(\frac{1}{k})$ is convergent and admits the exact limit. Then, there exist a holomorphic function $\widetilde{f}$ on a neighborhood of 0 and a real sequence $(\alpha_n)_{n\geq 0}$ such that $\lim_{\text{exact}} \sum_{k=1}^{n} g\left(\frac{1}{k}\right) = \widetilde{f}(\delta)$ and $\widetilde{f}(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$ on a neighborhood of 0. The real sequence $(\alpha_n)_{n\geq 0}$ is given by

$$\alpha_0 = \sum_{k=1}^{+\infty} g\left(\frac{1}{k}\right), \qquad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix},$$

and we have $\beta_0 = \beta_1 = 0$.

Example 11.13. We have

$$(n=2) \quad \psi^{(2)} = \begin{pmatrix} -1 & 0 \\ 1/2 & -1/2 \end{pmatrix},$$

$$(n=3) \quad \psi^{(3)} = \begin{pmatrix} -1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -1/6 & 1/2 & -1/3 \end{pmatrix},$$

$$(n=5) \quad \psi^{(5)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 \end{pmatrix},$$

$$(n=8) \quad \psi^{(8)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 \end{pmatrix}.$$

Finally, for $(n=11)$, $\psi^{(11)}$ is given by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 & 0 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 & 0 & 0 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 & 0 & 0 & 0 \\ 1/30 & 0 & -2/9 & 0 & 7/15 & 0 & -2/3 & 1/2 & -1/9 & 0 & 0 \\ 0 & 3/20 & 0 & -1/2 & 0 & 7/10 & 0 & -3/4 & 1/2 & -1/10 & 0 \\ -5/66 & 0 & 1/2 & 0 & -1 & 0 & 1 & 0 & -5/6 & 1/2 & -1/11 \end{pmatrix}.$$

Theorem 11.14. *(Calculation of the coefficients of $(\psi_{i,j})$ by induction) For every $1 \leq j \leq n$, we have*

$$\psi_{n+1,j} = \frac{X_n}{n+1} \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \vdots \\ \psi_{n,j} \end{pmatrix},$$

where $X_n = -\left(\binom{n+1}{0}, \binom{n+1}{1}, \ldots, \binom{n+1}{n-1}\right)$, and we have $\psi_{n+1,n+1} = \dfrac{-1}{n+1}$.

Proof. There exists a row vector $Y_n = (y_1, y_2, \ldots, y_n)$, such that

$$\psi^{(n+1)} = \begin{pmatrix} \psi^{(n)} & 0 \\ Y_n & -\frac{1}{n+1} \end{pmatrix}.$$

On the other hand, $Y_n = \frac{X_n \psi^{(n)}}{n+1}$, then

$$y_j = \psi_{n+1,j} = Y_n e_j = \frac{X_n}{n+1}\psi^{(n)} e_j = \frac{X_n}{n+1}\begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \vdots \\ \psi_{n,j} \end{pmatrix},$$

where $(e_1, e_2, \ldots, e_n)$ is the canonical base of $\mathbb{R}^n$.

□

Remark 11.15. For every $n \geq 1$, we have

$$X_n = (0, X_{n-1}) + (X_{n-1}, -n).$$

11.3. **The Relationship Between $\psi_{i,j}$ and the Bernoulli Numbers.** The Bernoulli numbers are defined as

$$\begin{cases} B_0 = 1, \\ B_0 + 2B_1 = 0, \\ B_0 + 3B_1 + 3B_2 = 0, \\ B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \\ \vdots \\ B_0 + \binom{n}{1} B_1 + \cdots + \binom{n}{n-1} B_{n-1} = 0. \end{cases}$$

Then,

$$M_n \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, we deduce that

$$\psi^{(n)} \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{pmatrix}.$$

Finally, we get the following result.

Property 11.16. For every natural number $k \geq 1$, we have

$$\psi_{k,1} = -B_{k-1}.$$

Then, the first column of $\psi^{(n)}$ is given by

$$\begin{pmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \vdots \\ \psi_{n,1} \end{pmatrix} = - \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{pmatrix}.$$

Remark 11.17. We deduce from Property 11.6 that $\psi_{2k+2,1} = 0$ for every natural number $k \geq 1$, because $B_{2k+1} = 0$.

