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Duals of Some Constructed $*$-Frames by Equivalent $*$-Frames

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Abstract. Hilbert frames theory have been extended to frames in Hilbert C^*-modules. The paper introduces equivalent $*$-frames and presents ordinary duals of a constructed $*$-frame by an adjointable and invertible operator. Also, some necessary and sufficient conditions are studied such that $*$-frames and ordinary duals or operator duals of another $*$-frames are equivalent under these conditions. We obtain a $*$-frame by an orthogonal projection and a given $*$-frame, characterize its duals, and give a bilateral condition for commutating frame operator of a primary $*$-frame and an orthogonal projection. At the end of paper, pre-frame operator of a dual frame is computed by pre-frame operator of a general $*$-frame and an orthogonal projection.

1. Introduction

Dual frames and operators corresponding to a frame play an important role in decomposition of vectors in a Hilbert space or a Hilbert C^*-module. In this paper, we consider some properties about dual frames and operators of a $*$-frame in a Hilbert C^*-module.

The manuscript contains the results that hold for classical frames, but we present them for the largest family of frames, $*$-frames. However, the proving methods applied in the manuscript are operator-theorical methods and are not involving boundary $\mathcal{A}$-valued.

Throughout the paper, we fix the notations $\mathcal{A}$ and J for a unital C^*-algebra and a finite or countably infinite index set, respectively. Also, the set $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module.

Some definitions of Hilbert modules and frames and their properties are recalled in the following.

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Suppose $\mathcal{A}$ is a C^*-algebra. A linear space $\mathcal{H}$ which is also an algebraic left $\mathcal{A}$-module together with an $\mathcal{A}$-inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{A}$ is called a left pre-Hilbert C^*-module if it satisfies the following properties.

(i) ${}_{\mathcal{A}}\langle f, f\rangle \geq 0$, for any $f \in \mathcal{H}$.
(ii) ${}_{\mathcal{A}}\langle f, f\rangle = 0$ if and only if $f = 0$.
(iii) ${}_{\mathcal{A}}\langle f, g\rangle = {}_{\mathcal{A}}\langle g, f\rangle^*$, for any $f, g \in \mathcal{H}$.
(iv) ${}_{\mathcal{A}}\langle \lambda f, h\rangle = \lambda {}_{\mathcal{A}}\langle f, h\rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$.
(v) ${}_{\mathcal{A}}\langle af + bg, h\rangle = a {}_{\mathcal{A}}\langle f, h\rangle + b {}_{\mathcal{A}}\langle g, h\rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

If $\mathcal{H}$ is a Banach space with respect to the induced norm by the $\mathcal{A}$-valued inner product $\|\cdot\| = \sqrt{\|{}_{\mathcal{A}}\langle\cdot,\cdot\rangle\|_{\mathcal{A}}}$, then $(\mathcal{H}, {}_{\mathcal{A}}\langle\cdot,\cdot\rangle)$ or ${}_{\mathcal{A}}\mathcal{H}$ is called a left Hilbert C^*-module over $\mathcal{A}$ or, simply, a left Hilbert $\mathcal{A}$-module. Similarly, a right Hilbert C^*-module has been defined.

The notion of frames for Hilbert spaces had been extended by Frank-Larson [5] to the notion of frames in Hilbert C^*-modules. Then Alijani-Dehghan [2] considered frames in Hilbert $\mathcal{A}$-modules with $\mathcal{A}$-valued bounds as a countable family $\{f_j\}_{j\in J}$ of a Hilbert $\mathcal{A}$-module $\mathcal{H}$ satisfying

$$A\langle f, f\rangle \mathcal{A}^* \leq \sum_{j\in J} \langle f, f_j\rangle \langle f_j, f\rangle \leq B\langle f, f\rangle B^*, \tag{1.1}$$

for all $f \in \mathcal{H}$ and strictly nonzero elements A and B in the $\mathcal{A}$ independent of f. A frame with $\mathcal{A}$-valued bounds in a Hilbert $\mathcal{A}$-module is called to be a $*$-frame. If $A = B = 1_{\mathcal{A}}$, then the $*$-frame is called a Parseval frame (or Parseval $*$-frame). Also, their operators are introduced and studied in [2]. If the sum in the middle of the inequalities (1.1) is convergence in the norm of $\mathcal{H}$, then the $*$-frame is called a standard $*$-frame. In this paper, all $*$-frames are standard but "standard" is omitted for rest.

Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with $\mathcal{A}$-valued bounds A and B. The pre-frame operator $\theta_{\mathcal{F}} : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ defined by $\theta_{\mathcal{F}}(f) = \{\langle f, f_j\rangle\}_{j\in J}$ is an injective and closed range adjointable $\mathcal{A}$-module map and $\|\theta_{\mathcal{F}}\| \leq \|B\|$. Also, the frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$Sf = \sum_{j\in J} \langle f, f_j\rangle f_j,$$

that is positive, invertible and adjointable, and the reconstruction formula $f = \sum_{j\in J} \langle f, S^{-1} f_j\rangle f_j$ holds for all $f \in \mathcal{H}$ [2].

Every element of a Hilbert space (or a Hilbert C^*-module) has a decomposition with respect to every its frame. Dual frames are important in this decomposition because coefficients are given from a dual frame.

In [1, 4], the authors extended dual frames to generalized dual frames in Hilbert spaces and Hilbert C^*-modules.

Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$. If there exists a $*$-frame $\{g_j\}_{j\in J}$ for $\mathcal{H}$ such that

$$f = \sum_{j\in J} \langle f, g_j \rangle f_j, \quad \forall f \in \mathcal{H},$$

then $*$-frame $\{g_j\}_{j\in J}$ is called a dual frame of $\{f_j\}_{j\in J}$. Also, assume that $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are two $*$-frames for $\mathcal{H}$. If there exists an invertible and adjointable $\mathcal{A}$-module map Γ on $\mathcal{H}$ such that

$$f = \sum_{j\in J} \langle \Gamma f, g_j \rangle f_j, \quad \forall f \in \mathcal{H},$$

then $(\{g_j\}_{j\in J}, \Gamma)$ is said to be an operator dual of $\{f_j\}_{j\in J}$.

In this study, we will consider some properties about ordinary duals and operator duals of $*$-frames. The proves of presented results are given only about operator duals if they are the same with ordinary cases, else every two cases are investigated independently.

The paper is organized as follows. The first section introduces equivalent $*$-frames and shows that ordinary duals of a $*$-frame $\{\xi f_j\}_{j\in J}$ are equivalent to ordinary duals of $\{f_j\}_{j\in J}$, where ξ is an adjointable and invertible operator. A necessary and sufficient condition is given for a relation between two arbitrary $*$-frames and also a relation between ordinary frames and operator duals is obtained by the concept of equivalent frames. Moreover, it is shown that the Grammian matrices of equivalent $*$-frames are equal. The second section presents some properties of some constructed $*$-frames. All operator duals of $*$-frames $\{\xi f_j\}_{j\in J}$ are characterized where ξ is an adjointable and invertible operator. An example is given for equivalent $*$-frames and their operator duals. Also, a $*$-frame is constructed by an orthogonal projection and its duals are studied. It is seen that an orthogonal projection and the frame operator of a primary $*$-frame commute under a bilateral condition. Finally, the paper calculates the pre-frame operator of a dual of a given $*$-frame by the pre-frame operator of the primary $*$-frame and an orthogonal projection.

2. Equivalent $*$-frames

In [3], equivalent sequences have been defined for sequences (frames) in Hilbert spaces. In this section, we extend this notion for sequences (frames) in Hilbert C^*-modules. Also, some properties of them will be studied.

Definition 2.1. Two sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ in $\mathcal{H}$ are said to be equivalent sequences if there exists an adjointable and invertible operator Λ on $\mathcal{H}$ such that $\Lambda f_j = g_j$, for $j \in J$.

For every $*$-frame, we have a family of $*$-frames by invertible and adjointable operators on $\mathcal{H}$. Now, the set of duals of these $*$-frames is characterized with respect to the primary $*$-frame. Firstly, we see the result about $*$-frames and then we investigate their duals.

