A Class of Hereditarily $\ell_p(c_0)$ Banach Spaces

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Abstract. We extend the class of Banach sequence spaces constructed by Ledari, as presented in “A class of hereditarily $\ell_1$ Banach spaces without Schur property” and obtain a new class of hereditarily $\ell_p(c_0)$ Banach spaces for $1 \leq p < \infty$. Some other properties of this spaces are studied.

1. Introduction

We follow the same notations and terminology as in [5]. Let $Y$ be a subspace of $X$. Then we say that $X$ contains $Y$ hereditarily if every infinite dimensional subspace of $X$ contains an isomorphic copy of $Y$. Thus, if $X$ hereditarily contains $Y$, then we naturally expect to have the interior properties of $X$ to be close to those of $Y$. Any exception may be of interest. For example, it is well known that $\ell_1$ possesses the Schur property, while there are hereditarily $\ell_1$ Banach spaces without the Schur property [1, 2, 3, 4, 7].

In this paper, we use $\ell_{w,p}$ spaces to introduce and study a new class of hereditarily $\ell_p(c_0)$ spaces. Indeed, if $p_1 > p_2 > \cdots > 1$, the subspace $Z_p$ for $p \in [1, \infty) \cup \{0\}$ of

$$X_p = \left( \sum_{n=1}^{\infty} \oplus \ell_{w,p_n} \right)_p ,$$

is hereditarily $\ell_p(c_0)$. Other properties of these spaces are investigated. In this article, we show that under some conditions for $p \in [1, \infty) \cup \{0\},$

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the natural operator from $\ell_{w,p}$ to $Z_p$ is unbounded. Also the natural operator from $Z_p$ to $\ell_{w,p}$ is unbounded.

Let $w = (w_n)$ be a fixed nonnegative real sequence. We recall the definition of $\ell_{w,p}$ ($1 \leq p < \infty$), the weighted $\ell_p$ Banach sequence space. We know

$$\ell(w,p) = \left\{ x = (x_1, x_2, \ldots) : x_i \in \mathbb{R}, \sum_{i=1}^{\infty} w_i |x_i|^p < \infty \right\}.$$ 

For any $x \in \ell(w,p)$, define

$$\|x\|_{w,p} = \left( \sum_{i=1}^{\infty} w_i |x_i|^p \right)^{\frac{1}{p}}.$$

For any $i$, let $e_i = \left( 0, \ldots, 0, \left( \frac{1}{w_i} \right)^{\frac{1}{p}}, 0, \ldots \right)$. We know that \{ $e_i : i \in \mathbb{N}$\} is a normalized basis for $\ell(w,p)$. Now we go through the construction of the spaces $X_p$ analogous of the space of Popov. Let $w = (w_n)$ be a fixed sequence, and $(\ell_{w_p^n})_{n=1}^{\infty}$ a sequence of Banach spaces as above with $\infty > p_1 > p_2 > \cdots > 1$. The direct sum of these spaces in the sense of $\ell_p$ is defined as the linear space

$$X_p = \left( \sum_{n=1}^{\infty} \oplus \ell_{w_p^n} \right)_p,$$

with $p \in [1, \infty)$ which is the space of all sequences $x = (x^1, x^2, \ldots)$, $x^n \in \ell_{w_p^n}$, $n = 1, 2, \ldots$, with

$$\|x\|_p = \left( \sum_{n=1}^{\infty} \|x^n\|_{w_p^n}^p \right)^{\frac{1}{p}} < \infty.$$

The direct sum of the spaces $(\ell_{w_p^n})$ in the sense of $c_0$ is the linear space

$$X_0 = \left( \sum_{n=1}^{\infty} \oplus \ell_{w_p^n} \right)_0,$$

of all sequences $x = (x^1, x^2, \ldots)$, $x^n \in \ell_{w_p^n}$, $n = 1, 2, \ldots$, for which $\lim_n \|x^n\|_{w_p^n} = 0$ with norm

$$\|x\|_0 = \max_n \|x^n\|_{w_p^n}.$$

The construction and idea of the proof follow from [8], but the nature of these spaces is different. So for similar results, we omit the details of the proofs.
In fact these spaces are a rich class of spaces which depend on the sequences \( w = (w_i) \) and \( (p_n) \) as above. Fix a sequence \( w = (w_i) \) of reals which satisfies the above conditions and a sequence \( (p_n) \) of reals with \( \infty > p_1 > p_2 > \cdots > 1 \). Consider the sequence space \( X_p \) as above. For each \( n \geq 1 \), denote by \( (e_{i,n})_{i=1}^{\infty} \) the unit vector basis of \( \ell_{w,p} \) and by \( (e_{i,n})_{i=1}^{\infty} \) its natural copy in \( X_p \):

\[
e_{i,n} = \left( 0, \ldots, 0, \frac{e_{i,n}}{\|e_{i,n}\|_{w,p}}, 0, \ldots \right) \in X_p.
\]

