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Fuzzy Best Simultaneous Approximation of a Finite Numbers of Functions

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Abstract. Fuzzy best simultaneous approximation of a finite number of functions is considered. For this purpose, a fuzzy norm on $C(X, Y)$ and its fuzzy dual space and also the set of subgradients of a fuzzy norm are introduced. Necessary case of a proved theorem about characterization of simultaneous approximation will be extended to the fuzzy case.

1. Introduction

In this paper, we consider fuzzy normed spaces in the sense of Cheng and Mordeson [3].

Recently, S.M. Vaezpour and et. al. have introduced the concept of t-best approximation and best simultaneous approximation in fuzzy normed spaces in [4, 8] and proved several theorems pertaining to these spaces.

Here, fuzzy best simultaneous approximation of a finite number of functions is considered.

The organization of the paper is as follows. Section 2 comprises the preliminaries on fuzzy normed space. In Section 3 by using of the relation between seminorms and fuzzy normed spaces, some fuzzy norms on the linear space of continuous functions and its dual space have been made. In Section 4, we prove some results by using of the notion of fuzzy subgradient and deduce the main Theorem of paper about fuzzy simultaneously approximating of a finite number of functions.

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2. Preliminaries

According to [7], a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for every $a \in [0,1]$;
(iv) $a * b \leq c * d$

whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$. Three examples of continuous t-norms are $\wedge, ., *_L$(the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}, a.b = ab, a *_L b = \max\{a + b - 1, 0\}$, respectively. Recall that $*_L \leq . \leq \wedge$. In fact, $* \leq \wedge$ for every t-norm $*$.

Definition 2.1 ([8]). Let Y be a linear space over real numbers. A fuzzy subset N of $Y \times \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is called a fuzzy norm on Y if for all $x, y \in Y$ and $s, t > 0$, the following conditions are satisfied:

(N1) $N(x, 0) = 0$;
(N2) $N(x, t) = 1$ for all $t > 0$ iff $x = 0$;
(N3) $N(sx, t) = N\left(x, \frac{t}{|s|}\right)$ if $s \neq 0$;
(N4) $N(x + y, s + t) \geq N(x, s) \wedge N(y, t)$;
(N5) $N(x, .) : [0,) \to [0, 1]$ is continuous;
(N6) $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(Y, N, *)$, will be referred to as a fuzzy normed linear space.

If $(Y, N, *)$ is a fuzzy normed space, the open ball $B_N(x, r, t)$ with center x, radius r, $0 < r < 1$ and $t > 0$ is defined as follows:

$$B_N(x, r, t) = \{y \in Y : N(y - x, t) > 1 - r\}.$$

We note that $B_N(x, r, t) = x + B_N(0, r, t)$, for all $x \in X$ and $0 < r < 1$, $t > 0$. The closed ball $\bar{B}_N(x, r, t)$ with center x, radius r, $0 < r < 1$ and $t > 0$ is defined as follows :

$$\bar{B}_N(x, r, t) = \{y \in Y : N(y - x, t) \geq 1 - r\}.$$

If $(Y, N, *)$ is a fuzzy normed space, the fuzzy set M_N in $Y \times Y \times [0, \infty)$ given by $M_N(x, y, t) = N(y - x, t)$ is a fuzzy metric on Y in the sense of Kramosil and Michálek [5]. This fuzzy metric induces a topology on Y, which has as a base of the collection $\{B_N(x, r, t) : x \in Y, 0 < r < 1, t > 0\}$. It is well known that a fuzzy normed space is a topological vector space and Hausdorff locally convex space. We need the following propositions which is proved by Bag and Samanta in Theorems 2.1, 2.2 of [2]:

Proposition 2.2. *Let $(Y, N, \wedge)$ be a fuzzy normed space and let $\alpha \in (0, 1)$. Then the following hold:*

(a) *The function* $\|.\|_\alpha \to [0,\infty)$ *given by*

$$\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\},$$

is a seminorm. In fact, it is the Minkowski functional of the ball $\bar{B}(0, 1-\alpha, 1)$.