Theorem 2.2 ([2])**.** *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with the frame operator S and lower and upper $*$-frame bounds A and B, respectively. Then an adjointable operator ξ on $\mathcal{H}$ is surjective if and only if $\{\xi f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$. In this case, $S_\xi := \xi S \xi^*$, $A\|(\xi\xi^*)^{-1}\|^{-\frac{1}{2}}$, and $B\|\xi\|$ are the frame operator and lower and upper frame bounds for $\{\xi f_j\}_{j\in J}$, respectively.*

Theorem 2.3. *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ and let ξ be an adjointable and invertible operator on $\mathcal{H}$. Then every dual of the $*$-frame $\{\xi f_j\}_{j\in J}$ is equivalent to a dual of $\{f_j\}_{j\in J}$, and converse of the relation is valid.*

Proof. First, suppose that $\{g_j\}_{j\in J}$ is a dual of $\{f_j\}_{j\in J}$. Then for $f \in \mathcal{H}$, we obtain

$$\begin{aligned} f &= \xi(\xi^{-1})f \\ &= \xi\left(\sum_{j\in J}\left\langle \xi^{-1}f, g_j\right\rangle f_j\right) \\ &= \sum_{j\in J}\left\langle f, (\xi^{-1})^* g_j\right\rangle \xi f_j. \end{aligned}$$

So $\{(\xi^{-1})^* g_j\}$ is a dual for $\{\xi f_j\}_{j\in J}$, and it is also equivalent to $\{g_j\}_{j\in J}$. Now, Suppose that $\{h_j\}_{j\in J}$ is a dual frame for $\{\xi f_j\}_{j\in J}$. Set $g_j = \xi^* h_j$, for $j \in J$. Then for $f \in \mathcal{H}$,

$$\begin{aligned} \sum_{j\in J}\langle f, g_j\rangle f_j &= \sum_{j\in J}\langle f, \xi^* h_j\rangle \xi^{-1}\xi f_j \\ &= \xi^{-1}\left(\sum_{j\in J}\langle \xi f, h_j\rangle \xi f_j\right) \\ &= \xi^{-1}\xi f \\ &= f. \end{aligned}$$

Then $\{g_j\}_{j\in J}$ is a dual for $\{f_j\}_{j\in J}$ and $h_j = (\xi^{-1})^* g_j$. $\square$

The following theorem constructs an equivalent $*$-frame by a combination of frame operators of two given $*$-frames that is equivalent to one of the given $*$-frames and its frame operator is the same as the frame operator of another $*$-frame.

Theorem 2.4. *If $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are $*$-frames with the frame operators $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$, respectively, then there exists a $*$-frame that is equivalent to $\{g_j\}_{j\in J}$ and its frame operator is $S_{\mathcal{F}}$.*

Proof. For the adjointable and invertible operator

$$\xi = S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}},$$

the sequence $\{\xi g_j\}_{j\in J}$ is a $*$-frame with the frame operator $S_\xi = \xi S_{\mathcal{G}} \xi^*$, by Theorem 2.2. So

$$\begin{aligned} S_\xi &:= \xi S_{\mathcal{G}} \xi^* \\ &= \left(S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}}\right) S_{\mathcal{G}} \left(S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}}\right)^* \\ &= S_{\mathcal{F}}. \end{aligned}$$

□

In the frame theory, the question to be considered is that "Dose the relation between two frames in a space exist". Up to now, duals or operator duals corresponding to a given frame are defined that are related to the primary frame. Here, we give an answer to the question, and characterize a lager family than duals or operator duals. On the other hand, in [1], it is shown that every dual frame is an operator dual but we now see an another relation between dual frames and operator dual frames of a $*$-frame by the equivalency relation between $*$-frames.

Theorem 2.5. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be $*$-frames for $\mathcal{H}$. Then the following statements are valid.*

(i) *$\{g_j\}_{j\in J}$ is equivalent to a dual frame of $\{f_j\}_{j\in J}$ if and only if there exists an adjointable and invertible operator ξ on $\mathcal{H}$ such that*

$$\xi f = \sum_{j\in J} \langle f, f_j \rangle g_j, \quad \forall f \in \mathcal{H}.$$

(ii) *$\{g_j\}_{j\in J}$ is equivalent to a dual frame of $\{f_j\}_{j\in J}$ if and only if there exists an adjointable and invertible operator Γ such that $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$.*

Proof. (i) First, assume that $\{g_j\}_{j\in J}$ is equivalent to a dual frame of $\{f_j\}_{j\in J}$. Then an adjointable and invertible operator Γ on $\mathcal{H}$ exists such that $\{\Gamma g_j\}_{j\in J}$ is a dual for $\{f_j\}_{j\in J}$. Now, for $f \in \mathcal{H}$,

$$f = \sum_{j\in J} \langle f, f_j\rangle \Gamma g_j.$$

Set $\xi = \Gamma^{-1}$, and it concludes

$$\begin{aligned} \xi f &= \Gamma^{-1} f \\ &= \Gamma^{-1} \left(\sum_{j\in J} \langle f, f_j\rangle \Gamma g_j \right) \\ &= \sum_{j\in J} \langle f, f_j\rangle g_j. \end{aligned}$$

