Common Fixed Point in Cone Metric Space for $s - \varphi$-contractive

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ABSTRACT. Huang and Zhang $^4$ have introduced the concept of cone metric space where the set of real numbers is replaced by an ordered Banach space. Shojaei $^5$ has obtained points of coincidence and common fixed points for $s$-Contraction mappings which satisfy generalized contractive type conditions in a complete cone metric space.

In this paper, the notion of complete cone metric space has been introduced. We have defined $s - \varphi$-contractive and obtained common fixed point theorem for a mapping $f, s$ which satisfies $s - \varphi$-contractive.

1. INTRODUCTION

Huang and Zhang $^4$ have introduced the concept of cone metric space where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors $^1, 12$ have generalized the results of Huang and Zhang $^4$ and studied the existence of common fixed points of a pair of self-mappings satisfying a contractive type condition in the framework of normal cone metric spaces. In $^3$ Bari and Vetro obtained some results on the points of coincidence and common fixed points in non-normal cone metric spaces. Shojaei $^5$ obtained points of coincidence and common fixed points for $s - contraction$ mappings satisfying generalized contractive type conditions in a complete cone metric space.
In 1969, Boyd and Wong [2] have introduced the notion of \( \varphi \) contraction. Generalization of the above Banach contraction principle has been a heavily investigated branch research, (see, e.g., [2, 10, 4]). In 2003, Kirk et. al., [7] introduced the notion of cyclic representation.

We have introduced the notion of \( s - \varphi \)-contractive mappings in a con-metric space and proved some propositions. Throughout this paper, we have denoted by \( \mathbb{N} \) the set of positive integers, by \( \mathbb{R} \) the set of real numbers and \( E \) will be a Real Banach Space.

**Definition 1.1.** Suppose \( E \) is a Real Banach Space and \( P \) is a subset of \( E \). \( P \) is called a Cone if and only if:

(i) \( P \) is nonempty, closed and satisfies \( P \neq \{0\} \),
(ii) If \( a, b \in \mathbb{R} \), such that, \( a, b \geq 0 \) and \( x, y \in P \), then \( ax + by \in P \),
(iii) If \( x \in P \) and \( -x \in P \) then \( x = 0 \).

For a Cone \( P \subseteq E \), we defined a partial ordering \( \preceq \) with respect to \( P \) by \( x \preceq y \) iff \( y - x \in P \). We shall write \( x < y \) iff \( x \preceq y \) and \( x \neq y \), and \( x \ll y \) iff \( y - x \in \text{int}P \), where \( \text{int}P \) is the interior of \( P \). From now on, it is assumed that \( \text{int}P \neq \emptyset \). The cone \( P \) is called normal if there is a number \( K \geq 1 \) such that, for all \( x, y \in E \), \( 0 \preceq x \preceq y \) implies \( \|x\| \leq K \|y\| \).

Here, the least positive integer \( K \) satisfying this inequality is called the normal constant of \( P \). \( P \) is said to be regular if every increasing sequence which is bounded from above is convergent, that is, if \( \{x_n\}_{n \geq 1} \) is a sequence such that \( x_1 \preceq x_2 \preceq \cdots \preceq y \) for some \( y \in E \), then there is \( x \in E \) such that \( \lim_{n \to \infty} \|x_n - x\| = 0 \).

Equivalently, the cone \( P \) is regular if and only if every decreasing sequence which is bounded from below is convergent.

**Lemma 1.2.** Suppose that \( E \) is a real Banach space with a cone \( P \). Then,

(i) If \( x \preceq y \) and \( 0 \leq a \leq b \) then \( ax \preceq by \),
(ii) If \( x \preceq y \) and \( u \preceq v \) then \( x + u \preceq y + v \),
(iii) If \( x_n \preceq y_n \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y \) then \( x \preceq y \).

**Proof.** The proof is simple. \( \square \)

**Lemma 1.3.** If \( P \) is a cone, \( x \in P, \alpha \in \mathbb{R}, 0 \leq \alpha < 1 \) and \( x \preceq \alpha x \) then \( x = 0 \).