Property 11.18. For every $k \in \mathbb{N}$ and $s \in \mathbb{N}^*$, we have

$$\psi_{k+s,s} = -\frac{B_k}{k!} \prod_{i=1}^{k-1} (s+i).$$

Proof. Let $\begin{pmatrix} 0 \\ \vdots \\ x_s \\ \vdots \\ x_n \end{pmatrix}$ be the s^{th} column of the matrix $\psi^{(n)}$. To find this column it is sufficient to determine the values of the real numbers (x_i) which verify

$$-\left(\binom{j}{0}, \binom{j}{1}, \binom{j}{j-1}, 0, \dots, 0\right) \begin{pmatrix} 0 \\ \vdots \\ x_s \\ \vdots \\ x_n \end{pmatrix} = \delta_{s,j}.$$

On the other hand, we know that $B_{2k+1} = 0$ for every natural number $k \geq 1$. To prove $\psi_{s+2k+1,s} = 0$, it is sufficient to show that the above column has the following form

$$\begin{pmatrix} 0 \\ \vdots \\ x_s \\ x_{s+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \alpha_s B_0 \\ \alpha_{s+1} B_1 \\ \vdots \\ \alpha_n B_{n-s} \end{pmatrix},$$

where $\alpha_s, \alpha_{s+1}, \dots, \alpha_n$ are real numbers. Then

$$\left(\binom{j}{0}, \binom{j}{1}, \binom{j}{j-1}, 0, \dots, 0\right) \begin{pmatrix} 0 \\ \vdots \\ \alpha_s B_0 \\ \vdots \\ \alpha_n B_{n-s} \end{pmatrix} = -\delta_{s,j}.$$

If $s \neq j$ and $s > j$, then this product is zero. If $s < j$, we find $\alpha_s B_0 \binom{j}{s-1} + \alpha_{s+1} B_1 \binom{j}{s} + \cdots + \alpha_j B_{j-s} \binom{j}{j-1} = 0$.
From Property 11.16, we know that

$$\binom{j-s+1}{0} B_0 + \binom{j-s+1}{1} B_1 + \cdots + \binom{j-s+1}{j-s} B_{j-s} = 0.$$

To find the sequence of the real numbers $(\alpha_i)_{i\geq 0}$, it is sufficient to determine $\lambda \in \mathbb{R}$, such that

$$\alpha_{s+k} B_k \binom{j}{s+k-1} = \lambda \binom{j-s+1}{k} B_k, \quad \text{for } s+k \leq j.$$

Then,

$$\alpha_{s+k} = \lambda \frac{\binom{j-s+1}{k}}{\binom{j}{s+k-1}},$$

$$\alpha_{s+k} = \frac{\lambda}{k!} \frac{(s+k-1)!(j-s+1)!}{j!}.$$

In the case of $k = 0$, we get

$$\alpha_s = \lambda \frac{(s-1)!(j-s+1)!}{j!}.$$

Then,

$$\lambda = \frac{j!\alpha_s}{(s-1)!(j-s+1)!}.$$

On the other hand, we know that $B_0 = 1$, then $\alpha_s = \frac{-1}{s}$ and we get

$$\lambda = -\frac{j!}{s!(j-s+1)!}.$$

We replace λ by its value, and define the real α_{s+k} as

$$\begin{aligned} \alpha_{s+k} &= -\frac{(s+k-1)!}{s!k!} \\ &= -\frac{1}{k!}(s+1)(s+2)\cdots(s+k-1). \end{aligned}$$

Finally, we deduce that

$$\psi_{k+s,s} = -\frac{B_k}{k!}(s+1)(s+2)\cdots(s+k-1).$$

□

Corollary 11.19. *For every* $(l,m) \in \mathbb{N}^2$ *such that* $m \geq 2$ *and* $2m \leq l$, *we have*

$$\psi_{l,l-2m+1} = 0.$$

Remark 11.20. From the above results, we deduce that the Property 11.6 is true, and we can show the following theorem.

Theorem 11.21. *The coefficients* $\psi^{(n)} = (\psi_{i,j})_{1\leq i,j\leq n}$ *are given by*

$$\psi_{i,j} = \begin{cases} -\frac{\binom{i}{j}B_{i-j}}{i}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

11.4. **The Black Magic Matrix with the Riemann Zeta Function.**

11.4.1. *The Radius of Convergence of* $\sum_{k=1}^{+\infty} \psi_{k,s-1} z^k$.

Lemma 11.22. *The radius of convergence of the series*

$$\sum_{k=1}^{+\infty} \psi_{k,s-1} z^k,$$

is zero.