Let \( \delta_n > 0 \) and \( \Delta = (\delta_n) \) such that

\[
\sum_{n=1}^{\infty} \delta_n^p = 1, \quad \text{if } p \geq 1,
\]

and \( \lim_n \delta_n = 0 \) and \( \max_n \delta_n = 1 \) if \( p = 0 \). For each \( i \geq 1 \) put

\[
z_i = \sum_{n=1}^{\infty} \delta_n e_{i,n}.
\]

Then

\[
\|z_i\|^p_p = \left( \sum_{n=1}^{\infty} \|\delta_n e_{i,n}\|_{w,p}^p \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} \delta_n^p \right)^{\frac{1}{p}} = 1,
\]

since \( \|e_{i,n}\|_{w,p} = 1 \) and

\[
\|z_i\|_0 = \max_n \|\delta_n e_{i,n}\|_{w,p} = 1.
\]

It is clear that for any sequence \( (t_i)_{i=1}^{m} \) of scalars,

\[
\left\| \sum_{i=1}^{m} t_i z_i \right\|_p^p = \sum_{n=1}^{\infty} \delta_n^p \left\| \sum_{i=1}^{m} t_i e_{i,n} \right\|_{w,p}^p, \quad \text{if } 1 \leq p < \infty,
\]

and

\[
\left\| \sum_{i=1}^{m} t_i z_i \right\|_0 = \max_n \left\| \sum_{i=1}^{m} t_i e_{i,n} \right\|_{w,p} = 1, \quad \text{if } p = 0.
\]

Let \( Z_p \) be the closed linear space of \( (z_i)_{i=1}^{\infty} \). For each \( I \subseteq \mathbb{N} \) the projection \( P_I \) denotes the natural projection of \( X_p \) on to \( [e_{i,n} : i \in \mathbb{N}, n \in I] \). Denote also \( Q_n = P_{\{n,n+1,\ldots\}} \).

We recall the main properties of \( \ell_{w,p} \) (\( 1 \leq p < \infty \)) and \( Z_1 \) spaces [3].
Theorem 1.1. For $1 \leq p < \infty$, $\ell(w, p)$ is hereditarily isometrically isomorphic to $\ell^p$,
\[
\left\| \sum_{i=1}^{n} t_i v_i \right\|_{w, p}^p = \sum_{i=1}^{n} |t_i|^p.
\]

Theorem 1.2. Let $w_i \geq 1$ for any $i \in \mathbb{N}$ and $w = (w_i)$. For $1 \leq p_{n+1} \leq p_n < \infty$, $\ell(w, p_{n+1}) \subseteq \ell(w, p_n)$. In particular $\|x\|_{w, p_n} \leq \|x\|_{w, p_{n+1}}$.

Theorem 1.3. $Z_1$ is a hereditarily $\ell_1$ Banach space which fails the Schur property.

2. The Result

Now, we show that $Z_p$ is hereditarily $\ell_p(c_0)$ for $p \in [1, \infty) \cup \{0\}$. But first, we collect some basic facts about our spaces in the following lemmas.

Lemma 2.1. Let $E_0$ be an infinite dimensional subspace of $Z_p$, $n, m, j \in \mathbb{N}(n > 1)$, and $\varepsilon > 0$. Then there are $\{x_i\}_{i=1}^{m} \subseteq E_0$ and $\{u_i\}_{i=1}^{m} \subseteq Z_p$ such that the $k$th component of $u_i$ is of the form
\[
u_{i,k} = \delta_k \sum_{s=j_1+1}^{j_{i+1}} a_{i,s} v_s,
\]
where $j = j_1 < j_2 < \cdots < j_{m+1}$. The $v_i$'s are obtained from the proof of Theorem 1.1, for $p = p_n$ such that
\[
\sum_{s=j_1+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} = 1, \quad \|u_i - x_i\| < \frac{\varepsilon}{m} \|u_i\|,
\]
for each $i = 1, \ldots, m$.