**Proof.** Since \( x \preceq \alpha x \) then \( \alpha x - x = (\alpha - 1)x \in P \). Since \( x \in P \), \( 0 \leq \alpha < 1 \), we have by Definition (1.1)(ii) that \( (1 - \alpha)x \in P \). It follows by Definition (1.1)(iii) that \( x = 0 \). \( \square \)

**Lemma 1.4.** see [4, 6]
(a) Every regular cone is normal.
(b) For each \( k > 1 \), there is a normal cone with normal constant \( K > k \).
(c) The cone \( P \) is regular if every decreasing sequence which is bounded from below is convergent.

**Definition 1.5.** Let \( X \) be a non-empty set. Suppose that the mapping \( h : X \times X \to E \) satisfies:

(a) \( 0 \preceq h(x, y) \) for all \( x, y \in X \),
(b) \( h(x, y) = 0 \) if and only if \( x = y \),
(c) \( h(x, y) \preceq h(x, z) + h(z, y) \) for all \( x, y \in X \),
(d) \( h(x, y) = h(y, x) \) for all \( x, y \in X \).

Then \( h \) is called a cone metric on \( X \), and the pair \((X, h)\) is a cone metric space, (CMS).

It is quite natural to consider a cone normed space, (CNS).

**Definition 1.6.** Let \( X \) be a vector space over \( \mathbb{R} \). Suppose that the mapping \( \| \cdot \|_P : X \to E \) satisfies:

(a) \( \| x \|_P \succeq 0 \) for all \( x \in X \),
(b) \( \| x \|_P = 0 \) if and only if \( x = 0 \),
(c) \( \| x + y \|_P \preceq \| x \|_P + \| y \|_P \) for all \( x, y \in X \),
(d) \( \| kx \|_P = |k| \| x \|_P \) for all \( k \in \mathbb{R} \),

then \( \| \cdot \|_P \) is called a cone \( p \)-norm on \( X \), and the pair \((X, \| \cdot \|_P)\) a cone \( p \)-normed space, (CNS).

Note that each CNS is a CMS, Indeed, \( h(x, y) = \| x - y \|_P \).

**Definition 1.7.** Suppose that \((X, h)\) is a cone metric space. A sequence \( \{x_n\} \) in \( X \) is said to be:

(i) convergent to \( x \in X \) if for every \( c \in E \) with \( 0 \preceq c \), there is \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( h(x_n, x) \preceq c \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) when \( n \to \infty \).
(ii) Cauchy sequence if for every \( c \in E \) with \( 0 \preceq c \), there is \( n_0 \in \mathbb{N} \) such that for all \( n, m \geq n_0 \), we have \( h(x_n, x_m) \preceq c \).
(iii) A cone metric space \((X, h)\) is said to be complete if every Cauchy sequence is convergent in \( X \).

**Lemma 1.8.** Suppose that \((X, h)\) is a cone metric space. If \( \{x_n\} \) is a convergent sequence in \( X \) then the limit of \( \{x_n\} \) is unique.

**Proof.** The proof of the following lemma is straightforward and is omitted. \( \square \)

**Lemma 1.9.** Suppose that \((X, h)\) is a cone metric space and \( \{x_n\} \) be a sequence in \( X \). If \( \{x_n\} \) converges to \( x \) and \( \{x_{n_k}\} \) is any subsequence of \( \{x_n\} \) then \( \{x_{n_k}\} \) converges to \( x \).
Lemma 1.10. Every regular cone is normal.

Proof. Suppose that $P$ is a regular cone which is not normal. For all $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2 \| t_n \| < \| s_n \|$. For each $n \geq 1$ put $y_n = \frac{t_n}{\| t_n \|}$ and $x_n = \frac{s_n}{\| s_n \|}$.

Then $x_n, y_n, y_n - x_n \in P, \| y_n \| = 1$ and $n^2 < \| x_n \|$ for all $n \geq 1$. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} y_n,$$

is convergent and $P$ is closed, there is an element $y \in P$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y.$$

Now, note that

$$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \leq \cdots \leq y.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x_n,$$

is convergent because $P$ is regular. Hence

$$\lim_{n \to \infty} \frac{\| x_n \|}{n^2} = 0,$$

which is a contradiction. \qed

Definition 1.11. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called a comparison function if it satisfies:

(i) $\varphi$ is increasing,

(ii) $\{ \varphi^n(t) \}_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t \in (0, \infty)$.

