Sahand Communications in Mathematical Analysis (SCMA) Vol. 14 No. 1 (2019), 117-126
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.61771.226

Some Observations on Dirac Measure-Preserving Transformations and their Results

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Abstract. Dirac measure is an important measure in many related branches to mathematics. The current paper characterizes measure-preserving transformations between two Dirac measure spaces or a Dirac measure space and a probability measure space. Also, it studies isomorphic Dirac measure spaces, equivalence Dirac measure algebras, and conjugate of Dirac measure spaces. The equivalence classes of a Dirac measure space and its measure algebras are presented. Then all of measure spaces that are isomorphic with a Dirac measure space are characterized and the concept of a Dirac measure class is introduced and its elements are characterized. More precisely, it is shown that every absolutely continuous measure with respect to a Dirac measure belongs to the Dirac measure class. Finally, the relation between Dirac measure preserving transformations and strong-mixing is studied.

1. Introduction

Ergodic theory is a property on measure preserving transformations over probability measure spaces, for more details one may refer to [8, 9, 14]. It utilizes techniques and examples from various branches such as mathematics, physics, probability theory, statistical mechanics and etc. (see [6, 11, 14] and reference therein). For instance the concept of ergodicity is one of key applications of ergodic theory which describes the long term average behaviour of systems evolving in time ([5, 7]). The mean and pointwise ergodic theorems are respectively established

2010 *Mathematics Subject Classification.* 28Dxx, 28D05.
Key words and phrases. Measure-preserving transformation, Dirac measure, Measure algebra, Measure-preserving transformation.
Received: 10 April 2017, Accepted: 05 November 2017.
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by Neumann and Birkhhoff in 1931 and 1932. The theorems play an important role in both mathematics and statistical mechanics, for further details see ([1, 11, 13]).

Before starting the main contribution of the current work, we briefly express the concept of the Dirac measure in the following paragraph.

The Dirac measure is a probability measure. In terms of probability, it represents the almost sure outcome x in the sample space X [3, 4, 10]. In fact, the idea of a Dirac measure is given from the well-known Dirac delta function. We would like to comment here that one of the important and most used probability measure spaces is a Dirac measure space and it has some applications in physics [12, 15]. The importance of the ergodic theory and Dirac measure space motivate us to study the existing results in ergodic theory with respect to the Dirac measure in more details. During the paper, a Dirac measure concentrated on x is denoted by μ_x.

The rest of this paper is organized as follows. In the next section, we review some basic definitions and notions of measure and Ergodic theories. In Section 3, we characterize measure-preserving transformations on two types of Dirac measure spaces. More precisely, measure-preserving transformations are considered on two different Dirac measure spaces $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu_\beta)$. It is shown that the set of all measure preserving transformations $T : (X, \mathcal{A}, \mu_\alpha) \to (Y, \mathcal{B}, \mu)$ is empty, where μ is not a Dirac measure and $\mathcal{B}$ is a nontrivial σ-algebra. In the sequel, we present the equivalence classes of a Dirac measure space and their measure algebras. Then all of measure spaces that are isomorphic with a Dirac measure space are characterized. Using the established results, we introduce the Dirac measure class.

At the end of this section, we would like to point out that one of the main contribution of the current work is showing that every absolutely continuous measure with respect to a Dirac measure belongs to our defined Dirac measure class. Also, the form of Brikhoff Ergodic Theorem for a Dirac measure space is presented.

2. Preliminaries

In this section, we give a brief review on the definitions and properties which are useful for establishing our main results.

In what follows, the triple $(X, \mathcal{B}, m)$ is called a measurable space if $\mathcal{B}$ is a σ-algebra on X and m is a measure defined on X. A measurable space $(X, \mathcal{B}, m)$ is said to be a probability space if $m(X) = 1$.

In the rest of this section, we recall some definitions from [14].

