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Common Fixed Point in Cone Metric Space for $\mathbf{s}-\varphi$-contractive

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Abstract. Huang and Zhang [4] have introduced the concept of cone metric space where the set of real numbers is replaced by an ordered Banach space. Shojaei [9] has obtained points of coincidence and common fixed points for s-Contraction mappings which satisfy generalized contractive type conditions in a complete cone metric space.

In this paper, the notion of complete cone metric space has been introduced. We have defined $s-\varphi$-contractive and obtained common fixed point theorem for a mapping f, s which satisfies $s-\varphi$-contractive.

1. Introduction

Huang and Zhang [4] have introduced the concept of cone metric space where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 12] have generalized the results of Huang and Zhang [4] and studied the existence of common fixed points of a pair of self-mappings satisfying a contractive type condition in the framework of normal cone metric spaces. In [3] Bari and Vetro obtained some results on the points of coincidence and common fixed points in non-normal cone metric spaces. Shojaei [9] obtained points of coincidence and common fixed points for $s-\textit{contraction}$ mappings satisfying generalized contractive type conditions in a complete cone metric space.

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In 1969, Boyd and Wong [2] have introduced the notion of $\varphi contraction$. Generalization of the above Banach contraction principle has been a heavily investigated branch research, (see, e.g., [2, 10, 4]). In 2003, Kirk et. al., [7] introduced the notion of cyclic representation.

We have introduced the notion of $s-\varphi$-contractive mappings in a con metric space and proved some propositions.
Throughout this paper, we have denoted by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers and E will be a Real Banach Space.

Definition 1.1. Suppose E is a Real Banach Space and P is a subset of E. P is called a *Cone* if and only if:

(i) P is nonempty, closed and satisfies $P \neq \{0\}$,
(ii) If $a, b \in \mathbb{R}$, such that, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$,
(iii) If $x \in P$ and $-x \in P$ then $x = 0$.

For a Cone $P \subseteq E$, we defined a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ iff $x \preceq y$ and $x \neq y$, and $x \ll y$ iff $y - x \in intP$, where $intP$ is the interior of P. From now on, it is assumed that $intP \neq \emptyset$. The cone P is called normal if there is a number $K \geq 1$ such that, for all $x, y \in E$, $0 \preceq x \preceq y$ implies $\|x\| \preceq K \parallel y \parallel$.

Here, the least positive integer K satisfying this inequality is called the normal constant of P. P is said to be regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\geq1}$ is a sequence such that $x_1 \preceq x_2 \preceq \cdots \preceq y$ for some $y \in E$, then there is $x \in E$ such that $lim_{n\to\infty}\|x_n - x\| = 0$.

Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 1.2. *Suppose that E is a real Banach space with a cone P. Then;*

(i) *If $x \preceq y$ and $0 \leq a \leq b$ then $ax \preceq by$,*
(ii) *If $x \preceq y$ and $u \preceq v$ then $x + u \preceq y + v$,*
(iii) *If $x_n \preceq y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$ then $x \preceq y$.*

Proof. The proof is simple. □

Lemma 1.3. *If P is a cone, $x \in P, \alpha \in R, 0 \leq \alpha < 1$ and $x \preceq \alpha x$ then $x = 0$.*

Proof. Since $x \preceq \alpha x$ then $\alpha x - x = (\alpha - 1)x \in P$. Since $x \in P$, $0 \leq \alpha < 1$, we have by Definition (1.1)(ii) that $(1 - \alpha)x \in P$. It follows by Definition (1.1)(iii) that $x = 0$. □

Lemma 1.4. *see* [4, 6]

(a) *Every regular cone is normal.*
(b) *For each $k > 1$, there is a normal cone with normal constant $K > k$.*
(c) *The cone P is regular if every decreasing sequence which is bounded from below is convergent.*

Definition 1.5. Let X be a non-empty set. Suppose that the mapping $h : X \times X \to E$ satisfies:

(a) $0 \preceq h(x, y)$ for all $x, y \in X$,
(b) $h(x, y) = 0$ if and only if $x = y$,
(c) $h(x, y) \preceq h(x, z) + h(z, y)$ for all $x, y \in X$,
(d) $h(x, y) = h(y, x)$ for all $x, y \in X$.

Then h is called a *cone metric* on X, and the pair (X, h) is a *cone metric space*, (CMS).

It is quite natural to consider *cone normed space*, (CNS).