If condition (ii) replaced by

(iii) $\sum_{k=1}^{\infty} \varphi^k(t) < \infty$ for all $t \in (0, \infty)$ then $\varphi$ is called a strong comparison function.

Every strong comparison function is a comparison function.

Example 1.12. Let $\varphi : [0, +\infty) \to [0, +\infty)$ defined by $\phi(t) = \frac{t}{1+t^2}$. Then $\varphi$ is a comparison function, since $\varphi^n(t) = \frac{t}{1+nt}$ converges to 0 as $n \to \infty$. On the other hand,

$$\sum_{k=1}^{\infty} \varphi^k(t) = \infty,$$

which shows that $\varphi$ is not a strong comparison function.
Definition 1.13. Suppose that $X$ is a nonempty set $p \in \mathbb{N}$, and $f : X \to X$ is a mapping. Then $X = \bigcup_{i=1}^{p} A_{i}$ is called a cyclic representation of $X$ with respect to $f$ if:

(i) Every $A_{i}$, $1 \leq i \leq p$ is a non empty subset of $X$,
(ii) $f(A_{i}) \subseteq A_{(i+1)}$, $1 \leq i \leq p$ and $A_{p+1} = A_{1}$.

Definition 1.14. Suppose that $X$ is a nonempty set, $p \in \mathbb{N}$, $A_{1}, A_{2}, \ldots, A_{p}$ are closed nonempty subsets of $X$, and $X = \bigcup_{i=1}^{p} A_{i}$. A mapping $f : X \to X$ is called cyclic weaker $\varphi$-contraction if:

(i) $X = \bigcup_{i=1}^{p} A_{i}$ is a cyclic representation of $X$ with respect to $f$,
(ii) There exists a continuous non decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(t) \preceq t$, $f(0) = 0$.

Lemma 1.15. Suppose that $(X, h)$ is a cone metric space and $P$ is a normal cone with normal constant $k$. Let $(x_{n}), (y_{n})$ be two sequence in $X$ and $x_{n} \to x$, $y_{n} \to y$ when $n \to \infty$. Then $h(x_{n}, y_{n}) \to h(x, y)$ when $n \to \infty$.

Lemma 1.16. Suppose that $(X, h)$ is a cone metric space. Then for each $C \succeq 0$, $C \in E$, there exists $\delta > 0$ such that $(c - x) \in \text{Int}P$, (i.e. $x \preceq c$), whenever $\|x\| < \delta$, $x \in E$.

2. Main Results

Lemma 2.1. Suppose that $(X, h)$ is a cone metric space. Then for each $c_{1}, c_{2} \in E$, $c_{1}, c_{2} \succeq 0$, there exists $c \in E$, $c \succeq 0$ such that $c \preceq c_{1}$, $c \preceq c_{2}$.

Proof. Since $c_{2} \succeq 0$, by Lemma 1.17, there is $\delta > 0$ such that $\|x\| < \delta$ implies $x \preceq c_{2}$. Choose $n$ such that $\frac{1}{n_{0}} < \frac{\delta}{\|c_{1}\|}$. Let $c = \frac{c_{1}}{n_{0}}$. Then

$$\|c\| = \left\|\frac{c_{1}}{n_{0}}\right\| = \frac{\|c_{1}\|}{n_{0}} < \delta,$$

and hence $c \preceq c_{2}$. But also $c \preceq c_{1}$ and $c \succeq 0$.