Proof. Put $E_1 = E_0 \cap [z_i]_{i=j+1}^{\infty}$. Since $E_0$ is infinite dimensional and $[z_i]_{i=j+1}^{\infty}$ has finite codimension in $Z_p$, $E_1$ is infinite dimensional as well. Put $j_1 = j$ and choose any $\bar{x}_1 \in E_1 \setminus \{0\}$ such that the $k$th component of $\bar{x}_1$ has the form
\[
\bar{x}_{1,k} = \delta_k \sum_{s=j_1+1}^{\infty} a_{1,s} v_s.
\]
Take $\bar{x}_1$ and use Lemma 2.2 of [S] to obtain $x_1$ and $u_1$ with above properties and continue the procedure of that lemma to construct the desired sequence. \qed

Lemma 2.2. Let $E_0$ be an infinite dimensional subspace of $Z_p$, $j, n \in \mathbb{N}$, and $\varepsilon > 0$. Then, there exist $x \in E_0$, $x \neq 0$, and $u \in Z_p$ such that
(i) $\|Q_n u\| \geq (1 - \varepsilon) \|u\|$, 
(ii) $\|u\|_{E_0, p} \neq \|u\|_{E_0, p}$.
(ii) $\|x - u\| < \varepsilon \|u\|$. 

Proof. Choose $m \in \mathbb{N}$ so that 

$$\frac{1}{\delta_m} m^{\frac{1}{p_n} - \frac{1}{p_n}} < \varepsilon.$$ 

Using Lemma 2.1, choose $\{x_i\}_{i=1}^m \subseteq E_0$ and $\{u_i\}_{i=1}^m \subseteq Z_p$ so that satisfy the claims of lemma and put 

$$x = \sum_{i=1}^m x_i \text{ and } u = \sum_{i=1}^m u_i.$$ 

First, we prove (ii). We know that $\|u_i\| \leq \|u\|$ for $i = 1, \ldots, m$ and 

$$\|x - u\| \leq \sum_{i=1}^m \|x_i - u_i\| < \sum_{i=1}^m \frac{\varepsilon \|u_i\|}{m} < \sum_{i=1}^m \frac{\varepsilon \|u\|}{m} = \varepsilon \|u\|.$$ 

To prove (i), we first show that 

$$\|u\| - \|Q_n u\| < m^{\frac{1}{p_n - 1}}.$$ 

Anyway, $\|u\| - \|Q_n u\| \leq \|p_{\{1, \ldots, n-1\}} u\|$. Hence, by Theorem 1.1 and Theorem 1.2 for $p \geq 1$, we have 

$$(\|u\| - \|Q_n u\|)^p \leq \sum_{k=1}^{n-1} \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s^{p_{w.p_k}} \left( m \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{w.p_{n-1}}} \right)^{\frac{p}{p_{n-1}}} \leq \sum_{k=1}^{n-1} \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s^{p_{w.p_{n-1}} \left( m \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{w.p_{n-1}}} \right)^{\frac{p}{p_{n-1}}}}$$ 

$$= \sum_{k=1}^{n-1} \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s^{p_{w.p_{n-1}}} \left( m \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{w.p_{n-1}}} \right)^{\frac{p}{p_{n-1}}}$$ 

$$= \sum_{k=1}^{n-1} \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s^{p_{w.p_{n-1}}} \left( m \sum_{i=1}^{j_{i+1}} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{w.p_{n-1}}} \right)^{\frac{p}{p_{n-1}}}$$
\[ \frac{p}{m} \sum_{k=1}^{n-1} \delta_k^p \]

\[ < \frac{p}{m} \sum_{k=1}^{n-1} \delta_k^p. \]

And for \( p = 0 \), we have

\[ \|u\| - \|Q_n u\| \leq \max_{1 \leq k < n} \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_i+1} a_{i,s} v_s \right\| \]

\[ \leq \max_{1 \leq k < n} \left( \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_i+1} |a_{i,s}|^{p-1} \right)^{\frac{1}{p-1}} \]

\[ = \max_{1 \leq k < n} \left( \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_i+1} 1 \right)^{\frac{1}{p-1}} \]

\[ = \max_{1 \leq k < n} \delta_k \]

\[ \leq \frac{p}{m} \sum_{k=1}^{n-1} \delta_k^p. \]