Definition 2.1. Let $(X_1, \mathcal{A}, m_1)$ and $(X_2, \mathcal{B}, m_2)$ be two probability spaces. A measurable transformation $T : (X_1, \mathcal{A}, m_1) \to (X_2, \mathcal{B}, m_2)$ is measure-preserving if $m_1(T^{-1}(B)) = m_2(B)$ for all $B \in \mathcal{B}$.

Some of measure-preserving transformations on a measure space are ergodic transformations. In the following, the definition of ergodic transformation is given.

Definition 2.2. Let $(X, \mathcal{B}, m)$ be a probability space and $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ be a measure-preserving transformation. Then, T is called ergodic if every member B of $\mathcal{B}$ with $T^{-1}(B) = B$ satisfies $m(B) = 0$ or $m(B) = 1$.

The Brikhoff Ergodic Theorem is an important result in ergodic theory. We recall this theorem in the following.

Theorem 2.3 ([14, Brikhoff Ergodic Theorem])**.** *Suppose* $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ *is a measure-preserving transformation (where we allow* $(X, \mathcal{B}, m)$ *to be* σ*-finite) and* $f \in L^1(m)$*. Then*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x),$$

converges a.e. to a function $g \in L^1(m)$*. Also* $g \circ T = g$ *a.e. and if* $m(X) < \infty$*, then* $\int gdm = \int fdm$*.*

In the following, the concepts of weak and strong-mixing are given. It should be commented that the ergodic transformations and strong-mixing transformations are equivalent.

Definition 2.4. Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$. Then

i . T is weak-mixing if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|m(T^{-i}A \cap B) - m(A)m(B)\right| = 0.$$

ii . T is strong-mixing if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} m(T^{-n}A \cap B) = m(A)m(B).$$

Consider the probability measure space $(X, \mathcal{B}, m)$ where $\mathcal{B}$ is a σ-algebra on X. Let A and B belong to $\mathcal{B}$. We say that A and B are equivalent if and only if $m(A \triangle B) = 0$ where $\triangle$ stands for the well-known symmetric difference, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and suppose that $\tilde{\mathcal{B}}$ consists of the collection of corresponding equivalence classes. It is known that $\tilde{\mathcal{B}}$ is a Boolean σ-algebra under the operations of complementation, union and intersection inherited from $\mathcal{B}$. Here, the measure m induces a measure $\tilde{m}$ on $\tilde{\mathcal{B}}$, i.e., $\tilde{m}(\tilde{B}) = m(B)$ where $\tilde{B}$ is the equivalence class to which B belongs. The pair $(\tilde{\mathcal{B}}, \tilde{m})$ is called a measure algebra.

Two measure spaces are equivalent if their corresponding measure algebras are isomorphic; see [14] for further details. Two probability spaces are said to be conjugate if their measure algebras are isomorphic.

3. Main results

In view of the earlier importance of a Dirac measure, this section deals with scrutinizing some results of ergodic theory on Dirac measure spaces. In addition,we introduce the concept of Dirac measure classes. In this section, we focus on the related properties of Dirac measure, ergodic and mixing transformations.

First, we characterize all of measure-preserving transformations between Dirac measure spaces and another probability spaces.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu_\alpha)$ and $(X, \mathcal{B}, \mu_\alpha)$ be two Dirac measure spaces where $\alpha \in X$, and $T : (X, \mathcal{A}, \mu_\alpha) \to (X, \mathcal{B}, \mu_\alpha)$ be a transformation. Then the following statements are valid.*

- i . *If "α" is a fixed point for T, then T is a measure-preserving transformation.*
- ii . *If "α" is not a fixed point for T; it means that $\beta = T\alpha$ and $\beta \neq \alpha$, and every element of $\mathcal{B}$ consists both "α" and "β" or none of them, then T is a measure-preserving transformation.*
- iii . *If "α" is not a fixed points for T; it means that $\beta = T\alpha$ and $\beta \neq \alpha$ and there exists an element of $\mathcal{B}$ that is only contained one of the point α or β, then T is not a measure-preserving transformation.*

Proof. i . Assume that $T\alpha = \alpha$, and E is an arbitrary element of $\mathcal{B}$. We have the following cases.