Definition 2.2. Let $(X, h)$ be a topological space. We define

$$B(x, c) = \{y \in x : h(x, y) \preceq c\}, \quad \dot{B} = \{B(x, c) : x \in X, C \succeq 0\},$$

and,

$$\tau_{c} = \left\{U \subseteq X : \text{for all } x \in U, \exists B \in \dot{B} \text{ such that } x \in B \subseteq U\right\}.$$

Proposition 2.3. In every cone metric space $(X, h)$, $\tau_{c}$ is a Topological space.
Proof. It is obvious that \( \emptyset, X \in \tau_c \). Now put \( U, V \in \tau_c \) and \( x \in U \cap V \). Then \( x \in U, x \in V \) and there exists \( c_1 \geq 0, c_2 \geq 0 \) such that \( x \in B(x, c_1) \subseteq U \) and \( x \in B(x, c_2) \subseteq V \). By Lemma (2.1), there is \( c \geq 0 \) such that \( c \leq c_1, c \leq c_2 \). Therefore \( x \in B(x, c) \subseteq B(x, c_1) \cap B(x, c_2) \subseteq U \cap V \). Hence \( U \cap V \in \tau_c \).

Now, put \( U = \{ U_\alpha \}_{\alpha \in I} \) and \( U_\alpha \in \tau_c \) for each \( \alpha \in I \), and let \( x \in U = \bigcup_{\alpha \in I} U_\alpha \). Then there exists \( \alpha_0 \in I \) such that \( x \in U_{\alpha_0} \). Hence there exists \( c \geq 0 \) such that \( x \in B(x, c) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha \). This shows that \( U = \bigcup_{\alpha \in I} U_\alpha \in \tau_c \). \( \square \)

Every element of \( \tau_c \) is called open. A subset \( C \) is called closed iff \( X - C \) is open.

Note that every cone metric space \((X, h)\), is a Hausdorff space. Indeed, if \( x \neq y \) are two points in \( X \) then \( d(x, y) = c \geq 0 \) and \( B(x, \frac{c}{3}), B(y, \frac{c}{3}) \) are in \( \tau_c \) but \( B(x, \frac{c}{3}) \cap B(y, \frac{c}{3}) = \emptyset \).

**Definition 2.4.** Suppose that \((X, h)\) is a cone metric space. A subset \( A \) of \( X \) is called compact if each cover of \( A \) by subsets from \( \tau_c \) can be reduced to a finite subcover, i.e., if \( A \subseteq \bigcup_{\alpha \in I} U_\alpha \), where \( U_\alpha \in \tau_c \) for all \( \alpha \in I \), then there is \( \alpha_1, \alpha_2, \ldots, \alpha_n \in I \) such that \( A \subseteq \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_n \).

**Definition 2.5.** Suppose that \((X, h)\) is a cone metric space. A subset \( A \) of \((X, h)\) is called totally bounded if for each \( c \gg 0, c \in \mathbb{E} \), \( A \) can be composed into union of sets \( N_i, i = 1, 2, \ldots, n \), \( A \subseteq \bigcup_{i=1}^{n} N_i \), where \( \delta(N_i) \leq c (\delta(K) = \sup\{h(x, y) : x, y \in K\}) \).

**Proposition 2.6.** Let \((X, h)\) be a complete cone metric space, \( p \in \mathbb{N} \), and \( A_1, A_2, \ldots, A_p \) are closed non empty subsets of \( X \) and \( X = \bigcup_{i=1}^{p} A_i \). Suppose that \( f, s : X \to X \) satisfies the following conditions:

(i) \( f(A_i) \subseteq f(A_{i+1}), s(A_i) \subseteq s(A_{i+1}) \), for \( 1 \leq i \leq n, A_{p+1} = A_p \), (we say, \( \bigcup_{i=1}^{p} A_i \) is a cyclic representation of \( X \) with respect to \( f \) and \( s \)).

(ii) \( h(f(x), f(y)) \leq kh(s(x), s(y)) \) where \( 0 < k < 1 \) and \( x \in A_i, y \in A_{i+1} \).

Then \( f, s \) has a unique common fixed point in \( \bigcap_{i=1}^{p} A_i \).