Definition 1.6. Let X be a vector space over $\mathbb{R}$. Suppose that the mapping $\| \,.\, \|_P$: $X \to E$ satisfies:

(a) $\| x \|_P \succeq 0$ for all $x \in X$,
(b) $\| x \|_P = 0$ if and only if $x = 0$,
(c) $\| x + y \|_P \preceq \| x \|_P + \| y \|_P$ for all $x, y \in X$,
(d) $\| kx \|_P = | k | \| x \|_P$ for all $k \in R$,

then $\| \,.\, \|_P$ is called a *cone p-norm* on X, and the pair $(X, \| \,.\, \|_P)$ *a cone p-normed space*, (CNS).

Note that each CNS is a CMS, Indeed, $h(x, y) = \| x - y \|_P$.

Definition 1.7. Suppose that (X, h) is a cone metric space. A sequence $\{x_n\}$ in X is said to be:

(i) convergent to $x \in X$ if for every $c \in E$ with $0 \preceq c$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $h(x_n, x) \preceq c$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$, when $n \to \infty$.
(ii) Cauchy sequence if for every $c \in E$ with $0 \preceq c$, there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$, we have $h(x_n, x_n) \preceq c$.
(iii) A cone metric space (X, h) is said to be complete if every Cauchy sequence is convergent in X.

Lemma 1.8. *Suppose that (X, h) is a cone metric space. If $\{x_n\}$ is a convergent sequence in X then the limit of $\{x_n\}$ is unique.*

Proof. The proof of the following lemma is straight forward and is omitted. □

Lemma 1.9. *Suppose that (X, h) is a cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ then $\{x_{n_k}\}$ converges to x.*

Lemma 1.10. *Every regular cone is normal.*

Proof. Suppose that P is a regular cone which is not normal. For all $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2 \parallel t_n \parallel < \parallel s_n \parallel$. For each $n \geq 1$ put $y_n = \frac{t_n}{\|t_n\|}$ and $x_n = \frac{s_n}{\|s_n\|}$.
Then $x_n, y_n, y_n - x_n \in P, \parallel y_n \parallel = 1$ and $n^2 < \parallel x_n \parallel$ for all $n \geq 1$. Since the series

$$\sum_{n=1} \frac{1}{n^2} y_n,$$

is convergent and P is closed, there is an element $y \in P$ such that

$$\sum_{n=1} \frac{1}{n^2} y_n = y.$$

Now, note that

$$0 \preceq x_1 \preceq x_1 + \frac{1}{2^2} x_2 \preceq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \preceq \ldots \preceq y.$$

Thus

$$\sum_{n=1} \frac{1}{n^2} x_n,$$

is convergent because P is regular. Hence

$$\lim_{n \to \infty} \frac{\parallel x_n \parallel}{n^2} = 0,$$

which is a contradiction. □

Definition 1.11. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called a comparison function if it satisfies:

(i) φ is increasing,
(ii) $\{\varphi^n(t)\}_{n \in N}$ converges to 0 as $n \to \infty$, for all $t \in (0, \infty)$.

If condition (ii) replaced by

(iii) $\sum_{k=1}^{\infty} \varphi^k(t) < \infty$ for all $t \in (0, \infty)$ then φ is called a strong comparison function.

Every strong comparison function is a comparison function.

Example 1.12. Let $\varphi : [0, +\infty) \to [0, +\infty)$ defined by $\phi(t) = \frac{t}{1+t}$. Then φ is a comparison function, since $\varphi^n(t) = \frac{t}{1+nt}$ converges to 0 as $n \to \infty$. On the other hand,

$$\sum_{k=1}^{\infty} \varphi^k(t) = \infty,$$

which shows that φ is not a strong comparison function.

Definition 1.13. Suppose that X is a nonempty set $p \in \mathbb{N}$, and $f : X \to X$ is a mapping. Then $X = \bigcup_{i=1}^{p} A_i$ is called a cyclic representation of X with respect to f if:

(i) Every A_i, $1 \le i \le p$ is a non empty subset of X,
(ii) $f(A_i) \subseteq A_{(i+1)}$, $1 \le i \le p$ and $A_{p+1} = A_1$.

Definition 1.14. Suppose that X is a nonempty set, $p \in N$, $A_1, A_2, \dots, A_p$ are closed nonempty subsets of X, and $X = \bigcup_{i=1}^{p} A_i$. A mapping $f : X \to X$ is called cyclic weaker $\varphi-$controuction if:

(i) $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to f,
(ii) There exists a continuous non decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(t) \preceq t$, $f(0) = 0$.