Case 1. If $\alpha \in E$, then $\alpha \in T^{-1}(E)$ and $\mu_\alpha(T^{-1}(E)) = 1 = \mu_\alpha(E)$.
Case 2. If $\alpha \notin E$, then $\alpha \notin T^{-1}(E)$, and $\mu_\alpha(T^{-1}(E)) = 0 = \mu_\alpha(E)$.
Therefore, T is measure-preserving in these two cases.

ii . Let $\beta = T\alpha \neq \alpha$ and E be an arbitrary element of $\mathcal{B}$. The following cases are obtained.

Case 1. If $\{\alpha, \beta\} \subseteq E$, then $\alpha \in T^{-1}(E)$ and $\mu_\alpha(T^{-1}(E)) = 1 = \mu_\alpha(E)$.
Case 2. If $\{\alpha, \beta\} \cap E = \phi$, then $\alpha \notin T^{-1}(E)$ and $\mu_\alpha(T^{-1}(E)) = 0 = \mu_\alpha(E)$.
So T is measure-preserving in these two cases.

iii . If there exists an element E of $\mathcal{B}$ that $E \cap \{\alpha, \beta\} = \beta$, then

$$\alpha \in T^{-1}(E) \quad \Rightarrow \quad \mu_\alpha(T^{-1}(E)) = 1 \neq 0 = \mu_\alpha(E).$$

It implies that T is not measure-preserving. Also, if there is an element E of $\mathcal{B}$ that $E \cap \{\alpha, \beta\} = \alpha$, then

$$\alpha \notin T^{-1}(E) \quad \Rightarrow \quad \mu_\alpha(T^{-1}(E)) = 0 \neq 1 = \mu_\alpha(E).$$

It implies that T is not measure-preserving.

□

Now, we consider a Dirac measure-preserving transformation T when two Dirac measure spaces are on different sets. The following theorem illustrates it.

Theorem 3.2. *Let $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu_\beta)$ be two Dirac measure spaces where $\alpha \in X$, $\beta \in Y$, and let $T : (X, \mathcal{A}, \mu_\alpha) \to (Y, \mathcal{B}, \mu_\beta)$ be a measurable transformation. Then the following statements are valid.*

i . *If $T\alpha = \beta$, then T is measure-preserving.*

ii . *If $T\alpha \neq \beta$; it means that $\gamma = T\alpha$ and $\gamma \neq \beta$, and every element of $\mathcal{B}$ consists both "β" and "γ" or none of them, then T is measure-preserving.*

Proof. i. Suppose that $T\alpha = \beta$, and E is an arbitrary element of $\mathcal{B}$. We have the following cases.

Case 1. If $\beta \in E$, then $\alpha \in T^{-1}(E)$, and $\mu_\alpha(T^{-1}(E)) = 1 = \mu_\beta(E)$.
Case 2. If $\beta \notin E$, then $\alpha \notin T^{-1}(E)$, and $\mu_\alpha(T^{-1}(E)) = 0 = \mu_\beta(E)$.

Consequently, T is a measure-preserving transformation.
ii. Assume that $\gamma = T\alpha \neq \beta$, and for $E \in \mathcal{B}$ consider the following two cases.

Case 1. If $\{\beta, \gamma\} \subseteq E$, then $\alpha \in T^{-1}(E)$ and $\mu_\alpha(T^{-1}(E)) = 1 = \mu_\beta(E)$.
Case 2. If $\{\beta, \gamma\} \cap E = \phi$, then $\alpha \notin T^{-1}(E)$ and $\mu_\alpha(T^{-1}(E)) = 0 = \mu_\beta(E)$.

□

Remark 3.3. In the general case, if T is a measurable transformation from a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$ to a measure space $(Y, \mathcal{B}, \mu)$, it seems that we can not characterize all of measure-preserving transformations, unless $T : (X, \mathcal{A}, \mu_\alpha) \to (Y, \{\phi, Y\}, \mu)$ where μ is a probability measure.