**Proof.** Given \( x_0 \in X \), let

\[
x_1 = f(x_0), sx_2 = f^2(x_0), \ldots, sx_n = f(x_n) = f^{(n+1)}(x_0).
\]

From (ii),

\[
h(f_{n+1}, f_n) \leq kh(s_{n+1}, s_n) = kh(f_{n+1}, f_{n+1}) \leq \cdots \leq k^n h(s_1, s_0).
\]
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Put $n > m$ such that $n \equiv_p m + 1$. Then;

$$h(fx_{n-1}, fx_{n-2}) \leq h(fx_n, fx_{n-1}) + h(fx_{n-2}, fx_{n-1}) + \cdots + h(fx_m, fx_{m+1})$$

\[ \leq k^n h(sx_1, sx_0) + k^{n-1} h(sx_1, sx_0) + \cdots + k^m h(sx_1, sx_0) \]

\[ \leq \frac{k^m}{1 - k} h(sx_1, sx_0). \]

Then if $m \to \infty$, we have, $h(fx_n, fx_m) \to 0$. Therefore, $\{fx_n\}_{n=1}^\infty$ is a cauchy sequence in the complete cone metric space $X$ and then there exists $z \in X$ such that $fx_n \to z$. Hence $sx_n \to z$.

Since many infinite sequences $\{fx_n\}_{n=1}^\infty$ lie in $A_i$ and every $A_i$ is closed, $z \in A_i$ for all $1 \leq i \leq p$ hence $z \in \bigcap_{i=1}^p A_i$. So there is $u \in \bigcap_{i=1}^p A_i$ such that $su = z$.

We have

$$h(z, fu) \leq h(fx_n, fu) + h(z, fx_n) \leq kh(sx_n, su) + h(z, fx_n),$$

hence

$$h(z, z) + kh(z, z) = 0,$$ when $n \to \infty$.

Therefore $h(z, fu) = 0$ or $fu = z$. Then $fu = z = su$.

So, $f, s$ have one common point in $\bigcap_{i=1}^p A_i$. On the other hand,

$$h(z, fz) = h(fu, fz) \leq kh(su, sz) = kh(z, fz).$$

Since $0 < k < 1$, $sz = fz = z$. So, $z$ is a common fixed point of $f, s$ in $\bigcap_{i=1}^p A_i$.

Now let $z, z_0 \in \bigcap_{i=1}^p A_i$, such that $fz_0 = sz_0 = z_0$ and $fz = sz = z$.

By assumption, we have $h(z, z_0) = h(fz, fz_0) \leq kh(sz, sz_0) = kh(z, z_0)$.

Since $0 < k < 1$, $z = z_0$, and this shows that the common fixed point is unique.

\[ \Box \]

**Definition 2.7.** Two self maps $f$ and $s$ of a cone metric space $(X, h)$ are called reciprocal continuous if and only if

\[ \lim_{n \to \infty} sfx_n = sz, \quad \lim_{n \to \infty} fsx_n = fz, \]

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} sx_n = z$ for some $z \in X$.

**Definition 2.8.** Suppose that $s, f$ are self-mappings on a CMS $(X, h)$. A point $z \in X$ is called a coincidence point of $s, f$ if $sz = fz$ and it is called a common fixed point of $s, f$ if $sz = z = fz$.

Moreover, a pair of self-mappings $(s, f)$ will be called weakly compatible on $X$ if they commute at their coincidence points, that is, if $z \in X$ and $sz = fz$ then $sfz = fsz$. 
Definition 2.9. Let \((X, h)\) be a complete cone metric space. Suppose \(f, s : X \to X\) be mappings. If there exists a function \(\varphi : E \to E\) with \(\varphi(t) \prec t\) and \(t - \varphi(t)\) is non-decreasing for all \(t > 0\), \(\varphi(0) = 0\) and for any \(x, y \in X\)
\[
 (2.1) \quad h(fx, fy) \leq \varphi(h(sx, sy)).
\]
we say \(f, s\) are \(s - \varphi - \text{contractive}\).

In additional if there is, \(p \in \mathbb{N}\) and \(A_1, A_2, \ldots, A_p\) are closed non empty subsets of \(X\) such that \(X = \bigcup_{i=1}^{p} A_i\) and \(\bigcup_{i=1}^{p} A_i\) is a cyclic representation of \(X\) with respect to \(f, s\) and for all \(x \in A_i, y \in A_{i+1}, (1 \leq i \leq p)\) and \(A_{p+1} = A_1\), \((2.1)\) holds we say \(f, s\) are cyclic \(s - \varphi - \text{contractive}\) on \(X\).