Lemma 1.15. *Suppose that (X, h) is a cone metric space and P is a normal cone with normal constant k. Let $\{x_n\}, \{y_n\}$ be two sequence in X and $x_n \to x$, $y_n \to y$ when $n \to \infty$. Then $h(x_n, y_n) \to h(x, y)$ when $n \to \infty$.*

Lemma 1.16. *Suppose that (X, h) is a cone metric space. Then for each $C \succeq 0$, $C \in E$, there exists $\delta > 0$ such that $(c - x) \in \mathrm{Int}P$, (i.e. $x \ll c$), whenever $\| x \| < \delta$, $x \in E$.*

2. Main Results

Lemma 2.1. *Suppose that (X, h) is a cone metric space. Then for each $c_1, c_2 \in E$, $c_1, c_2 \succeq 0$, there exists $c \in E$, $c \succeq 0$ such that $c \preceq c_1$, $c \preceq c_2$.*

Proof. Since $c_2 \succeq 0$, by Lemma 1.16, there is $\delta > 0$ such that $\| x \| < \delta$ implies $x \preceq c_2$. Choose n such that $\frac{1}{n_0} < \frac{\delta}{\|c_1\|}$. Let $c = \frac{c_1}{n_0}$. Then

$$\| c \| = \left\| \frac{c_1}{n_0} \right\| = \frac{\| c_1 \|}{n_0} < \delta,$$

and hence $c \preceq c_2$. But also $c \preceq c_1$ and $c \succeq 0$. □

Definition 2.2. Let (X, h) be a topological space. We define

$$B(x, c) = \{y \in x : h(x, y) \preceq c\}, \qquad \hat{B} = \{B(x, c) : x \in X, C \succeq 0\},$$

and,

$$\tau_c = \left\{ U \subseteq X : \text{ for all } x \in U, \exists B \in \hat{B} \text{ such that } x \in B \subseteq U \right\}.$$

Proposition 2.3. *In every cone metric space (X, h), τ_c is a Topological space.*

Proof. It is obvious that $\emptyset, X \in \tau_c$. Now put $U, V \in \tau_c$ and $x \in U \cap V$. Then $x \in U$, $x \in V$ and there exists $c_1 \succeq 0, c_2 \succeq 0$ such that $x \in B(x, c_1) \subseteq U$ and $x \in B(x, c_2) \subseteq V$. By Lemma (2.1), there is $c \succeq 0$ such that $c \preceq c_1$, $c \preceq c_2$. Therefore $x \in B(x, c) \subseteq B(x, c_1) \cap B(x, c_2) \subseteq U \cap V$. Hence $U \cap V \in \tau_c$.

Now, put $U = \{U_\alpha\}_{\alpha \in I}$ and $U_\alpha \in \tau_c$ for each $\alpha \in I$, and let $x \in U = \bigcup_{\alpha \in I} U_\alpha$. Then there exists $\alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Hence there exists $c \succeq 0$ such that $x \in B(x, c) \subseteq U_{\alpha_0} \subseteq \cup_{\alpha \in I} U_\alpha$. This shows that $U = \cup_{\alpha \in I} U_\alpha \in \tau_c$. □

Every element of τ_c is called open. A subset C is called closed iff $X - C$ is open.

Note that every cone metric space (X, h), is a Hausdorff space. Indeed, if $x \neq y$ are two points in X then $d(x, y) = c \succeq 0$ and $B(x, \frac{c}{3}), B(y, \frac{c}{3})$ are in τ_c but $B\left(x, \frac{c}{3}\right) \cap B\left(y, \frac{c}{3}\right) = \emptyset$.

Definition 2.4. Suppose that (X, h) is a cone metric space. A subset A of X is called compact if each cover of A by subsets from τ_c can be reduced to a finite subcover, *i.e.* if $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ where $U_\alpha \in \tau_c$ for all $\alpha \in I$, then there is $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $A \subseteq \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_n$.

Definition 2.5. Suppose that (X, h) is a cone metric space. A subset A of (X, h) is called totally bounded if for each $c \gg 0$, $c \in E$, A can be composed into union of sets N_i, $i = 1, 2, , n$, $(A \subseteq \bigcup_{i=1}^n N_i)$, where $\delta(N_i) \preceq c$ $(\delta(K) = sup\{h(x, y) : x, y \in K\})$.