(b) *The family* $\{\|.\|_\alpha : \alpha \in (0,1)\}$ *is separating and ascending.*

This family will be called the seminorms corresponding to the fuzzy norm $(N, \wedge)$.

Proposition 2.3. *Let* $\{\|.\|_\alpha : \alpha \in (0,1)\}$ *be an ascending family of separating seminorms on a real linear space* Y*, and let* $\|x\|_0 = 0$*, for all* $x \in Y$*. Then, the pair* $(N, \wedge)$ *is a fuzzy norm on* Y*,where* $N : Y \times [0,\infty) \to [0,1]$ *is given by* $N(x, 0) = 0$*, for all* $x \in Y$ *and*

$$N(x,t) = \sup\{\alpha \in [0,0) : \|x\|_\alpha < t\},$$

for all $x \in Y$ *and* $t > 0$.

It is clear that $\|.\|_\alpha$ is the corresponding family of seminorms to the $(N, \wedge)$. We recall that a family $\{\|.\|_\alpha : \alpha \in (0,1)\}$ of seminorms is said to be separating if $\|x\|_\alpha = 0,\ \forall\ \alpha \in (0,1)$, then $x = 0$.

3. Fuzzy Space of Functions

In the remainder of paper, we suppose that X be a Hausdorff compact space and $C(X,Y)$ be the set of all continuous mappings from X to the fuzzy normed space (Y, N).

Proposition 3.1. *Let* X *be a Hausdorff compact space. Let* Y *be a fuzzy normed space with fuzzy norm* N *and* $f \in C(X,Y)$*. Define*

$$\|f\|_\alpha^A = \max_{x\in D} \inf\{t : N(f(x),t) > \alpha\} = \max_{x\in D} \|f(x)\|_\alpha,$$

where

$$D = f^{-1}(\bar{B}_N(0, 1-\alpha, 1)).$$

Then $\left\{\|.\|_\alpha^A : \alpha \in (0,1)\right\}$ *is an ascending family of separating seminorms on* $C(X,Y)$.

Proof. Since N is a fuzzy norm on Y, by Proposition 2.2, $\|f(x)\|_\alpha$ is an ascending family of seminorms on $Y, \forall x \in X,\ \alpha \in (0,1)$. Therefore $\|f\|_\alpha^A$ is a seminorm on $C(X,Y)$. Now, if $\|f\|_\alpha^A = 0$ for all $\alpha \in (0,1)$, then $\|f(x)\|_\alpha \leq \|f\|_\alpha^A = 0$. So, for all $\alpha \in (0,1); \|f(x)\|_\alpha = 0$. By Proposition 2.2 , $\|.\|_\alpha$ is separating so that $f(x) = 0,\ \forall x \in X$. That is the family $\left\{\|.\|_\alpha^A : \alpha \in (0,1)\right\}$ is separating. If $\alpha \leq \beta$ then $1-\beta \leq 1-\alpha$ and then $\bar{B}_N(0, 1-\alpha, 1) \subseteq \bar{B}_N(0, 1-\beta, 1), \|f\|_\alpha^A \leq \|f\|_\beta^A$. That is $\left\{\|.\|_\alpha^A : \alpha \in (0,1)\right\}$ is an ascending family of seminorms on $C(X,Y)$. $\square$

Corollary 3.2. *The pair* $(N_A, \wedge)$ *is a fuzzy norm on* $C(X,Y)$, *where* $N_A : C(X,Y) \times [0,\infty) \to [0,1]$ *is given by* $N_A(f,0) = 0$ *and* $N_A(f,t) = \sup\left\{\alpha \in [0,1) : \|f\|_\alpha^A < t\right\}, \ \forall\ f \in C(X,Y)$.

Proof. Corollary follows from the above Proposition and Proposition 2.3. □

Denote by $C^*(X,Y)$ the set of all continuous linear mappings from the pair $(C(X,Y), N_A)$ to $\mathbb{R}$.