Now let $T : (Y, \{\phi, Y\}, \mu) \to (X, \mathcal{A}, \mu_\alpha)$ be a transformation. Then T can not preserve measure because if $E \in \mathcal{A}$ such that $E \cap Rang(T) \neq \phi$ and $\alpha \notin E$, then

$$T^{-1}(E) \neq \phi \quad \Rightarrow \quad T^{-1}(E) = Y.$$

So $\mu_\alpha(E) = 0 \neq 1 = \mu(T^{-1}(E))$.

Therefore, the following problem remains:

"It is possible to characterize the Dirac measure-preserving of T is a measurable transformation between a Dirac measure space and another measure space that is not Dirac? "

Now, we consider the Dirac measure algebras and their equivalence classes. Suppose that $(X, \mathcal{A}, \mu_\alpha)$ is a Dirac measure space.

For a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$, there are two equivalence classes with respect to this equivalence relation. To see this, let $A, B \in \mathcal{A}$. It can be seen that

$$\begin{aligned} A \sim B \quad & \leftrightarrow \quad \mu(A \triangle B) = 0 \\ & \leftrightarrow \quad \mu_\alpha(A - B) + \mu_\alpha(B - A) = 0 \\ & \leftrightarrow \quad \mu_\alpha(A - B) = \mu_\alpha(B - A) = 0 \\ & \leftrightarrow \quad \alpha \notin A \cup B \quad or \quad \alpha \in A \cap B. \end{aligned}$$

Therefore, for $A \in \mathcal{A}$,

1. If $\alpha \in A$, then $[A] = \{B \in \mathcal{A}; \alpha \in B\}$, and this class is denoted by $\tilde{X}$.
2. If $\alpha \notin A$, then $[A] = \{B \in \mathcal{A}; \alpha \notin B\}$, and in this case, we show $[A] = \tilde{\phi}$.

Moreover, the measure μ_α induces a measure $\tilde{\mu_\alpha}$ on σ-algebra $\{\tilde{\phi}, \tilde{X}\}$, such that

$$\tilde{\mu_\alpha}(\tilde{X}) = \mu_\alpha(X) = 1, \qquad \tilde{\mu_\alpha}(\tilde{\phi}) = \mu_\alpha(\phi) = 0,$$

and it shows that $\tilde{\mu_\alpha}$ is a Dirac measure too. Then the measure algebra corresponding to a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$ is Dirac measure algebra $(\{\tilde{\phi}, \tilde{X}\}, \tilde{\mu_\alpha})$.

More precisely, the notion of isomorphism between measure spaces is defined as follows ([14]).

Definition 3.4. The probability spaces $(X, \mathcal{B}_1, m_1)$ and $(Y, \mathcal{B}_2, m_2)$ are said to be isomorphic if there exist $M_1 \in \mathcal{B}_1$ and $M_2 \in \mathcal{B}_2$ with $m_1(M_1) = 1 = m_2(M_2)$ and an invertible measure-preserving transformation $\Phi : M_1 \to M_2$ (The space M_i is assumed to be equipped with the σ-algebra $M_i \cap \mathcal{B}_i = \{M_i \cap B : B \in \mathcal{B}_i\}$ and the restriction of the measure m_i to this σ-algebra).

In the following results, we study all of measure spaces that are isomorphic with a Dirac measure space and so we characterize all of measure algebras that are equivalent with a Dirac measure algebra.

