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Some Results on Polynomial Numerical Hulls of Perturbed Matrices

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Abstract. In this paper, the behavior of the pseudopolynomial numerical hull of a square complex matrix with respect to structured perturbations and its radius is investigated.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices equipped with the operator norm $\|.\|$ induced by the Euclidean vector norm $\|x\| = (x^*x)^{1/2}$ on $\mathbb{C}^n$, i.e., $\|A\| = \max\left\{\|Ax\| : x \in S^1\right\}$, where $A \in \mathbb{M}_n$ and $S^1 = \{x \in \mathbb{C}^n : \|x\| = 1\}$ is the unit sphere. For a positive integer k and $A \in \mathbb{M}_n$, the *polynomial numerical hull* of A of order k is defined (e.g., see [2] and references therein) as

$$V^k(A) = \{\lambda \in \mathbb{C} : |p(\lambda)| \leq \|p(A)\| \text{ for all } p \in \mathbb{P}_k\}, \tag{1.1}$$

where $\mathbb{P}_k$ is the set of all scalar polynomials with degree at most k. It is known that

$$\sigma(A) = V^m(A) \subseteq \cdots \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq \cdots \subseteq V^1(A) = W(A), \tag{1.2}$$

where $m \geq n, \sigma(A)$ and $W(A)$ are the spectrum and the numerical range of A, respectively.

In the following proposition, we recall properties of polynomial numerical hulls of matrices which will be useful in our discussion. For more details, see [2].

Proposition 1.1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

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(i) $V^k(\alpha A) = \alpha V^k(A)$, *where* $\alpha \in \mathbb{C}$*;*
(ii) $V^k(A + \beta I) = V^k(A) + \beta$, *where* $\beta \in \mathbb{C}$*;*
(iii) $V^k(A^t) = V^k(A)$ *and* $V^k(A^*) = \overline{V^k(A)} := \left\{\overline{\lambda} : \lambda \in V^k(A)\right\}$;
(iv) *If* A *is Hermitian or skew Hermitian, then* $V^k(A) = \sigma(A)$*, for* $k \geq 2$.

Let $\epsilon \geq 0$ and $A \in \mathbb{M}_n$. The $\epsilon-$pseudopolynomial numerical hull (pseudopolynomial numerical hull for short) of order k of A is defined and denoted, (e.g., see [4]) by

(1.3)
$$V_\epsilon^k(A) = \Big\{\lambda \in \mathbb{C} : \lambda \in V^k(A + E) \text{ for some } E \in \mathbb{M}_n \text{ with } \|E\| \leq \epsilon\Big\} = \bigcup_{\|E\| \leq \epsilon} V^k(A + E).$$

It is easy to see that $V_0^k(A) = V^k(A)$, and

(1.4)
$$\sigma_\epsilon(A) = V_\epsilon^n(A) \subseteq \cdots \subseteq V_\epsilon^{k+1}(A) \subseteq V_\epsilon^k(A) \subseteq \cdots \subseteq V_\epsilon^1(A) = W_\epsilon(A),$$

where $\sigma_\epsilon(A)$ and $W_\epsilon(A)$ are the pseudospectrum and the pseudonumerical range of A, which are defined, respectively, by

$$\sigma_\epsilon(A) = \{\lambda \in \sigma(A + E) : \exists E \in \mathbb{M}_n \text{ s.t. } \|E\| \leq \epsilon\}, \tag{1.5}$$

$$W_\epsilon(A) = \{\lambda \in W(A + E) : \exists E \in \mathbb{M}_n \text{ s.t. } \|E\| \leq \epsilon\}. \tag{1.6}$$

It is clear that $\sigma_0(A) = \sigma(A)$ and $W_0(A) = W(A)$. Moreover, in the following proposition we state some properties of pseudospectrum and pseudonumerical range of matrices which will be useful in our discussion. For more details, see [4] and [7].