In the second step, suppose an adjointable and invertible operator ξ on $\mathcal{H}$ with the following property

$$\xi f = \sum_{j\in J} \langle f, f_j\rangle g_j, \quad \forall f \in \mathcal{H}.$$

Since ξ is invertible,

$$f = \sum_{j\in J} \langle f, f_j\rangle \xi^{-1} g_j, \quad \forall f \in \mathcal{H}.$$

It shows that $\{\xi^{-1} g_j\}_{j\in J}$ is a dual for $\{f_j\}_{j\in J}$, and is equivalent to $\{g_j\}_{j\in J}$.

(ii) For the proof of "if" part, assume that there exists a dual frame $\{h_j\}_{j\in J}$ for $\{f_j\}_{j\in J}$ such that $\{h_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are equivalent. Therefore, there is an adjointable and invertible operator $\Lambda : \mathcal{H} \longrightarrow \mathcal{H}$, with $\Lambda g_j = h_j$, for all $j \in J$. By Theorem 2.2, the sequence $\{g_j\}_{j\in J}$ is a $*$-frame. On the other hand, for $f \in \mathcal{H}$,

$$\begin{aligned} f &= \sum_{j\in J} \langle f, h_j\rangle f_j \\ &= \sum_{j\in J} \langle f, \Lambda g_j\rangle f_j \\ &= \sum_{j\in J} \langle \Lambda^* f, g_j\rangle f_j, \end{aligned}$$

then

$$f = \sum_{j\in J} \langle \Lambda^* f, g_j\rangle f_j.$$

It shows that $\{(g_j, \Lambda^*)\}$ is an operator dual for $\{f_j\}_{j\in J}$. The converse part is clear by the last equalities. □

By a brief modification in the proof of the first part of Theorem 2.5, this subject can be considered for operator duals in the following form.

Theorem 2.6. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be $*$-frames for $\mathcal{H}$. Then $\{g_j\}_{j\in J}$ is equivalent to an operator dual frame of $\{f_j\}_{j\in J}$ if and only if there exists an adjointable and invertible operator ξ on $\mathcal{H}$ such that*

$$\xi f = \sum_{j\in J} \langle f, g_j\rangle f_j, \quad \forall f \in \mathcal{H}.$$

Proof. Suppose $\{g_j\}_{j\in J}$ is equivalent to $\{h_j\}_{j\in J}$ where $(\{h_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. Then there exists an adjointable and invertible operator θ on $\mathcal{H}$ such that $\theta g_j = h_j$, for all $j \in J$, and for $f \in \mathcal{H}$

$$\begin{aligned} f &= \sum_{j\in J} \langle \Gamma f, h_j\rangle f_j \\ &= \sum_{j\in J} \langle \Gamma f, \theta g_j\rangle f_j \\ &= \sum_{j\in J} \langle \theta^*\Gamma f, g_j\rangle f_j. \end{aligned}$$

Set $\xi = (\theta^*\Gamma)^{-1}$ and then the result is obtained. For converse, let ξ be an adjointable and invertible operator on $\mathcal{H}$ that

$$\xi f = \sum_{j\in J} \langle f, g_j\rangle f_j.$$

Then

$$f = \sum_{j\in J} \left\langle \xi^{-1} f, g_j\right\rangle f_j, \quad \forall f \in \mathcal{H},$$

and $(\{g_j\}_{j\in J}, \xi^{-1})$ is an operator dual of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ is equivalent to itself. □

Moreover, some equivalence frames have the same Grammian matrices. These frames are introduced in the following proposition.

Proposition 2.7. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be equivalent Parseval frames for $\mathcal{H}$, and let $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_{\mathcal{G}}$ are Grammian matrices of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, respectively. Then $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{G}}$.*

Proof. Scince two frames $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are equivalence, there exists an adjointable and invertible operator $\xi : \mathcal{H} \longrightarrow \mathcal{H}$ by $\xi f_j = g_j$ for $j \in J$. Since their frame operators are the identity operator on $\mathcal{H}$, then by Theorem 2.2

$$\xi\xi^* = \xi id \xi^* = id.$$

So ξ is a unitary operator and then for $i, j \in J$,

$$\langle g_i, g_j \rangle = \langle \xi f_i, \xi f_j \rangle = \langle f_i, f_j \rangle ,$$

which shows that $\mathcal{G}_{\mathcal{F}} = [\langle f_i, f_j \rangle]_{i,j\in J} = [\langle g_i, g_j \rangle]_{i,j\in J} = \mathcal{G}_{\mathcal{G}}$. □

3. Constructed $*$-frames and Some their Properties

In Theorem 2.2, a family of $*$-frames have been obtained with respect to a given $*$-frame. Now, we can characterize all operator duals of every element of this family. This characterization is given in the following.