Proposition 2.6. *Let (X, h) be a complete cone metric space, $p \in \mathbb{N}$, and $A_1, A_2, \ldots, A_p$ are closed non empty subsets of X and $X = \bigcup_{i=1}^p A_i$. Suppose that $f, s : X \to X$ satisfies the following conditions:*

(i) $f(A_i) \subseteq f(A_{i+1})$, $s(A_i) \subseteq s(A_{i+1})$, *for* $1 \leq i \leq n$, $A_{p+1} = A_p$. *(we say, $\bigcup_{i=1}^p A_i$ is a cyclic representation of X with respect to f and s).*

(ii) $h(f(x), f(y)) \preceq kh(s(x), s(y))$ *where* $0 < k < 1$ *and* $x \in A_i, y \in A_{i+1}$.

Then f, s has a unique common fixed point in $\bigcap_{i=1}^p A_i$.

Proof. Given $x_0 \in X$, let

$$sx_1 = fx_0, sx_2 = fx_1 = f^2 x_0, \ldots, sx_n = fx_n = f^{(n+1)} x_0.$$

From (ii),

$$h(fx_{n+1}, fx_n) \preceq kh(sx_{n+1}, sx_n) = kh(fx_n, fx_{n-1}) \preceq \cdots \preceq k^n h(sx_1, sx_0).$$

Put $n > m$ such that $n \equiv_p m+1$. Then;

$$\begin{aligned} h(fx_{n-1}, fx_{n-2}) &\preceq h(fx_n, fx_{n-1}) + h(fx_{n-1}, fx_{n-2}) + \cdots + h(fx_{m+1}, fx_m) \\ &\preceq k^n h(sx_1, sx_0) + k^{n-1} h(sx_1, sx_0) + \cdots + k^m h(sx_1, sx_0) \\ &= (k^n + k^{n-1} + \cdots + k^m) h(sx_1, sx_0) \\ &\preceq \frac{k^m}{1-k} h(sx_1, sx_0). \end{aligned}$$

Then if $m \to \infty$, we have, $h(fx_n, fx_m) \to 0$. Therefore, $\{fx_n\}_{n=1}$ is a cauchy sequence in the complete cone metric space X and then there exists $z \in X$ such that $fx_n \to z$. Hence $sx_n \to z$.
Since many infinite sequences $\{fx\}_{n=1}$ lie in A_i and every A_i is closed, $z \in A_i$ for all $1 \leq i \leq p$ hence $z \in \bigcap_{i=1}^p A_i$. So there is $u \in \bigcap_{i=1}^p A_i$ such that $su = z$.
We have

$$h(z, fu) \preceq h(fx_n, fu) + h(z, fx_n) \preceq kh(sx_n, su) + h(z, fx_n),$$

hence

$$h(z, z) + kh(z, z) = 0, \quad \text{when} \quad n \to \infty.$$

Therefore $h(z, fu) = 0$ or $fu = z$. Then $fu = z = su$.

So, f, s have one common point in $\bigcap_{i=1}^p A_i$. On the other hand,

$$h(z, fz) = h(fu, fz) \preceq kh(su, sz) = kh(z, fz).$$

Since $0 < k < 1$, $sz = fz = z$. So, z is a common fixed point of f, s in $\bigcap_{i=1}^p A_i$.

Now let $z, z_0 \in \bigcap_{i=1}^p A_i$, such that $fz_0 = sz_0 = z_0$ and $fz = sz = z$. By assumption, we have $h(z, z_0) = h(fz, fz_0) \preceq kh(sz, sz_0) = kh(z, z_0)$. Since $0 < k < 1$, $z = z_0$, and this shows that the common fixed point is unique. □

Definition 2.7. Two self maps f and s of a cone metric space (X, h) are called reciprocal continuous if and only if

$$\lim_{n\to\infty} sfx_n = sz, \qquad \lim_{n\to\infty} fsx_n = fz,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} sx_n = z$ for some $z \in X$.

Definition 2.8. Suppose that s, f are self-mappings on a CMS (X, h). A point $z \in X$ is called a coincidence point of s, f if $sz = fz$ and it is called a common fixed point of s, f if $sz = z = fz$.

Moreover, a pair of self-mappings (s, f) will be called weakly compatible on X if they commute at their coincidence points, that is, if $z \in X$ and $sz = fz$ then $sfz = fsz$.