Proposition 3.3. *Let* N_A *be the fuzzy norm on* $C(X,Y)$ *and* $\|.\|_\alpha^A$ *be the family of seminorms on* $C(X,Y)$ *as Proposition 3.1. For each* $l \in C^*(X,Y)$, *define* $\|l\|_0^{*A} = 0$ *and*

$$\|l\|_\alpha^{*A} = \sup\left\{|l(f)| : \|f\|_\alpha^A \le 1\right\},$$

whenever $\alpha \in (0,1)$. *Then,* $\left\{\|.\|_\alpha^{*A} : \alpha \in (0,1)\right\}$ *is an ascending family of separating seminorms on* $C^*(X,Y)$ *which we call it dual seminorms.*

Proof. It is easy to show that $\|.\|_\alpha^{*A}$ is a family of seminorms on $C^*(X,Y)$. If $\|l\|_\alpha^{*A} = 0$ for all $\alpha \in (0,1)$, then $|l(f)| = 0$ for all $f \in U(1-\alpha,1)$, where $U(1-\alpha,1) = \{f \in C(X,Y) : \|f\|_{1-\alpha} \le 1\}$. Since $U(1-\alpha,1)$ is absorbent [1], for given $f \in C(X,Y)$, there exists $k > 0$ such that $kf \in U(1-\alpha,1)$. Then, $l(kf) = kl(f) = 0$ and so $l(f) = 0$. Thus $\|.\|_\alpha^{*A}$ is separating.
Since $\|.\|_\alpha^A$ is ascending, therefore if $\alpha \le \beta$, then $\|f\|_\alpha^A \le \|f\|_\beta^A$ and then $\|l\|_\alpha^* \le \|l\|_\beta^*$. Therefore, $\left\{\|.\|_\alpha^{*A} : \alpha \in (0,1)\right\}$ is an ascending family of seminormes on $C^*(X,Y)$. □

Similarly to the Corollary 3.2 we have:

Corollary 3.4. *The pair* $(N_A^*, \wedge)$ *is a fuzzy norm on* $C^*(X,Y)$, *where* $N_A^* : C^*(X,Y) \times [0,\infty) \to [0,1]$ *is given by* $N_A^*(l,0) = 0$, *and* $N_A^*(l,t) = \sup\left\{\alpha \in [0,1) : \|l\|_\alpha^{*A} < t\right\}, \ \forall\ l \in C^*(X,Y)$.

The family of seminorms corresponding to the pair $(N_A^*, \wedge)$ is $\|.\|_\alpha^{*A}$. Since $(C(X,Y), N_A)$ is a separated locally convex space then the polar of any neighborhood in $(C(X,Y), N_A)$ is compact in the weak* topology. So that we have:

Corollary 3.5. *Let* $(C(X,Y), N_A)$ *and* $(C^*(X,Y), N_A^*)$ *are the above fuzzy normed spaces. Then theorem of Alaoglu-Bourbaki be hold.*

Finally, we need to the following easy remark.

Remark 3.6 ([1])**.** Let $(Y, N, \wedge)$ be a fuzzy normed space. Let $\{\|.\| : \alpha : \alpha \in (0,1)\}$ be the seminorms corresponding to the fuzzy norm $(N, \wedge)$.

(a) If $\|x\|_\alpha < t$, then $N(x,t) \ge \alpha$.
(b) If $N(x,t) > \alpha$, then $\|x\|_\alpha < t$, where $x \in Y$.

4. Fuzzy Best Simultaneous Approximation

Here, we continue to use of notions and notations in the previous section.

Definition 4.1 ([4]). Let $(Y, N, \wedge)$ be a fuzzy normed space. A subset $A \subseteq Y$ is called F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $N(x, t) > 1 - r$ for all $x \in A$.

Definition 4.2 ([4]). Let $(Y, N, *)$ be a fuzzy normed space over real numbers, W be a subset of Y and S be a F-bounded subset in Y. For $t > 0$, we define,

$$d(S, W, t) = \sup_{s \in S} \inf_{w \in W} N(s - w, t), \ d(S, w, t) = \sup_{s \in S} N(s - w, t), \ w \in W.$$

An element $w_0 \in W$ is called a t-best simultaneous approximation to S from W if for $t > 0$,

$$d(S, W, t) = \sup_{s \in S} N(s - w_0, t).$$

The set of all t-best simultaneous approximation to S from W will be denoted by $S_W^t(S)$ and we have,

$$S_W^t(S) = \left\{ w \in W : \sup_{s \in S} N(s - w, t) = d(S, W, t) \right\}.$$

For any $\mu_1, \ldots, \mu_n$ in $(C(X, Y), N_A)$, let

$$S_\mu = \left\{ \sum_{i=1}^{n} a_i \mu_i : \|a\|_B = 1 \right\},$$

where $\|.\|_B$ is that a given norm on $\mathbb{R}^n$ and $a = (a_1, \ldots, a_n)^T$.