Proposition 1.2. *Let* $A \in \mathbb{M}_n$*. Then the following assertions are true:*
(i) *For every* $\epsilon \geq 0$, $\sigma(A) + D(0, \epsilon) \subseteq \sigma_\epsilon(A)$*, where* $D(a, \epsilon)$ *is the closed disk centered at* a *with radius* ϵ*.*
(ii) A *is normal (i.e.,* $A^*A = AA^*$*, where* $A^* = \overline{A}^T$*) if and only if* $\sigma_\epsilon(A) = \sigma(A) + D(0, \epsilon)$ *for every* $\epsilon \geq 0$;
(iii) *For every* $\epsilon \geq 0, \sigma_\epsilon(A) \subseteq W_\epsilon(A) = W(A) + D(0, \epsilon)$.

The matrix $A + E$, where $\|E\| \leq \epsilon$, is called a perturbation of A, and the parameter ϵ is said to be a tolerance for this perturbation. The theory of perturbations has many applications in quantum, control theory, numerical computation of eigenvalues and also in industrial areas; see [1]. In this paper, we are going to study some properties of pseudopolynomial numerical hulls of perturbed matrices. For this, in Section 2, we present some properties of pseudopolynomial numerical hulls of matrices. Then we study the structured pseudospectrum for perturbations of sets of all normal, Hermitian, skew-Hermitian, real symmetric and real skew-symmetric matrices. In Section 3, we investigate some inequalities for the pseudospectral numerical radius of matrices.

2. Main Results

At first, we state some algebraic properties of the pseudopolynomial numerical hulls of matrices.

Theorem 2.1. *Let $k \in \mathbb{N}$ and $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) $V^k_{\epsilon_1}(A) \subseteq V^k_{\epsilon_2}(A)$, *where* $0 \leq \epsilon_1 \leq \epsilon_2$;

(ii) $V^k(A) = \bigcap_{\epsilon \geq 0} \bigcup_{\|E\| \leq \epsilon} V^k(A+E)$;

(iii) $V^k_\epsilon(A) + D(0,\delta) \subseteq V^k_{\epsilon+\delta}(A)$, *where* $\epsilon, \delta \geq 0$;

(iv) $V^k_\epsilon(A) \subseteq D(0, r(A)+\epsilon)$, *where* $\epsilon \geq 0$ *and* $r(A) = \max\limits_{z \in W(A)} |z|$;

(v) $\bigcup_{\|E\| \leq \epsilon} V^k(E) = D(0,\epsilon)$.

Proof. The assertion in (i) follows easily from (1.3). By (1.3), that $V^k(A) \subseteq V^k_\epsilon(A)$ for every $\epsilon \geq 0$. This proves $\subseteq$. Using the fact that

$$\bigcap_{\epsilon \geq 0} \bigcup_{\|E\| \leq \epsilon} V^k(A+E) = \bigcap_{\epsilon \geq 0} V^k_\epsilon(A) \subseteq V^k_0(A) = V^k(A),$$

the proof of (ii) is complete.

To prove (iii), let $\epsilon, \delta \geq 0$, and $w = z + \xi$, where $z \in V^k_\epsilon(A)$ and $|\xi| \leq \delta$. Then there exists $E \in \mathbb{M}_n$ such that $\|E\| \leq \epsilon$ and $z \in V^k(A+E)$. By Proposition 1.1 (i),(ii), we have

$$V^k(A+E+\xi I) = V^k(A+E) + \xi.$$

So, $w = z + \xi \in V^k(A + (E + \xi I))$. Since $\|E + \xi I\| \leq \epsilon + \delta$, by (1.3), the result in (iii) holds.

The result in (iv) follows from the fact that $V^k_\epsilon(A) \subseteq W(A) + D(0,\epsilon)$. Using [4, Theorem 3.4(vi)] and (1.3), the result in (v) holds. So the proof is complete. □

Theorem 2.2. *Let $A, B \in \mathbb{M}_n$ and $\delta = \|A - B\|$. Then for all $\epsilon \geq 0$,*

$$V^k_\epsilon(A) \subseteq V^k_{\epsilon+\delta}(B).$$

In particular, if $\epsilon \geq \delta$, then

$$V^k_{\epsilon-\delta}(B) \subseteq V^k_\epsilon(A) \subseteq V^k_{\epsilon+\delta}(B),$$

and so, $V^k(B) \subseteq V^k_\delta(A)$.