On the other hand, for \( p \geq 1 \), we have

\[ \|u\|^p \geq \delta_n^p \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_i+1} a_{i,s} v_s \right\|^p \]

\[ = \delta_n^p \left( \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_i+1} |a_{i,s}|^{p-1} \right)^{\frac{p}{p-1}} \]

\[ \geq \delta_n^p \left( \sum_{i=1}^{m} \left( \sum_{s=j_i+1}^{j_i+1} |a_{i,s}|^{p-1} \right)^{\frac{p}{p-1}} \right)^{\frac{p}{p-1}} \]

\[ = \delta_n^p \left( \sum_{i=1}^{m} 1 \right)^{\frac{p}{p-1}} \]

\[ = \delta_n^p \frac{p}{m} \sum_{k=1}^{n-1} \delta_k^p. \]
And for $p = 0$, we can write

$$
\|u\| = \max_{k \in \mathbb{N}} \left\| \sum_{j=1}^m \sum_{s=j_i+1}^{j_{i+1}} a_{i,s}v_s \right\|_{w.p_k} \\
\geq \delta_n \left( \sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n} \right)^{\frac{1}{p_n}} \\
= \delta_n \left( \sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n} \right)^{\frac{1}{p_n}} \\
= \delta_n \left( \frac{1}{m} \right)^{\frac{1}{p}} \\
= \delta_n m^{\frac{1}{p}}.
$$

Thus, anyway $\|u\| \geq \delta_n m^{\frac{1}{p}}$, and hence,

$$
1 - \frac{\|Q_n u\|}{\|u\|} \leq \frac{m^{\frac{1}{p_n}-1}}{\delta_n m^{\frac{1}{p}}} = \frac{1}{\delta_n m^{\frac{1}{p_n}}-1} < \varepsilon,
$$

and $\|Q_n u\| \geq (1 - \varepsilon) \|u\|$. □

The following theorem is the main result of this paper:

**Theorem 2.3.** (i) The Banach space $Z_p$ is hereditarily $\ell_p$ for $1 \leq p < \infty$.

(ii) The space $Z_0$ is hereditarily $c_0$.

**Proof.** For the proof of our main results we need the following Lemma from Popov [8] (see Lemma 2.4 and Theorem 2.5). □

**Lemma 2.4.** Suppose $\varepsilon > 0$ and $\varepsilon_s$ for $s \in \mathbb{N}$ are such that

(i) $2\varepsilon_s \leq \varepsilon$ if $p = 1$,

(ii) $\sum_{s=1}^{\infty} (2\varepsilon_s)^q \leq \varepsilon^q$ if $1 \leq p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$,

(iii) $\sum_{s=1}^{\infty} (2\varepsilon_s) \leq \varepsilon$ if $p = 0$. 

If for given vectors \( \{u_s\}_{s=1}^{\infty} \subset S(Z_p) \), where \( Z_p = Z_p(P) \), there is a sequence of integers \( 1 \leq n_1 < n_2 < \cdots \) such that, for each \( s \in \mathbb{N} \), one has

\[
\begin{align*}
(i) \quad & \|u_s - Q_n u_s\| \leq \varepsilon_s, \\
(ii) \quad & \|Q_{n+1} u_s\| \leq \varepsilon_s,
\end{align*}
\]

then \( \{u_s\}_{s=1}^{\infty} \) is \((1 + \varepsilon)(1 - 3\varepsilon)^{-1}\)-equivalent to the unit vector basis of \( \ell_p \) (respectively, \( c_0 \)).

**Theorem 2.5.** The Banach space \( Z_p = Z_p(P) \) is hereditarily \( \ell_p \) if \( 1 \leq p < \infty \) and is hereditarily \( c_0 \) if \( p = 0 \).

The proofs of Lemma 2.4 and Theorem 2.5 are based on the definition of \( Q_i \) and the norm on \( Z_p \). In fact by the conditions of this lemma and for any sequence \( (a_s)_{s=1}^{m} \) of scalars, it follows that

\[
(1 - 3\varepsilon) \left( \sum_{s=1}^{m} |a_s|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{s=1}^{m} a_s u_s \right\| \leq (1 + \varepsilon) \left( \sum_{s=1}^{m} |a_s|^p \right)^{\frac{1}{p}},
\]

for \( 1 \leq p < \infty \), and

\[
(1 - 3\varepsilon) \max_{1 \leq s < m} |a_s| \leq \left\| \sum_{s=1}^{m} a_s u_s \right\| \leq (1 + \varepsilon) \max_{1 \leq s < m} |a_s|,
\]

for \( p = 0 \). Then by using the stability properties of the bases (5, p. 5) and Lemma 2.2, we conclude the proofs.