Theorem 3.5. *Let $(Y, \mathcal{B}, \mu)$ be a measure space and let $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu_\beta)$ be two Dirac measure spaces. Then the following statements are valid.*

i. *If μ is a probability measure, $\mathcal{B} = \{\phi, Y\}$ and there exists an invertible transformation $\theta : A \to Y$ such that A is an element of $\mathcal{A}$ containing α, then $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu)$ are isomorphic.*

ii. *Two Dirac measure spaces $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu_\beta)$ are isomorphic in two following cases. Assume that A is an element of $\mathcal{A}$ containing "α" and B is an element of $\mathcal{B}$ including "β".*
   1. *If there exists an invertible transformation $\theta : A \to B$ such that $\theta(\alpha) = \beta$.*
   2. *If there exists an invertible transformation $\theta : A \to B$ such that $\gamma := \theta(\alpha) = \beta$ and for every $E \in \mathcal{B}$,*

$$\{\gamma, \beta\} \subseteq E \quad \textit{or} \quad \{\gamma, \beta\} \cap E = \phi.$$

iii. *Suppose that $Y = X$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\alpha \in A \cap B$. Then, the Dirac measure spaces $(X, \mathcal{A}, \mu_\alpha)$ and $(X, \mathcal{B}, \mu_\alpha)$ are isomorphic in the following states:*
   1. *If there exists an invertible transformation $\theta : A \to B$ such that "α" is a fixed point for θ.*
   2. *If there exists an invertible transformation $\theta : A \to B$ such that $\gamma = \theta(\alpha) \neq \beta$ and for $E \in \mathcal{B}$*

$$\{\gamma, \beta\} \subseteq E \quad \textit{or} \quad \{\gamma, \beta\} \cap E = \phi.$$

Proof. i. According to Definition 3.4, it is enough to find an invertible measure-preserving transformation. By Remark 3.3, $\theta : A \to Y$ is a measure-preserving transformation. Also $\mu_\alpha(A) = 1 = \mu(Y)$. Then the conditions of Definition 3.4 are valid and two measure spaces $(X, \mathcal{A}, \mu_\alpha)$ and $(Y, \mathcal{B}, \mu)$ are isomorphic. For proof of (ii) and (iii), let θ be an invertible transformation that has one of properties in "1" and "2". Then Theorems 3.1 and 3.2 conclude that θ is measure-preserving. □

Theorem 3.6. *If $(Y, \mathcal{B}, m)$ is a measure space which is conjugate with a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$, then $(Y, \mathcal{B}, m)$ is a probability measure space and $\tilde{\mathcal{B}}$, the measue algebra of $\mathcal{B}$, is $\{\tilde{\phi}, \tilde{Y}\}$.*

Proof. If $(X, \mathcal{A}, \mu_\alpha)$ is conjugate with $(Y, \mathcal{B}, m)$, then there exists a bijective $\theta : \tilde{\mathcal{B}} \to \tilde{\mathcal{A}}$ such that

1. $\theta((\tilde{E})^c) = \theta(\tilde{E})^c, \quad \forall \tilde{E} \in \tilde{\mathcal{B}}$,
2. $\theta(\cup_{i=1}^{\infty} \tilde{E}_i) = \cup_{i=1}^{\infty} \theta(\tilde{E}_i), \quad \forall \tilde{E}_i \in \tilde{\mathcal{B}}, \quad i \in \mathbb{N}$,
3. $\mu_\alpha(\theta(\tilde{E})) = m(\tilde{E}), \quad \forall \tilde{E} \in \tilde{\mathcal{B}}$.

Also, since θ is bijective, two possible cases for θ are as follows:

i. $\theta(\tilde{Y}) = \tilde{\phi}, \quad \theta(\tilde{\phi}) = \tilde{X}$,

ii. $\theta(\tilde{Y}) = \tilde{X}, \quad \theta(\tilde{\phi}) = \tilde{\phi}$.

By the first property of θ, the item (i) can not be true and by the third property of θ, $(Y, \mathcal{B}, m)$ is a probability measure space. □

In the above proof, if $\theta : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$, then the same result is given. Here, we recall a Dirac measure class for every measure space that is conjugate with a Dirac measure space. It means that every measure function $\mu : (Y, \mathcal{B}) \to \{0,1\}$ belongs to the Dirac measure class. The equivalence relation between the sets of a σ-algebra gives an interesting result for a class of measures that are absolutely continuous with respect to a Dirac measure.