Lemma 2.10. Put \(\varphi : E \to E\) with \(\varphi(t) \prec t\) for all \(t > 0\) and \(\varphi(0) = 0\) then;

(i) \(\varphi^k(t) \prec t\) for all \(t \in (0, +\infty)\) and \(k \in \mathbb{N}\).

(ii) \(\lim_{k \to \infty} \varphi^k(t) = 0\) for all \(t > 0\).

Proof. Proof of (i) is by induction.

Now, we prove (ii). Since \(\varphi^{k+1}(t) < \varphi^k(t)\), the sequence \(\{\varphi^k(t)\}\) is decreasing and bounded from below by 0, therefore \(\lim_{k \to \infty} \varphi^k(t) = l \geq 0\) and \(l \leq \varphi^k(t)\) for all \(k\). If \(l > 0\) then \(l - \varphi(l) > 0\). We have;
\[
  0 < l - \varphi(l) \leq \varphi^k(t) - \varphi(\varphi^k(t)) \to 0,
\]
when \(k \to \infty\). This contradicts with \(l - \varphi(l) > 0\). So \(l = 0\). \(\square\)

Proposition 2.11. Let \((X, h)\) be a complete cone metric space, \(p \in \mathbb{N}\), \(A_1, A_2, \ldots, A_p\) be closed non empty subsets of \(X\), and \(X = \bigcup_{i=1}^{p} A_i\). Suppose that \(f, s : X \to X\) are mappings. Assume that \(f, s\) satisfy the following:

(i) \(\bigcup_{i=1}^{p} A_i\) is a cyclic representation of \(X\) with respect to \(f, s\).

(ii) There exists a function \(\varphi : E \to E\) with \(\varphi(t) \prec t\) for all \(t > 0\) and \(\varphi(0) = 0\) such that \(h(fx, fy) \leq \varphi(h(sx, sy))\), for any \(x \in A_i, y \in A_{i+1}\) where \(A_{p+1} = A_1\).

(iii) \(f, s\) are reciprocal continuous and weakly compactable.

Then \(f, s\) have a unique common fixed point in \(\bigcap_1^{p} A_i\).

Proof. Put \(x_0 \in X\) and let;
\[
 sx_1 = fx_0, sx_2 = fx_1 = f^2 x_0, \ldots, sx_{n+1} = fx_n = f^{n+1} x_0.
\]
If there exists \(z_0\) such that \(fz_0 = z_0 = sz_0\) then the existence of the fixed point is proved.

We assumed that \(sx_n \neq x_n\) for all \(n\) (This implies \(fx_n \neq x_n\)).
First we show that \( sx_n \neq sx_m \) for all \( n \neq m \). Suppose that \( sx_n = sx_m \) for some \( n \neq m \) (By contrary hypothesis). We can suppose \( m > n \). Then
\[
h(sx_n, sx_{n+1}) = h(sx_n, f x_n) = h(sx_m, f sx_m) = h(f x_{m-1}, sx_{m+1}) = h(f x_{m-1}, f x_m) \leq \varphi(h(sx_{m-1}, sx_m)) \leq \cdots \leq \varphi^{m-n}(h(sx_n, sx_{n+1})),
\]
which is in contradiction with (ii) Lemma (2.1). Thus \( sx_n \neq sx_m \) for all \( n \neq m \).