Theorem 3.1. *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ and let ξ be an adjointable and invertible operator on $\mathcal{H}$. Then the set $(\{g_j\}_{j\in J}, \Gamma\xi^{-1})$ is all of operator duals of $\{\xi f_j\}_{j\in J}$, where $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$.*

Proof. Let $(\{g_j\}_{j\in J}, \Gamma)$ be an operator dual of $\{f_j\}_{j\in J}$. Then for $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in J} \left\langle \Gamma\xi^{-1} f, g_j \right\rangle \xi f_j &= \xi \left(\sum_{j\in J} \left\langle \Gamma\xi^{-1} f, g_j \right\rangle f_j \right) \\ &= \xi(\xi^{-1}) f \\ &= f.\end{aligned}$$

It shows that $(\{g_j\}_{j\in J}, \Gamma\xi^{-1})$ is an operator dual of $\{\xi f_j\}_{j\in J}$. Now, if $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{\xi f_j\}_{j\in J}$, then it is enough to set $\Gamma := \Gamma\xi$ in the last equalities which follows that $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. □

Example 3.2. Assume that $\mathcal{H}$ is the Hilbert C^*-module of all diagonal matrices in $M_{2\times 2}(\mathbb{C})$ over itself. Let two series

$$\sum_{i\in\mathbb{N}} |a_i|^2, \qquad \sum_{i\in\mathbb{N}} |b_i|^2,$$

be converge to α and β, respectively. The sequence

$$\{F_i\}_{i\in\mathbb{N}} = \left\{ \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \right\}_{i\in\mathbb{N}},$$

is a $\begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix}$-tight $*$-frame for $\mathcal{H}$. If

$$E = \begin{bmatrix} e_{11} & 0 \\ 0 & e_{22} \end{bmatrix},$$

is a real element of $\mathcal{H}$ such that $\det E \neq 0$, then the operator $\Lambda M = ME$ is invertible and adjointable on $\mathcal{H}$. So the sequence

$$\{G_i\}_{i\in\mathbb{N}} = \{\Lambda F_i\}_{i\in\mathbb{N}} = \left\{ \begin{bmatrix} a_i e_{11} & 0 \\ 0 & b_i e_{22} \end{bmatrix} \right\}_{i\in\mathbb{N}},$$

is a $*$-frame for $\mathcal{H}$ that is equivalent to $\{F_i\}_{i\in\mathbb{N}}$. Now, if $(\{H_i\}_{i\in\mathbb{N}}, \Gamma)$ is an operator dual of $\{F_i\}_{i\in\mathbb{N}}$, then $(\{H_i\}_{i\in\mathbb{N}}, \Gamma)$ is an operator dual of $\{G_i\}_{i\in\mathbb{N}}$. To see this fact, from commutativity of $\mathcal{H}$, for any $M \in \mathcal{H}$, we have

$$\begin{aligned} M &= \sum_{i\in\mathbb{N}} \langle \Gamma M, H_i \rangle F_i \\ &= \sum_{i\in\mathbb{N}} (\Gamma M)\overline{H_i} F_i \\ &= \sum_{i\in\mathbb{N}} (\Gamma M)\overline{H_i} F_i E E^{-1} \\ &= \sum_{i\in\mathbb{N}} (\Gamma M) E^{-1} \overline{H_i} G_i \\ &= \sum_{i\in\mathbb{N}} (\Lambda^{-1}\Gamma M)\overline{H_i} G_i \\ &= \sum_{i\in\mathbb{N}} \left\langle \Lambda^{-1}\Gamma M, H_i \right\rangle G_i. \end{aligned}$$

By an orthogonal projection, a $*$-frame will be obtained and a relation will be also given for this projection. To see this, we must show that the inverse of frame operator is unique in the reconstruction formula. So, firstly this fact will be considered.