Proof. The radius of convergence of the series $\sum_{k=1}^{+\infty} \psi_{k,s-1} z^k$ is given by

$$\frac{1}{R} = \limsup_{k\to+\infty} \sqrt[k]{\mid \psi_{k,s-1} \mid}.$$

We have,

$$\begin{aligned} \psi_{k,s-1} &= -\frac{B_{k-s+1}}{(k-s+1)!} s(s+1)(s+2)\cdots(k-1) \\ &= -\frac{B_{k-s+1}}{(k-s+1)!} \frac{(k-1)!}{(s-1)!}. \end{aligned}$$

For $k = 2m + s - 1$, we get

$$\psi_{2m+s-1,s-1} = -\frac{B_{2m}}{(s-1)!} \frac{(2m+s-2)!}{(2m)!}.$$

Since $2m! \sim \sqrt{4\pi m}(\frac{2m}{e})^{2m}$ *and* $\mid B_{2m} \mid \sim 4\sqrt{\pi m}(\frac{m}{\pi e})^{2m}$, then

$$\frac{\mid B_{2m} \mid}{2m!} \sim 2(\frac{1}{2\pi})^{2m}.$$

So, we deduce that

$$\sqrt[2m+s-1]{\frac{\mid B_{2m} \mid}{2m!}} \sim \frac{1}{2\pi}.$$

On the other hand, we have

$$(2m+s-1)! \sim \sqrt{2\pi(2m+s-1)}\left(\frac{2m+s-1}{e}\right)^{2m+s-1}.$$

Then,

$$\frac{(2m+s-1)!}{(s-1)!} \sim \frac{\sqrt{2\pi(2m+s-1)}}{(s-1)!}\left(\frac{2m+s-1}{e}\right)^{2m+s-1}.$$

So, we deduce that

$$\sqrt[2m+s-1]{\frac{(2m+s-1)!}{(s-1)!}} \sim (2m+s-1)^{\frac{1}{4m+2s-2}}\left(\frac{2m+s-1}{e}\right),$$

and

$$\sqrt[2m+s-1]{\frac{(2m+s-1)!}{(s-1)!}} \sim e^{\frac{1}{4m+2s-2}\ln(2m+s-1)}\left(\frac{2m+s-1}{e}\right).$$

Finally, we get

$$\sqrt[2m+s-1]{\mid \psi_{2m+s-1,s-1} \mid} \sim \frac{1}{2\pi}e^{\frac{1}{4m+2s-2}\ln(2m+s-1)}\left(\frac{2m+s-1}{e}\right),$$

$$\sqrt[2m+s-1]{\mid \psi_{2m+s-1,s-1} \mid} \sim \frac{m}{\pi e}.$$

Then $\lim\limits_{m\to+\infty} \sqrt[2m+s-1]{\mid \psi_{2m+s-1,s-1} \mid} = +\infty$, which implies that

$$\limsup_{k\to+\infty} \sqrt[k]{\mid \psi_{k,s-1} \mid} = +\infty.$$

So, we deduce that the radius of convergence of the series $\sum\limits_{k=1}^{+\infty} \psi_{k,s-1} z^k$, is zero. □

11.4.2. *The Riemann Zeta Function.* This section is concerned the Euler zeta series, which is the function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},$$

where s is a real number greater than 1.
For $s \in \mathbb{N} \setminus \{0, 1\}$, consider the real function $g : x \longrightarrow x^s$, and the series

$$s_N = \sum_{k=1}^{N} g\left(\frac{1}{k}\right).$$

Theorem 11.23. *The series* $s_N = \sum_{k=1}^{N} \frac{1}{k^s}$ *does not admit a exact limit.*

Proof. We assume that the series (s_n) admits the exact limit, then there exists a holomorphic function $\widetilde{f}$ on a neighborhood of 0 such that $\widetilde{f}\left(\frac{1}{N}\right) = s_N$ from a certain rank.

If $\widetilde{f}(z) = \sum\limits_{k=0}^{+\infty} \alpha_k z^k$, then

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix} = \psi^{(n)} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \psi^{(n)} e_{s-1}.$$

Then,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \psi_{1,s-1} \\ \psi_{2,s-1} \\ \vdots \\ \psi_{n,s-1} \end{pmatrix},$$

and $\alpha_0 = \sum_{k=1}^{+\infty} \frac{1}{k^s} = \zeta(s)$. Then,

$$\widetilde{f}(z) = \zeta(s) + \sum_{k=1}^{+\infty} \psi_{k,s-1} z^k.$$

Since $\lim\limits_{\text{exact}} \sum_{n=1}^{N} \frac{1}{n^s} = \widetilde{f}(\delta)$, then,

$$\begin{aligned} \widetilde{f}\left(\frac{1}{N}\right) &= \sum_{n=1}^{N} \frac{1}{n^s} \\ &= \zeta(s) + \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k}, \end{aligned}$$

from a certain rank N. Then,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k}.$$

The above result is not true, since the function $\widetilde{f}$ is not holomorphic on a neighborhood of 0 by Lemma 11.22. Then, we reached to obtain a contradiction and we deduce that the series (s_n) does not admit an exact limit.