Now, by (i) Lemma (2.1)
\[(2.2)\quad h(sx_n, sx_{n+1})h(f x_n, f x_n) \leq \varphi(h(sx_{n-1}, sx_n)) < h(sx_{n-1}, sx_n).
\]
So, the sequence \( \{h(sx_n, sx_{n+1})\} \) is decreasing and bounded from below. Therefore \( \lim_{n \to \infty} h(sx_n, sx_{n+1}) \) exists. Put \( \lim_{n \to \infty} h(sx_n, sx_{n+1}) = l. \) If \( l > 0 \), then by definition of \( \varphi \), \( \varphi(l) < l. \) Since \( \{h(sx_n, sx_{n+1})\} \) is decreasing, \( h(sx_n, sx_{n+1}) \geq l \), and we have;
\[(2.3)\quad 0 \leq l - \varphi(l) \leq h(sx_n, sx_{n+1}) - \varphi((h(sx_n, sx_{n+1}))) \quad \text{for all} \quad n \in \mathbb{N}.
\]
By (2.1) we have;
\[(2.4)\quad h(sx_{n+1}, sx_{n+2}) \leq \varphi(h(sx_n, sx_{n+1})).
\]
By (2.3) and (2.4) for all \( n \in \mathbb{N} \) we have;
\[
0 \leq l - \varphi(l) \leq h(sx_n, sx_{n+1}) - h(sx_{n+1}, sx_{n+2}) \to l - l = 0,
\]
when \( n \to \infty. \) We get, \( 0 \leq l - \varphi(l) \leq 0 \), which is in contradiction with \( \varphi(l) < l. \) Thus \( l = 0 \) or;
\[(2.5)\quad h(sx_n, sx_{n+1}) \to 0 \quad \text{when} \quad (n \to \infty).
\]
For all \( n, x_n, x_{n+p-1} \) lie in the different sets \( A_i \) and \( A_{i+1} \), for all \( 1 \leq i \leq p. \)
We have;
\[
h(sx_n, sx_{n+p-1}) = h(f x_{n-1}, f x_{n+p-2}) \leq \varphi(h(sx_{n-1}, sx_{n+p-2})).
\]
Similar to above, the sequence \( \{h(sx_n, sx_{n+p-1})\} \) is decreasing and converges to zero. Therefore,
\[(2.6)\quad h(sx_n, sx_{n+p-1}) \to 0 \quad \text{when} \quad n \to \infty.
\]
By (2.1) and (2.5) when \( n \to \infty \) and \( 1 \leq k \leq p \) we get;

\[
(2.7) \quad h(sx_n, sx_{n+k}) \preceq h(sx_n, sx_{n+1}) + h(sx_{n+1}, sx_{n+2}) + \cdots + h(sx_{n+k-1}, sx_{n+k}) \to 0,
\]

drawing a picture of this inequality.

when \( n \to \infty \).

Now, we show that, for every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that if \( n > m > n_0 \) with \( n \equiv_p m + 1 \) then;

\[
(2.8) \quad h(sx_n, sx_m) \leq \varepsilon.
\]

We prove this by contradict hypothesis.

If there exists \( \varepsilon_0 > 0 \) such that for all \( n > m > n_0, n \equiv_p m + 1 \),

\[
(2.9) \quad h(sx_n, sx_m) - h(sx_n, sx_m) \leq \varphi(h(sx_n, sx_m)).
\]

By (2.4) we have;

\[
(2.10) \quad h(sx_{n+1}, sx_{m+1}) \leq \varphi(h(sx_n, sx_m)).
\]

By (2.7), (2.10) and Triangle inequality, we get;

\[
(2.11) \quad \varepsilon_0 - \varphi(\varepsilon_0) \leq h(sx_n, sx_m) - h(sx_{n+1}, sx_{m+1})
\]

\[
\leq h(sx_n, sx_{n+1}) + h(sx_{n+1}, sx_{m+1})
\]

\[
+ h(sx_{n+1}, sx_m - h(sx_n, sx_{m+1})
\]

\[
= h(sx_n, sx_{n+1}) + h(sx_{m+1}, sx_m).
\]

By (2.2) and (2.11) it has been followed that,

\[
(2.12) \quad \varepsilon_0 - \varphi(\varepsilon_0) \leq 2h(sx_{m+1}, sx_m),
\]

or

\[
(2.13) \quad h(sx_{m+1}, sx_m) \geq \frac{\varepsilon_0 - \varphi(\varepsilon_0)}{2} > 0.
\]

which shows that the sequence \( h(sx_{m+1}, sx_m) \) does not converge to zero

when \( m \to \infty \), which contradicts (2.5), or (2.6) holds.

Now we prove that \( \{sx_n\} \) is a Cauchy sequence in \( X \).