**Definition 2.6.** Let \( X \) be an arbitrary Banach space. Then

a) \( X \) has the nowhere Schur property if \( X \) contains no infinite dimensional closed subspace with the Schur property.

b) \( X \) has the nowhere dual Schur property if \( X \) contains no infinite dimensional closed subspace such that its dual has the Schur property.

**Definition 2.7.** A Banach space \( X \) has the Schur property if every weak convergent sequence is norm convergent.

Theorem 1.3 in this paper and Theorem 1.3 of [5] have the following consequence.

**Theorem 2.8.** \( Z_1 \) possesses the nowhere dual Schur property.

### 3. Operators

**Definition 3.1.** Let \( X \) and \( Y \) be any of the spaces \( \ell_{w,p} \) (\( 1 \leq p < \infty \)), or \( Z_p \) (\( 1 \leq p < \infty \)) with their natural bases \( (x_n)_{n=1}^{\infty} \) and \( (y_n)_{n=1}^{\infty} \), respectively. The formal (possibly unbounded) operator \( T : X \rightarrow Y \) which extends by linearity and continuity the equality \( Tx_n = y_n \) is called the natural operator from \( X \) to \( Y \).
Theorem 3.2. Let \( p \in [1, \infty) \) and \( p_1 > p_2 > \cdots > 1 \).

(i) If \( \inf_n p_n < p \), then the natural operator from \( \ell_{w,p} \) to \( Z_p \) is unbounded.

(ii) If \( \inf_n p_n \geq p \), then the natural operator from \( Z_p \) to \( \ell_{w,p} \) is unbounded.

Proof. For constant scalars \( a_1 = a_2 = \cdots = a_m = 1 \), we have

\[
\left\| \sum_{i=1}^{m} z_i \right\|_p = \sum_{n=1}^{\infty} \delta_n^p \left\| \sum_{i=1}^{m} e_{i,n} \right\|_{w,p_n} = \sum_{n=1}^{\infty} \delta_n^p m^{\frac{p}{p_n}},
\]

if \( 1 \leq p < \infty \).

On the other hand,

\[
\left\| \sum_{i=1}^{m} e_i \right\|_{w,p} = m,
\]

if \( 1 \leq p < \infty \).

Therefore, for \( 1 \leq p < \infty \), we have

\[
\|T\|^p \geq \frac{\left\| \sum_{i=1}^{m} Te_i \right\|_p^p}{\left\| \sum_{i=1}^{m} e_i \right\|_{w,p}^p} = \frac{\sum_{i=1}^{m} z_i^p}{\sum_{i=1}^{m} e_i^p} = \sum_{n=1}^{\infty} \delta_n^p m^{\frac{p}{p_n} - 1}.
\]

If \( \inf_n p_n < p \), then there exists \( n_0 \) such that \( p_{n_0} < p \), and hence,

\[
\|T\|^p \geq \sum_{n=1}^{\infty} \delta_n^p m^{\frac{p}{p_n} - 1} \geq \delta_{n_0}^p m^{\frac{p}{p_{n_0}} - 1} \to \infty,
\]

as \( m \to \infty \).

Now assume that \( \inf_n p_n \geq p \). In this case, we have \( \frac{p}{p_n} - 1 < 0 \) for each \( n \). Given \( \varepsilon > 0 \), we choose \( n_0 \) so that

\[
\sum_{n=n_0}^{\infty} \delta_n^p < \frac{\varepsilon}{2}.
\]

Then we choose \( m_0 \) such that

\[
\left( \max_{1 \leq n \leq n_0} \delta_n \right)^p m^{\frac{p}{p_{n_0}} - 1} \leq \frac{\varepsilon}{2n_0},
\]

for \( m \geq m_0 \). So, for such \( m \), we have

\[
\|T\|^p \geq \frac{1}{\sum_{n=1}^{\infty} \delta_n^p m^{\frac{p}{p_n} - 1}}.
\]
\[ \sum_{n=1}^{n_0} \delta_n^p m^{\frac{p}{m}-1} + \sum_{n=n_0+1}^{\infty} \delta_n^p m^{\frac{p}{m}-1} \geq \sum_{n=1}^{n_0} \left( \max_{1 \leq n < n_0} \delta_n \right)^p m^{\frac{p}{m}-1} + \sum_{n=n_0+1}^{\infty} \delta_n^p m^{\frac{p}{m}-1} \geq \frac{1}{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}} \to \infty, \]
as \( m \to \infty \).

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