Theorem 3.3. *If $\{f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$, then there exists a unique adjointable operator Λ on $\mathcal{H}$ such that*

$$f = \sum_{j\in J} \langle f, \Lambda f_j \rangle f_j, \quad \forall f \in \mathcal{H}.$$

Proof. By the reconstruction formula, $\Lambda = S^{-1}$ exists. For the uniqueness of S^{-1} with this property, we know that $\{S^{-\frac{1}{2}} f_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}$, so set $g_j = S^{-\frac{1}{2}} f_j$, therefore $f_j = S^{\frac{1}{2}} g_j$. Now, suppose that Λ is an adjointable operator such that

$$f = \sum_{j\in J} \langle f, \Lambda f_j \rangle f_j, \quad \forall f \in \mathcal{H}.$$

Therefore, we have

$$f = \sum_{j\in J} \langle f, \Lambda f_j \rangle f_j$$

$$
\begin{aligned}
&= \sum_{j\in J} \left\langle f, \Lambda S^{\frac{1}{2}} g_j \right\rangle S^{\frac{1}{2}} g_j \\
&= S^{\frac{1}{2}} \left(\sum_{j\in J} \left\langle f, \Lambda S^{\frac{1}{2}} g_j \right\rangle g_j \right) \\
&= S^{\frac{1}{2}} \left(\sum_{j\in J} \left\langle S^{\frac{1}{2}} \Lambda^* f, g_j \right\rangle g_j \right) \\
&= S^{\frac{1}{2}} \left(S^{\frac{1}{2}} \Lambda^* f \right) \\
&= S\Lambda^* f, \quad \forall f \in \mathcal{H}.
\end{aligned}
$$

It concludes that $S\Lambda^* = id$ and then $\Lambda^* = S^{-1}$. More precisely, Λ is self-adjoint, positive and invertible. □

Now, a $*$-frame is constructed by an orthogonal projection.

Proposition 3.4. *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with the frame operator S and $*$-frame bounds A and B. Also, suppose P is an orthogonal projection on $\mathcal{H}$. Then $\{Pf_j\}_{j\in J}$ is a $*$-frame for R_P with $*$-frame bounds A and B. Moreover, if $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual of $\{f_j\}_{j\in J}$, then $\{P\Gamma^* g_j\}_{j\in J}$ is a dual frame for $\{Pf_j\}_{j\in J}$.*

Proof. For $f \in R_P$,

$$
\begin{aligned}
\sum_{j\in J} \langle f, Pf_j\rangle \langle Pf_j, f\rangle &= \sum_{j\in J} \langle Pf, f_j\rangle \langle f_j, Pf\rangle \\
&= \sum_{j\in J} \langle f, f_j\rangle \langle f_j, f\rangle ,
\end{aligned}
$$

and by the definition of the $*$-frame $\{f_j\}_{j\in J}$, we have

$$
A \langle f, f\rangle A^* \le \sum_{j\in J} \langle f, Pf_j\rangle \langle Pf_j, f\rangle \le B \langle f, f\rangle B^*.
$$

Now, if $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual of $\{f_j\}_{j\in J}$, then for $f \in R_P$,

$$
\begin{aligned}
f &= Pf \\
&= P \left(\sum_{j\in J} \langle \Gamma Pf, g_j\rangle f_j \right) \\
&= \sum_{j\in J} \langle f, P\Gamma^* g_j\rangle Pf_j.
\end{aligned}
$$

If $\{g_j\}_{j\in J}$ is also a dual of $\{f_j\}_{j\in J}$, then $\{Pg_j\}_{j\in J}$ is a dual of $\{Pf_j\}_{j\in J}$. So the result is clear by $\Gamma = id_{\mathcal{H}}$. □

By the last theorem, a necessary and sufficient condition is found for commutating a projection with the inverse of the frame operator of a given $*$-frame.