Now suppose that functions $F_1, \ldots, F_n$ in $(C(X, Y), N_A)$ are given, S_F is defined as above and W is an m-dimensional subspace of $(C(X, Y), N_A)$. For $t > 0$,

$$d(S_F, W, t) = \max_{s \in S_F} \inf_{w \in W} N_A(s - w, t).$$

Note that if $a \in \mathbb{R}^n$, $\|a\|_B = 1$, then N_A is attained.

We need to introduce the subdifferential or set of subgradients of N_A at any element of $(C(X, Y), N_A)$. This is the set defined at the point $f \in (C(X, Y), N_A)$ by

$$\begin{aligned} \partial N_A(f) = \{ & l \in (C^*(X, Y), N_A^*) : N_A\left(f, \|F\|_\alpha^A - l(F - f)\right) \geq \alpha, \\ & \forall\ F \in C(X, Y),\ \forall \alpha \in (0, 1) \}. \end{aligned}$$

From this, the subgradients of seminorms, corresponding to N_A can be defined as follows :

$$\partial\|f\|_\alpha^A = \left\{l \in C^*(X,Y) : l(F-f) + \|f\|_\alpha^A \leq \|F\|_\alpha^A,\right.$$
$$\left.\forall F \in C(X,Y)\right\}, \quad \alpha \in (0,1).$$

For more detail about subdifferential see, for example, Rockafellar [6].

Proposition 4.3. *Let $f \in (C(X,Y), N_A), \alpha \in (0,1)$. Then $l \in \partial\|f\|_\alpha^A$ if and only if*

(i) $l(f) = \|f\|_\alpha^A$.
(ii) $\|l\|_\alpha^{*A} \leq 1$, *where,* $\|.\|_\alpha^{*A}$, *where*

$$\|l\|_\alpha^{*A} = \max\{|l(f)| : \|f\|_\alpha^A \leq 1\}.$$

Proof. Let $l \in \partial\|f\|_\alpha^A$, then for each $F \in (C(X,Y), N_A)$

$$l(F-f) + \|f\|_\alpha^A \leq \|F\|_\alpha^A.$$

If $F = 2f$, then we have $l(f) \leq \|f\|_\alpha^A$ and if we get $F = \frac{1}{2}f$ then we have $\|f\|_\alpha^A \leq l(f)$. These together implies $l(f) = \|f\|_\alpha^A$. So (i) is follows. For (ii), let $F \in (C(X,Y), N_A)$ and $\|F\|_\alpha^A \leq 1$. By (i), we have $l(f) = \|f\|_\alpha^A$. Therefore

$$\begin{aligned} l(F) &= l(F - f + f) \\ &= l(F-f) + l(f) \\ &= l(F-f) + \|f\|_\alpha^A \\ &\leq \|F\|_\alpha^A - \|f\|_\alpha^A + \|f\|_\alpha^A \\ &\leq 1 - 0 \\ &= 1. \end{aligned}$$

So, $\max_{\|F\|_\alpha^A \leq 1} l(F) = \|l\|_\alpha^* \leq 1$.

Now, let (i) and (ii) hold for $l \in (C^*(X,Y), N_A^*)$ and $f \in (C(X,Y), N_A)$ and $F \in C(X,Y)$ be arbitrary.