Proof. Let $\lambda \in V^k_\epsilon(A)$. Then there exists $E \in \mathbb{M}_n$ with $\|E\| \leq \epsilon$ such that $\lambda \in V^k(A+E) = V^k(B + (A - B + E))$. Since $\|A-B+E\| \leq \delta+\epsilon$, we see, by (1.3), that $\lambda \in V^k_{\epsilon+\delta}(B)$.

The result in the second assertion follows easily from the first assertion and changing the roles of B and A, and also $\epsilon-\delta$ and ϵ. Since $V_0^k(B) = V^k(B)$, the final assertion also holds. □

Corollary 2.3. *Let $A, B \in \mathbb{M}_n$ and $\delta = \|AB - BA\|$. Then for all $\epsilon \geq 0$,*

$$V_\epsilon^k(AB) \subseteq V_{\epsilon+\delta}^k(BA).$$

It is known, (e.g., see [3]) that for a normal matrix $A \in \mathbb{M}_n$,

$$conv(\sigma(A)) = W(A), \tag{2.1}$$

where $conv(.)$ denotes the convex hull. Now, we prove a similar result for pseudopolynomial numerical hulls of normal matrices. For this, we need the following lemma.

Lemma 2.4. *Let $S, T \subseteq \mathbb{C}$ be such that T is a convex set. Then*

$$conv(S+T) = conv(S) + T.$$

Proof. Let $p \in conv(S+T)$. Then there exist nonnegative real numbers $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$ summing to 1 such that

$$p = \sum_{i=1}^{l} \alpha_i s_i + \sum_{i=1}^{l} \alpha_i t_i,$$

where $s_i \in S$ and $t_i \in T$ for $i = 1, \ldots, k$. Now, the convexity of T shows that $p \in conv(S) + T$.

To prove the opposite inclusion, let $p \in conv(S) + T$. Then there exist nonnegative real numbers $\alpha_1, \ldots, \alpha_k$ summing to 1 and $s_1, \ldots, s_k \in S$ and $t \in T$ such that

$$\begin{aligned} p &= \sum_{i=1}^{k} \alpha_i s_i + t \\ &= \sum_{i=1}^{k} \alpha_i (s_i + t) \in conv(S+T). \end{aligned}$$

So, the proof is complete. □

Theorem 2.5. *Let $A \in \mathbb{M}_n$ be normal and $k \in \mathbb{N}$. Then*

$$conv(\sigma_\epsilon(A)) = conv(V_\epsilon^k(A)) = W_\epsilon(A).$$

Proof. Using (1.4), Proposition 1.2(ii), Lemma 2.4 and (2.1), we have

$$\begin{aligned} W_\epsilon(A) &= W(A) + D(0,\epsilon) \\ &= conv(\sigma(A)) + D(0,\epsilon) \\ &= conv(\sigma(A) + D(0,\epsilon)) \\ &= conv(\sigma_\epsilon(A)) \end{aligned}$$

$$\subseteq conv(V_\epsilon^k(A))$$
$$\subseteq conv(W_\epsilon(A)) = W_\epsilon(A).$$

So, the proof is complete. □

Example 2.6. Consider the normal matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are $\left\{2, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$. In Figure 1, by using Theorem 2.5 and Proposition 1.2 (ii), the pseudonumerical range of A is plotted for $\epsilon = \frac{1}{2}$. Also, the circular disks are $\epsilon-$pseudospectrum of A.