□

11.5. **The Twelfth Property of the Matrix $\psi^{(n)}$.** From the above results, the formula $\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k}$ is false, but we can correct this equality by adding a new term $E(M, N, s)$ which is defined as

$$E(M, N, s) = \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k}.$$

In fact, we have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s),$$

which implies that

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{k=s-1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s) \\
&= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{\psi_{s-1,s-1}}{N^{s-1}} - \frac{\psi_{s,s-1}}{N^s} \\
&\quad - \frac{\psi_{s+1,s-1}}{N^{s+1}} - \sum_{k=s+2}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} - \frac{1}{2}\frac{1}{N^s} - \sum_{k=s+1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s) \\
&= \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} + \frac{1}{2}\frac{1}{N^s} - \sum_{k=s+1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).
\end{aligned}$$

For $r = k - s$, we obtain

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} + \frac{1}{2}\frac{1}{N^s} - \sum_{r=1}^{2M-1} \frac{\psi_{r+s,s-1}}{N^{r+s}} + E(M,N,s).$$

Then,

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} + \frac{1}{2}\frac{1}{N^s} + \sum_{r=1}^{2M-1} \frac{\binom{r+s}{s-1}B_{r+1}}{(r+s)N^{r+s}} + E(M,N,s).$$

On the other hand, we have $B_{2k+1} = 0$ for every natural $k \geq 1$, then

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} + \frac{1}{2}\frac{1}{N^s} + \sum_{m=1}^{M} \frac{\binom{2m+s-1}{s-1}B_{2m}}{(2m+s-1)N^{2m+s-1}} + E(M,N,s).$$

So, we get

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} + \frac{1}{2}\frac{1}{N^s} + \sum_{m=1}^{M} \prod_{i=0}^{2m-1}(s+i) \frac{B_{2m}}{(2m)!N^{2m+s-1}} + E(M,N,s).$$

Finally, we find the standard Euler-Maclaurin formula [6] applied to the zeta function $\zeta(s)$, where s is a natural number and $s \geq 2$. Then we deduce that the matrix of the black magic $\psi^{(n)}$ has a beautiful twelfth property which is given as follows.

Property 11.24. By using the black magic matrix, we can represent the Euler-maclaurin formula as

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \left\langle C_{s-1}, \widetilde{X}_{M,N,s} \right\rangle + E(M,N,s),$$

where

- $\langle .,. \rangle$ is the scalar product $\langle x, y \rangle = \sum x_i y_i$.
- $C_{s-1} = \psi^{(2M+s)} e_{s-1}$ is the (s-1)-th column of the matrix $\psi^{(2M+s)}$.
- $\widetilde{X}_{M,N,s}$ is the column vector defined as

$$\widetilde{X}_{M,N,s} = \begin{pmatrix} \frac{1}{N} \\ \frac{1}{N^2} \\ \vdots \\ \frac{1}{N^{2M+s}} \end{pmatrix}.$$

Example 11.25. We have

$$\psi^{(10)} = \begin{array}{cccccccccc} \zeta(2) & \zeta(3) & & \zeta(5) & & & & \zeta(9) & & \zeta(11) \end{array}$$
$$\psi^{(10)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 & 0 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 & 0 & 0 \\ 1/30 & 0 & -2/9 & 0 & 7/15 & 0 & -2/3 & 1/2 & -1/9 & 0 \\ 0 & 3/20 & 0 & -1/2 & 0 & 7/10 & 0 & -3/4 & 1/2 & -1/10 \end{pmatrix}.$$

We have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M,N,s).$$

The coefficients of the first column of $\psi^{(10)}$ are

$$-1, 1/2, -1/6, 0, 1/30, 0, -1/42, 0, 1/30, 0,$$

then,

$$\zeta(2) = \sum_{n=1}^{N} \frac{1}{n^2} + \frac{1}{N} - \frac{1}{2}\frac{1}{N^2} + \frac{1}{6}\frac{1}{N^3} - \frac{1}{30}\frac{1}{N^5} + \frac{1}{42}\frac{1}{N^7} - \frac{1}{30}\frac{1}{N^9} + E(4,N,2).$$