Then

$$\begin{aligned} l\left(\frac{F}{\|F\|_\alpha^A}\right) &= l\left(\frac{F}{\|F\|_\alpha^A} - \frac{f}{\|F\|_\alpha^A} + \frac{f}{\|F\|_\alpha^A}\right) \\ &= l\left(\frac{F}{\|F\|_\alpha^A} - \frac{f}{\|F\|_\alpha^A}\right) + \frac{1}{\|F\|_\alpha^A} l(f) \\ &= \frac{1}{\|F\|_\alpha^A} l(F-f) + \frac{\|f\|_\alpha^A}{\|F\|_\alpha^A} \\ &\leq 1. \end{aligned}$$

Last inequality is holds by (ii). Therefore $l(F-f) + \|f\|_\alpha^A \leq \|F\|_\alpha^A$. $\square$

Corollary 4.4. *Let* $f \in (C(X,Y), N_A)$ *and* $l \in (C^*(X,Y), N_A^*)$ *such that* $\forall\alpha \in (0,1), \|l\|_\alpha^* \leq 1$. *Then* $N_A(f,t) \geq l(f), t > 1$.

Proof. Under the assumptions,definition of $N(f,t)$ in Corollary 3.2 is especial case of Proposition 3.3. □

For a given n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of functions in $(C(X,Y), N_A)$ define the set

$$K_t(\mu) = \Big\{(a,l) : a \in \mathbb{R}^n, \|a\|_B = 1, \sum_{i=1}^n a_i\mu_i = d(S_\mu, 0, t)v,$$
$$l \in \partial\|v\|_\alpha^A, \|v\|_\alpha^A = 1\Big\}.$$

The following characterization of the directional derivatives of $d(S,0,t)$ generalizes a result for functions as well for matrices given in [9, 10].

Proposition 4.5. *Let* $\mu_1, \ldots, \mu_n, \eta_1, \ldots, \eta_n$ *be any elements in* $(C(X,Y), N_A)$. *Then*

$$\lim_{z\to 0^+} \frac{d(S_{\mu+z\eta}, 0, t) - d(S_\mu, 0, t)}{z} = \max_{(a,l)\in K_t(\mu)} l\left(\sum_{j=1}^n a_j\eta_j\right).$$

Proof. For any real z, and for any $(a(z), l(z)) \in K_t(\mu + z\eta)$,

$$\begin{aligned}
d(S_\mu, 0, t) &= \max_{s\in S_\mu} N_A(s,t) \\
&= \max_{\|a\|_B=1} N_A\left(\sum_{j=1}^n a_j\mu_j, t\right) \\
&\geq N_A\left(\sum_{j=1}^n a_j(z)\mu_j, t\right) \\
&\geq l(z)(\sum_{j=1}^n a_j(z)\mu_j) \\
&= l(z)\left(\sum_{j=1}^n a_j(z)(\mu_j + z\eta_j) - z\sum_{j=1}^n a_j(z)\eta_j\right) \\
&= d(S_{\mu+z\eta}, 0, t) - zl(z)\left(\sum_{j=1}^n a_j(z)\eta_j\right),
\end{aligned}$$

where second inequality is holds by Corollary 4.4.

Also, for any $(a,l) \in K_t(\mu)$

$$\begin{aligned} d(S_{\mu+z\eta},0,t) &= \max_{s\in S_{\mu+z\eta}} N_A(s,t) \\ &= \max_{\|a\|_B=1} N_A\left(\sum_{j=1}^{n} a_j(\mu_j+z\eta_j), t\right) \\ &\geq N_A\left(\sum_{j=1}^{n} a_j(\mu_j+z\eta_j), t\right) \\ &\geq l\left(\Sigma_{j=1}^{n} a_j(\mu_j+z\eta_j)\right) \\ &= d(S_\mu,0,t) + zl\left(\sum_{j=1}^{n} a_j\eta_j\right). \end{aligned}$$

It follows that for all $z > 0$, and all $(a,l) \in K_t(\mu), (a(z), l(z)) \in K_t(\mu+z\eta)$,

$$\begin{aligned} l\left(\sum_{j=1}^{n} a_j\eta_j\right) &\leq \frac{d(S_{\mu+z\eta},0,t) - d(S_\mu,0,t)}{z} \\ &\leq l(z)\left(\sum_{j=1}^{n} a_j(z)\eta_j\right). \end{aligned}$$

If z tends to zero, and uses the weak* compactness of the polar U^0 in the dual space by Corollary 3.5, the result follows. □