Let \( \varepsilon > 0 \), by (2.7), there exists \( n_1 \in \mathbb{N} \) such that if \( n > m > n_1 \) with

\[
(2.14) \quad n \equiv_p m + 1, \text{ then;}
\]

\[
(2.15) \quad h(sx_n, sx_m) < \frac{\varepsilon}{3}.
\]

On the other hand by (2.4) there exists \( n_2 \in \mathbb{N} \) such that for any \( n > n_2 \);

\[
(2.16) \quad h(sx_n, sx_{n+k}) < \frac{\varepsilon}{3}, \quad \text{for} \quad k \in \{1, 2, \ldots, p\}.
\]

Let \( n > m > \max(n_1, n_2) \) and then we can find \( u \in \{0, 1, 2, \ldots, p\} \) such

that \( n \equiv_p m + u + 1 \).

We consider two cases;
case i) If \( u = 0 \), we have by (2.13)
\[
h(sx_n, sx_m) < \frac{\varepsilon}{3} < \varepsilon.
\]

case ii) If \( u \geq 1 \), we have;
\[
2.14
h(sx_m, sx_n) \leq h(sx_m, sx_{m-1}) + h(sx_{m-1}, sx_{m+u}) + h(sx_{m+u}, sx_n)
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

This shows that \( \{sx_n\} \) is a Cauchy sequence. Since \( X \) is a complete cone metric space, there exists \( z \in X \) such that \( \lim_{n \to \infty} sx_n = z \). Since \( fx_n = sx_{n+1} \), \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} sx_n = z \). Since \( f \) is a cyclic representation of \( X \) with respect to \( f, s \), infinitely many members of \( \{sx_n\} \) lie in \( A_i \) for \( 1 \leq i \leq p \). Since \( A_i \) is closed, \( z \in A_i \) for all \( 1 \leq i \leq p \), we see that \( z \in \bigcap_{i=1}^{p} A_i \), and there is \( u \in \bigcap_{i=1}^{p} A_i \) such that \( su = z \).

We have
\[
h(z, fu) \leq h(fx_n, fu) + h(z, fx_n)
\leq kh(sx_n, su) + h(z, fx_n),
\]
hence \( h(z, z) + kh(z, z) = 0 \), when \( n \to \infty \). Therefore \( h(z, fu) = 0 \) or \( fu = z \). Then \( fu = z = su \). So, \( f, s \) have a common point in \( \bigcap_{i=1}^{p} A_i \).

If there exists \( x^* \in \bigcap_{i=1}^{p} A_i \) such that \( sx^* = z \), then
\[
h(z, fz) \leq h(z, fx_n) + h(fx_n, fx^*)
\leq h(z, fx_n + \varphi(h(sx_n, sx^*))
\leq h(z, fx_n) + h(sx_n, sx^*) \to 0, \text{ when } n \to \infty.
\]
This implies that \( sx^* = z = fx^* \).

On the other hand, If \( fz \neq z \) then;
\[
h(z, fz) = h(fu, fz)
\leq \varphi(h(su, sz))
\leq h(z, fz). \quad \text{(By definition of } \varphi).\]
This is a contradiction, therefore we have \( fz = z \) or \( sz = fz = z \). So, \( z \) is a common fixed point of \( f, s \) in \( \bigcap_{i=1}^{p} A_i \). Now let \( z_0 \in \bigcap_{i=1}^{p} A_i \), such that \( fz_0 = sz_0 = z_0 \) and \( fz = sz = z \). By assumption, we have;
\[
h(z, z_0) = h(fz, fz_0) \leq \varphi(h(sz, sz_0)) < h(z, z_0).
\]
Now, let \( z_1, z_2 \in \bigcap_{i=1}^{p} A_i \) be two common fixed points of \( f, s \). We show that \( z_1 = z_2 \). We have \( sz_1 = fz_1, sz_2 = fz_2 \). By (2.1),
\[ h(fz_1, fz_2) \preceq \varphi(h(sz_1, sz_2)). \]

Therefore,
\[ h(z_1, z_2) = h(fz_1, fz_2) \preceq \varphi(h(sz_1, sz_2)) = \varphi(h(z_1, z_2)) \prec h(z_1, z_2). \]

So \( z_1 = z_2 \) and the common fixed point is unique. \( \square \)

**References**