Theorem 3.5. *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with the frame operator S. Suppose P is an orthogonal projection on $\mathcal{H}$. Then $PS^{-1}f_j = S_P^{-1}Pf_j$, for all $j \in J$ if and only if $PS^{-1} = S^{-1}P$, where S_P is the frame operator of the $*$-frame $\{Pf_j\}_{j\in J}$.*

Proof. First, assume that $PS^{-1}f_j = S_P^{-1}Pf_j$, for all $j \in J$. Now, let $f \in \mathcal{H}$. Then we have

$$\begin{aligned} S_P^{-1}Pf &= S_P^{-1}P\left(\sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle f_j\right) \\ &= \sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle S_P^{-1}Pf_j \\ &= \sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle PS^{-1}f_j \\ &= PS^{-1}\left(\sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle f_j\right) \\ &= PS^{-1}f. \end{aligned}$$

Therefore $PS^{-1}f = S_P^{-1}Pf$, for all $f \in \mathcal{H}$, and so

$$S_P^{-1}P = PS^{-1}P \quad \Rightarrow \quad PS^{-1}P = PS^{-1}.$$

Also,

$$PS^{-1}P = (PS^{-1}P)^* = (PS^{-1})^* = S^{-1}P.$$

For the proof of converse, suppose $PS^{-1} = S^{-1}P$. Let $f \in R_P$. Then we have

$$\begin{aligned} f &= Pf \\ &= P\left(\sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle f_j\right) \\ &= \sum_{j\in J}\left\langle f, S^{-1}f_j\right\rangle Pf_j \\ &= \sum_{j\in J}\left\langle Pf, S^{-1}f_j\right\rangle Pf_j \\ &= \sum_{j\in J}\left\langle f, PS^{-1}f_j\right\rangle Pf_j \end{aligned}$$

$$= \sum_{j\in J} \left\langle f, S^{-1}Pf_j \right\rangle Pf_j.$$

By Theorem 3.3 and the assumption, for $j \in J$

$$S^{-1}Pf_j = S_P^{-1}f_j = S_P^{-1}Pf_j = PS^{-1}f_j,$$

and the proof is complete. □

Since the pre-frame operator of a given $*$-frame in a Hilbert $\mathcal{A}$-module has the closed range, the its range is orthogonal complement (complementable) and the orthogonal projection on its range is well-defined [6]. In the following theorem, the relation between pre-frame operators of a Parseval frame in a Hilbert $\mathcal{A}$-module and its dual is obtained.

Theorem 3.6. *Let $\{f_j\}_{j\in J}$ be a Parseval frame of $\mathcal{H}$ with pre-frame operator $\theta_{\mathcal{F}}$ and let $\{g_j\}_{j\in J}$ be a $*$-frame with the pre-frame operator $\theta_{\mathcal{G}}$. Then $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$ if and only if $P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = \theta_{\mathcal{F}}$, where $P_{\theta_{\mathcal{F}}}$ is the orthogonal projection on the range of $\theta_{\mathcal{F}}$.*

Proof. Suppose that $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. Since $\{f_j\}_{j\in J}$ is a Parseval frame, the pre-frame operator $\theta_{\mathcal{F}}$ is an isometry;

$$\langle \theta_{\mathcal{F}}f, \theta_{\mathcal{F}}f\rangle = \sum_{j\in J} \langle f, f_j\rangle \langle f_j, f\rangle = \langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

Then for $f \in \mathcal{H}$,

$$\begin{aligned}
\langle P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma f, \theta_{\mathcal{F}}f\rangle &= \langle \theta_{\mathcal{G}}\Gamma f, P_{\theta_{\mathcal{F}}}\theta_{\mathcal{F}}f\rangle \\
&= \langle \theta_{\mathcal{G}}\Gamma f, \theta_{\mathcal{F}}f\rangle \\
&= \left\langle \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, f \right\rangle \\
&= \langle f, f\rangle \\
&= \langle \theta_{\mathcal{F}}f, \theta_{\mathcal{F}}f\rangle .
\end{aligned}$$

Then $P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = \theta_{\mathcal{F}}$. Conversely, since $\theta_{\mathcal{F}}$ is an isometry, by a similar method, we have

$$\begin{aligned}
\langle f, g\rangle &= \langle \theta_{\mathcal{F}}f, \theta_{\mathcal{F}}g\rangle \\
&= \langle P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma f, \theta_{\mathcal{F}}g\rangle \\
&= \left\langle \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, g \right\rangle, \quad \forall f \in \mathcal{H},
\end{aligned}$$

then

$$f = \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, \quad \forall f \in \mathcal{H},$$

and $(\{g_j\}_{j\in J},\Gamma)$ is a dual for $\{f_j\}_{j\in J}$. □

Corollary 3.7. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be two $*$-frames for $\mathcal{H}$ with per-frame operators $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{G}}$, respectively. Then $(\{g_j\}_{j\in J},\Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$ if and only if $\theta^*_{\mathcal{F}}P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = id$*

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