Definition 2.9. Let (X, h) be a complete cone metric space. Suppose $f, s : X \to X$ be mappings. If there exists a function $\varphi : E \to E$ with $\varphi(t) \prec t$ and $t - \varphi(t)$ is non-decreasing for all $t > 0$, $\varphi(0) = 0$ and for any $x, y \in X$

$$h(fx, fy) \preceq \varphi(h(sx, sy)). \tag{2.1}$$

we say f, s are $s - \varphi -$ *contractive*.

In additional if there is, $p \in \mathbb{N}$ and $A_1, A_2, \ldots, A_p$ are closed non empty subsets of X such that $X = \bigcup_{i=1}^{p} A_i$ and $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to f, s and for all $x \in A_i, y \in A_{i+1}$, $(1 \leq i \leq p$ and $A_{p+1} = A_1)$, (2.1) holds we say f, s are *cyclic* $s{-}\varphi{-}$*contractive* on X.

Lemma 2.10. *Put* $\varphi : E \to E$ *with* $\varphi(t) \prec t$ *for all* $t \succ 0$ *and* $\varphi(0) = 0$ *then;*

(i) $\varphi^k(t) \prec t$ *for all* $t \in (0, +\infty)$ *and* $k \in \mathbb{N}$.
(ii) $\lim\limits_{k\to\infty} \varphi^k(t) = 0$ *for all* $t \succ 0$.

Proof. Proof of (i) is by induction.

Now, we prove (ii). Since $\varphi^{k+1}(t) < \varphi^k(t)$, the sequence $\{\varphi^k(t)\}$ is decreasing and bounded from below by 0, therefore $\lim\limits_{k\to\infty} \varphi^k(t) = l \succeq 0$ and $l \preceq \varphi^k(t)$ for all k. If $l \succ 0$ then $l - \varphi(l) \succ 0$. We have;

$$0 \preceq l - \varphi(l) \preceq \varphi k(t) - \varphi(\varphi^k(t)) \to 0,$$

when $k \to \infty$. This contradicts with $l - \varphi(l) \succ 0$. So $l = 0$. □

Proposition 2.11. *Let* (X, h) *be a complete cone metric space,* $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ *be closed non empty subsets of* X, *and* $X = \bigcup_{i=1}^{p} A_i$. *Suppose that* $f, s : X \to X$ *are mappings. Assume that* f, s *satisfy the following;*

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of* X *with respect to* f, s.
(ii) *There exists a function* $\varphi : E \to E$ *with* $\varphi(t) \prec t$ *for all* $t \succ 0$ *and* $\varphi(0) = 0$ *such that* $h(fx, fy) \preceq \varphi(h(sx, sy))$, *for any* $x \in A_i$, $y \in A_{i+1}$ *where* $A_{p+1} = A_1$.
(iii) f, s *are reciprocal continuous and weakly compactable.*

Then f, s *have a unique common fixed point in* $\bigcap_1^p A_i$.

Proof. Put $x_0 \in X$ and let;

$$sx_1 = fx_0, sx_2 = fx_1 = f^2x_0, \ldots, sx_{n+1} = fx_n = f^{n+1}x_0.$$

If there exists z_0 such that $fz_0 = z_0 = sz_0$ then the existence of the fixed point is proved.

We assumed that $sx_n \neq x_n$ for all n (This implies $fx_n \neq x_n$).

First we show that $sx_n \neq sx_m$ for all $n \neq m$. Suppose that $sx_n = sx_m$ for some $n \neq m$ (By contrary hypothesis). We can suppose $m > n$. Then

$$\begin{aligned} h(sx_n, sx_{n+1}) &= h(sx_n, fx_n) \\ &= h(sx_m, fsx_m) \\ &= h(fx_{m-1}, sx_{m+1}) \\ &= h(fx_{m-1}, fx_m) \\ &\preceq \varphi(h(sx_{m-1}, sx_m) \\ &\preceq \cdots \preceq \varphi^{m-n}(h(sx_n, sx_{n+1})), \end{aligned}$$

So,

$$h(sx_n, sx_{n+1}) \preceq \varphi^{m-n}(h(sx_n, sx_{n+1})).$$

which is in contradiction with (ii) Lemma (2.1). Thus $sx_n \neq sx_m$ for all $n \neq m$.