Similarly, we deduce the following formulas

$$\zeta(3) = \sum_{n=1}^{N} \frac{1}{n^3} + \frac{1}{2}\frac{1}{N^2} - \frac{1}{2}\frac{1}{N^3} + \frac{1}{4}\frac{1}{N^4} - \frac{1}{12}\frac{1}{N^6} + \frac{1}{12}\frac{1}{N^8} - \frac{3}{20}\frac{1}{N^{10}} + E(4,N,3),$$

$$\zeta(5) = \sum_{n=1}^{N} \frac{1}{n^5} + \frac{1}{4}\frac{1}{N^4} - \frac{1}{2}\frac{1}{N^5} + \frac{5}{12}\frac{1}{N^6} - \frac{7}{24}\frac{1}{N^8} + \frac{1}{2}\frac{1}{N^{10}} + E(3,N,5),$$

$$\zeta(9) = \sum_{n=1}^{N} \frac{1}{n^9} + \frac{1}{8}\frac{1}{N^8} - \frac{1}{2}\frac{1}{N^9} + \frac{3}{4}\frac{1}{N^{10}} + E(1,N,9).$$

12. The Relationship Between the Hyperreal Numbers and the Omicran-reals

Let u be an element of Δ, ${}^*\mathbb{R}$ be the field of hyperreal numbers and $u(\delta)$ be the Omicran defined by the sequence $\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}$. We use the symbol $\langle u(\frac{1}{i})\rangle$ to represent the hyperreal defined by the sequence $\left(u\left(\frac{1}{n}\right)\right)_{n\geq 1}$.

The map defined as

$$\begin{array}{rrcl} \iota: & \mathcal{O} & \longrightarrow & {}^*\mathbb{R}, \\ & u(\delta) & \longmapsto & \left\langle u(\frac{1}{i})\right\rangle, \end{array}$$

is a ring homomorphism. In addition, we have the following results:

- The map ι is injective, in fact, if $\iota(u(\delta)) = 0$ then $\left\langle u(\frac{1}{i})\right\rangle = 0$. We deduce that $\left\{i : u(\frac{1}{i}) = 0\right\} \in \mathcal{U}$. Then $u(\frac{1}{i})$ is zero for an infinity of indices i. From the properties of u as an element of Δ, we deduce that $\widetilde{u} = 0$. Finally $u(\delta) = 0$.
- From the above result, we deduce that the field $\mathcal{O}$ is isomorphic to a subfield of ${}^*\mathbb{R}$. More precisely, we have
$$\mathcal{O} \approx \iota(\mathcal{O}) \subseteq^* \mathbb{R}.$$
- The total order relation defined on ${}^*\mathbb{R}$ extends the total order relation defined on $\mathcal{O}$. In fact,
  (i) if $u(\delta) \leq v(\delta)$ then there exists n_0 such that $u(\frac{1}{i}) \leq v(\frac{1}{i})$, which implies that $\left\{i : u(\frac{1}{i}) \leq v(\frac{1}{i})\right\} \in \mathcal{U}$ (because, the finite sets are not elements of $\mathcal{U}$).
  Finally, we deduce that $\left\langle u\left(\frac{1}{i}\right)\right\rangle \leq \left\langle v\left(\frac{1}{i}\right)\right\rangle$.
  (ii) conversely, if $\left\langle u\left(\frac{1}{i}\right)\right\rangle \leq \left\langle v\left(\frac{1}{i}\right)\right\rangle$, then $\left\{i : u(\frac{1}{i}) \leq v(\frac{1}{i})\right\} \in \mathcal{U}$. From the properties of the elements of Δ, we deduce that there exists n_0 such that $u\left(\frac{1}{i}\right) \leq v\left(\frac{1}{i}\right)$ for every $i \geq n_0$. Finally, we get
$$u(\delta) \leq v(\delta) \quad \Leftrightarrow \quad \left\{i : u\left(\frac{1}{i}\right) \leq v\left(\frac{1}{i}\right)\right\} \in \mathcal{U}.$$

From the above results, we can justify the identification of the field of Omicran-reals $\mathcal{O}$ by a strict subset of the field of hyperreals, and we deduce that:

"Any property that is true for every hyperreal number is also true for every Omicran."

13. Concluding Remark

According to Robinson's approach, the construction of the hyperreal numbers is related to the existence of an ultrafilter with special properties. Within this ultrafilter we can find the element A such that the cardinal of the set $A \cap \{1, 2, \ldots, n\}$ is very small compared to the cardinal of $A^c \cap \{1, 2, \ldots, n\}$, from a certain rank n_0. Unfortunately, this property is not useful enough to obtain an effective approach in practice. In this work, we have proposed an explicit approach without using the ultrafilters and without adding any axiom. We have come up with new notions used to obtain more applications thanks to this new method. Finally, we believe that the new method becomes more usable for many researchers in all fields of mathematics not only for the specialists in model theory and mathematical logic.

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