Proposition 3.7. *Let $(X, \mathcal{A}, \mu_\alpha)$ be a Dirac measure space and λ be a probability measure which is absolutely continuous with respect to μ_α. Then not only λ is a Dirac measure class, but also $\lambda = \mu_\alpha$.*

Proof. Suppose that $\tilde{\mathcal{A}}^{\mu_\alpha} = \{\tilde{X}^{\mu_\alpha}, \tilde{\phi}^{\mu_\alpha}\}$ is the σ-algebras of equivalence classes corresponding to measure spaces $(X, \mathcal{A}, \mu_\alpha)$. Now, let A be an arbitrary element of $\mathcal{A}$. The following cases are satisfied.

Case 1. If $\alpha \in A$, then $\alpha \notin A^c$ and $\mu_\alpha(A^c) = 0$. By $\lambda \ll \mu_\alpha$, we have $\lambda(A^c) = 0$. Since λ is a probability measure, so $\lambda(A) = 1$.

Case 2. If $\alpha \notin A$, then $\mu_\alpha(A) = 0$. Since $\lambda \ll \mu_\alpha$, we have $\lambda(A) = 0$. So $\lambda(A^c) = 1$.

Case 1 and case 2 show that $\lambda = \mu_\alpha$ and λ is a Dirac measure class. □

By the above results, it is well-known that every measure-preserving transformation for a Dirac measure space is ergodic. In the following, the Brikhoff Ergodic Theorem for a Dirac measure space is given.

Proposition 3.8. *If T is a measure-preserving transformation on a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$, then for $f \in L^1(\mu_\alpha)$,*

$$\int \lim \frac{1}{n} \sum_{i=1}^{n-1} f(T^i(x)) d\mu_\alpha = f(\alpha).$$

Proposition 3.9. *A nontrivial measure space $(X, \mathcal{A}, \mu_\alpha)$ is a Dirac measure class if and only if every measure-preserving transformation on it is ergodic.*

Proof. It is not difficult to verify the validity of "if" part. For converse, suppose that every measure-preserving transformation is ergodic. Let A be an arbitrary element of $\mathcal{A}$. Since the identity transformation on $(X, \mathcal{A}, \mu_\alpha)$ is measure-preserving and then it is ergodic, then

$$I^{-1}(A) = A \quad \to \quad \mu(A) = 0 \quad or \quad \mu(A) = 1.$$

So, the range of μ is $\{0,1\}$ and the measure space is a Dirac measure class. □

One of the interesting subjects strong-mixing and weak-mixing transformation. It is well-known that every strong-mixing transformation is weak-mixing and every weak-mixing transformation is an ergodic transformations. But the converse is not true, it means that there exists a probability measure space that has an ergodic transformation such that it is not weak-mixing ([14]).

There is a useful property for a Dirac measure space about the converse of the above relation. The following proposition illustrates this property.

Proposition 3.10. *Every measure-preserving transformation T on a Dirac measure space $(X, \mathcal{A}, \mu_\alpha)$ is a strong-mixing transformation.*

Proof. Let T be measure-preserving on $(X, \mathcal{A}, \mu_\alpha)$ and $A, B \in \mathcal{A}$. If $\alpha \notin B$, then the result is clear. We must consider two cases in Theorem 3.1.

1. "α" is a fixed point for T and if $\alpha \notin A$, then $\alpha \notin T^{-n}(A)$, for all $n \in \mathbb{N}$, and
$$\lim_{n\to\infty} \mu_\alpha(T^{-n}(A) \cap B) = 0 = \mu_\alpha(A) \cap \mu_\alpha(B).$$
2. "α" is not fixed point for T and $T(\alpha) = \beta \neq \alpha$, then $\beta \notin A$ by Theorem 3.1 , $\alpha \notin T^{-n}(A)$, and we have
$$\lim_{n\to\infty} \mu_\alpha(T^{-n}(A) \cap B) = 0 = \mu_\alpha(A) \cap \mu_\alpha(B), \quad \forall n \in \mathbb{N}.$$

□

Acknowledgment. The authors are very grateful to the referees for their valuable suggestions and opinions.

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