Now, by (i) Lemma (2.1)

$$h(sx_n, sx_{n+1})h(fx_{n-1}, fx_n) \preceq \varphi(h(sx_{n-1}, sx_n) < h(sx_{n-1}, sx_n). \tag{2.2}$$

So, the sequence $\{h(sx_n, sx_{n+1})\}$ is decreasing and bounded from below. Therefore $\lim\limits_{n\to\infty} h(sx_n, sx_{n+1})$ exists. Put $\lim\limits_{n\to\infty} h(sx_n, sx_{n+1}) = l$. If $l > 0$, then by definition of φ, $\varphi(l) \prec l$. Since $\{h(sx_n, sx_{n+1})\}$ is decreasing, $h(sx_n, sx_{n+1}) \succeq l$, and we have;

(2.3)

$$0 \preceq l - \varphi(l) \preceq h(sx_n, sx_{n+1}) - \varphi((h(sx_n, sx_{n+1}))) \quad \text{for all} \quad n \in \mathbb{N}.$$

By (2.1) we have;

$$h(sx_{n+1}, sx_{n+2}) \preceq \varphi(h(sx_n, sx_{n+1})). \tag{2.4}$$

By (2.3) and (2.4) for all $n \in \mathbb{N}$ we have;

$$0 \preceq l - \varphi(l) \preceq h(sx_n, sx_{n+1}) - h(sx_{n+1}, sx_{n+2}) \to l - l = 0,$$

when $n \to \infty$. We get, $0 \preceq l - \varphi(l) \preceq 0$, which is in contradication with $\varphi(l) \prec l$. Thus $l = 0$ or,

$$h(sx_n, sx_{n+1}) \to 0 \quad \text{when} \quad (n \to \infty). \tag{2.5}$$

For all n, x_n, x_{n+p-1} lie in the different sets A_i and A_{i+1}, for all $1 \leq i \leq p$. We have;

$$h(sx_n, sx_{n+p-1}) = h(fx_{n-1}, fx_{n+p-2}) \preceq \varphi(h(sx_{n-1}, sx_{n+p-2})).$$

Similar to above, the sequence $\{h(sx_n, sx_{n+p-1})\}$ is decreasing and converges to zero. Therefore,

$$h(sx_n, sx_{n+p-1}) \to 0 \quad \text{when} \quad n \to \infty. \tag{2.6}$$

By (2.1) and (2.5) when $n \to \infty$ and $1 \leq k \leq p$ we get;

$$h(sx_n, sx_{n+k}) \preceq h(sx_n, sx_{n+1}) + h(sx_{n+1}, sx_{n+2}) + \cdots + h(sx_{n+k-1}, sx_{n+k}) \to 0, \tag{2.7}$$

when $n \to \infty$.

Now, we show that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n > m > n_0$ with $n \equiv_p m + 1$ then;

$$h(sx_n, sx_m) \preceq \varepsilon. \tag{2.8}$$

We prove this by contradict hypothesis.

If there exists $\varepsilon_0 > 0$ such that for all $n > m > n_0$, $n \equiv_p m + 1$, $h(sx_n and sx_m) \succeq \varepsilon_0$. By definition of φ, we get;

$$\varepsilon_0 - \varphi(\varepsilon_0) \preceq h(sx_n, sx_m) - \varphi(h(sx_n, sx_m)). \tag{2.9}$$

By (2.4) we have;

$$h(sx_{n+1}, sx_{m+1}) \preceq \varphi(h(sx_n, sx_m). \tag{2.10}$$

By (2.9), (2.10) and Triangle inequality, we get;

$$\begin{aligned} \varepsilon_0 - \varphi(\varepsilon_0) &\preceq h(sx_n, sx_m) - h(sx_{n+1}, sx_{m+1}) \\ &\preceq h(sx_n, sx_{n+1}) + h(sx_{n+1}, sx_{m+1}) \\ &\quad + h(sx_{n+1}, sx_m - h(sx_n, sx_{m+1}) \\ &= h(sx_n, sx_{n+1}) + h(sx_{m+1}, sx_m). \end{aligned} \tag{2.11}$$

By (2.2) and (2.11) it has been followed that,

$$\varepsilon_0 - \varphi(\varepsilon_0) \preceq 2h(sx_{m+1}, sx_m),$$

or

$$h(sx_{m+1}, sx_m) \succeq \frac{(\varepsilon_0 - \varphi(\varepsilon_0))}{2} > 0,$$

which shows that the sequence $h(sx_{m+1}, sx_m)$ does not converge to zero when $m \to \infty$, which contradicts (2.5), or (2.8) holds.