Let

$$\mu(w) = (\mu_1(w), \ldots, \mu_n(w)),$$

where

$$\mu_i(w) = F_i - w, i = 1, \ldots, n, \text{ for all } w \in W,$$

and let $L(w)$ denote the set of n-tuples $\{(g_1, \ldots, g_n)\}$ of elements in $(C^*(X,Y), N_A^*)$ defined by

$$L(w) = conv\{(a_1 l, \ldots, a_n l), (a,l) \in K_t(\mu(w))\},$$

where "conv" is used to denotes the convex hull. Note that

$$\begin{aligned} \sum_{i=1}^{n} g_i(\mu_i(w)) &= \sum_{i=1}^{n} a_i l(F_i - w) \\ &= l\left(\sum_{i=1}^{n} a_i F_i - w\right) \\ &= d\left(S_{\mu(w)},0,t\right), \quad \forall g \in L(w). \end{aligned} \tag{4.1}$$

The following Proposition is true, in both cases :"if and only if". "if" case is holds by Proposition 4.5 and Corollary 3.5 (Alaoglu-Bourbaki theorem) and no need to change in fuzzy case (see [9]).

Proposition 4.6. *Let there exists* $g = (g_1, \ldots, g_n) \in L(w_0)$ *such that*

$$\sum_{i=1}^{n} g_i(w) = l\left(\sum_{i=1}^{n} a_i(F_i - w)\right) = 0, \quad \text{for all } w \in W. \tag{4.2}$$

Then $w_0 \in S_W^t(S_F)$.

Proof. Let the conditions be satisfied at w_0 and let w be any element of W. Then

$$\begin{aligned} d(S_F, w, t) &= d\left(S_{\mu(w)}, 0, t\right) \\ &= \max_{\|a\|_B = 1} N_A\left(\sum_{i=1}^{n} a_i(F_i - w), t\right) \\ &\geq N_A\left(\sum_{i=1}^{n} a_i(F_i - w), t\right) \\ &\geq l\left(\sum_{i=1}^{n} a_i(F_i - w)\right), \quad \forall (a, l) \in K_t(\mu(w_0)). \end{aligned}$$

Suppose that $g \in L(w_0)$ satisfies (4.2). Then, by (4.1)

$$\begin{aligned} d(S_F, w, t) &\geq \max_{\|a\|_B = 1} N_A\left(\sum_{i=1}^{n} a_i(F_i - w), t\right) \\ &\geq N_A\left(\sum_{i=1}^{n} a_i(F_i - w), t\right) \\ &\geq \sum_{i=1}^{n} g_i(F_i - w - w_0 + w_0) \\ &= \sum_{i=1}^{n} g_i(F_i - w_0) + \sum_{i=1}^{n} g_i(F_i - w + w_0) \\ &= d(S_F, w_0, t). \end{aligned}$$

The proof is compelete. □

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References

1. C. Alegre and S. Romaguera, *The Hahn-Banach Extension Theorem for Fuzzy Normed Spaces Revisited*, Abstr. Appl. Anal., (2014), article ID 151472.
2. T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces,* Journal of Fuzzy Mathematics, 11 (2003), pp. 687-705.
3. S.C. Cheng and J.N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., 86 (1994), pp. 429-436.
4. M. Goudarzi and V.M. Vaezpour , *Best Simultaneous approximation in fuzzy normed spaces*, Iranian J. of Fuzzy Syst., 7 (2010), pp. 87-96.
5. I. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11 (1975), pp. 336-344.
6. R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, NJ, (1970).
7. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pac. J. Math., 10 (1960), pp. 313-334.
8. V.M. Vaezpour and F. Karimi, *t-best approximation in fuzzy normed spaces*, Iranian J. of Fuzzy Syst., 5 (2008), pp. 93-99.
9. G.A. Watson, *A Characterization of Best Simultaneous Approximations*, J. of Approximation Theory, 75 (1993), pp. 175-182 .
10. G.A. Watson, *Characterization of subdifferential of some matrix norm*, Linear Algebra Appl., 170 (1992), pp. 33-45.

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