Now we prove that $\{sx_n\}$ is a Cauchy sequence in X.

Let $\varepsilon > 0$, by (2.7), there exists $n_1 \in \mathbb{N}$ such that if $n > m > n_1$ with $n \equiv_p m + 1$, then;

$$h(sx_n, sx_m) < \frac{\varepsilon}{3}. \tag{2.12}$$

On the other hand by (2.7) there exists $n_2 \in \mathbb{N}$ such that for any $n > n_2$;

$$h(sx_n, sx_{n+k}) \prec \frac{\varepsilon}{3}, \quad \text{for} \quad k \in \{1, 2, \ldots, p\}. \tag{2.13}$$

Let $n > m > \max(n_1, n_2)$ and then we can find $u \in \{0, 1, 2, \ldots, p\}$ such that $n \equiv_p m + u + 1$.

We consider two cases;

case i) If $u = 0$, we have by (2.13)

$$h(sx_n, sx_m) \prec \frac{\varepsilon}{3} \prec \varepsilon.$$

case ii) If $u \geq 1$, we have;

(2.14)

$$\begin{aligned} h(sx_m, sx_n) &\preceq h(sx_m, sx_{m-1}) + h(sx_{m-1}, sx_{m+u}) + h(sx_{m+u}, sx_n) \\ &\prec \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

This shows that $\{sx_n\}$ is a Cauchy sequence. Since X is a complete cone metric space, there exists $z \in X$ such that $\lim_{n\to\infty} sx_n = z$. Since $fx_n = sx_{n+1}$, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} sx_n = z$. Since f is a cyclic representation of X with respect to f, s, infinitly many members of $\{sx_n\}$ lie in A_i for $1 \leq i \leq p$. Since A_i is closed, $z \in A_i$ for all $1 \leq i \leq p$, we see that $z \in \bigcap_1^p A_i$, and there is $u \in \bigcap_1^p A_i$ such that $su = z$.

We have

$$\begin{aligned} h(z, fu) &\preceq h(fx_n, fu) + h(z, fx_n) \\ &\preceq kh(sx_n, su) + h(z, fx_n), \end{aligned}$$

hence $h(z, z) + kh(z, z) = 0$, when $n \to \infty$. Therefore $h(z, fu) = 0$ or $fu = z$. Then $fu = z = su$. So, f, s have a common point in $\bigcap_{i=1}^p A_i$.

If there exists $x^* \in \bigcap_1^p A_i$ such that $sx^* = z$, then

$$\begin{aligned} h(z, fz) &\preceq h(z, fx_n) + h(fx_n, fx^*) \\ &\preceq h(z, fx_n + \varphi(h(sx_n, sx^*)) \\ &\prec h(z, fx_n) + h(sx_n, sx^*) \to 0, \quad \text{when} \quad n \to \infty. \end{aligned}$$

This implies that $sx^* = z = fx^*$.

On the other hand, If $fz \neq z$ then;

$$\begin{aligned} h(z, fz) &= h(fu, fz) \\ &\preceq \varphi(h(su, sz)) \\ &\prec h(z, fz). \quad \text{(By definition of } \varphi). \end{aligned}$$

This is a contradiction, therefore we have $fz = z$ or $sz = fz = z$. So,z is a common fixed point of f, s in $\bigcap_{i=1}^p A_i$. Now let $z, z_0 \in \bigcap_{i=1}^p A_i$, such that $fz_0 = sz_0 = z_0$ and $fz = sz = z$. By assumption, we have;

$$h(z, z_0) = h(fz, fz_0) \preceq \varphi(h(sz, sz_0)) \prec h(z, z_0).$$

Now, let $z_1, z_2 \in \bigcap_{i=1}^p A_i$ be two common fixed points of s, f. We show that $z_1 = z_2$. We have $sz_1 = z_1 = fz_1$, $sz_2 = z_2 = fz_2$. By (2.1),

$h(fz_1, fz_2) \preceq \varphi(h(sz_1, sz_2)$.
Therefore,

$$h(z_1, z_2) = h(fz_1, fz_2) \preceq \varphi(h(sz_1, sz_2)) = \varphi(h(z_1, z_2)) \prec h(z_1, z_2).$$

So $z_1 = z_2$ and the common fixed point is unique. □

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