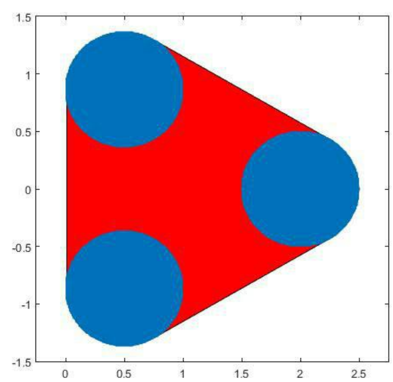

FIGURE 1. $\epsilon-$pseudonumerical range and $\epsilon-$pseudospectrum of A for $\epsilon = \frac{1}{2}$.

Now, we are going to introduce the notion of structured pseudopolynomial numerical hulls of matrices.

Definition 2.7. Let $\epsilon \geq 0$, $A \in \mathbb{M}_n$ and $\mathbb{S} \subseteq \mathbb{M}_n$. The *structured $\epsilon-$pseudopolynomial numerical hull of order k of A with respect to the structure* $\mathbb{S}$ (structured pseudopolynomial numerical hulls for short) is defined as

(2.2)
$$V_\epsilon^{k,\mathbb{S}}(A) = \left\{\lambda \in \mathbb{C} : \lambda \in V^k(A+E) \text{ for some } E \in \mathbb{S} \text{ with } \|E\| \leq \epsilon\right\}.$$

It is easy to see that

$$V_\epsilon^{k,\mathbb{S}}(A) = \bigcup_{E \in \mathbb{S} \|E\| \leq \epsilon} V^k(A+E) \subseteq V_\epsilon^k(A). \tag{2.3}$$

Moreover, we have $V_\epsilon^{n,\mathbb{S}}(A) = \sigma_\epsilon^{\mathbb{S}}(A) := \bigcup_{E\in\mathbb{S},\|E\|\leq\epsilon} \sigma(A+E)$ which is introduced and studied in [6]. Also, for $\mathbb{S} = \mathbb{M}_n$, we see that $V_\epsilon^{k,\mathbb{S}}(A) = V_\epsilon^k(A)$. In addition, we have

$$V_\epsilon^{1,\mathbb{S}}(A) = W_\epsilon^{\mathbb{S}}(A) := \bigcup_{E\in\mathbb{S},\|E\|\leq\epsilon} W(A+E),$$

which $\epsilon-$structured pseudonumerical range.

In the following theorem, we are going to state some basic properties of the structured pseudopolynomial numerical hulls of matrices. For this, if $\mathbb{S} \subseteq \mathbb{M}_n$ is a structure, $B \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$, then $B\mathbb{S}B = \{BXB : X \in \mathbb{S}\}, \alpha\mathbb{S} = \{\alpha X : X \in \mathbb{S}\}, \mathbb{S}^t = \left\{X^t : X \in \mathbb{S}\right\}$ and $\mathbb{S}^* = \{X^* : X \in \mathbb{S}\}$.

Theorem 2.8. *Let $\mathbb{S} \subseteq \mathbb{M}_n, \epsilon \geq 0$ and $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) *If $U^*\mathbb{S}U = \mathbb{S}$, where $U \in \mathbb{M}_n$ is a unitary matrix, then $V_\epsilon^{k,\mathbb{S}}(U^*AU) = V_\epsilon^{k,\mathbb{S}}(A)$;*

(ii) *$\sigma_\epsilon^{\mathbb{S}}(A) = V_\epsilon^{m,\mathbb{S}}(A) \subseteq \cdots \subseteq V_\epsilon^{k+1,\mathbb{S}}(A) \subseteq V_\epsilon^{k,\mathbb{S}}(A) \subseteq \cdots \subseteq V_\epsilon^{1,\mathbb{S}}(A) = W_\epsilon^{\mathbb{S}}(A)$ for all $m \geq n$;*

(iii) *$V_\epsilon^{k,\mathbb{S}}(\alpha A) = \alpha V_{\epsilon/|\alpha|}^{k,\mathbb{S}}(A)$ if $\alpha\mathbb{S} = \mathbb{S}$, where $\alpha \in \mathbb{C} \setminus \{0\}$;*

(iv) *$V_\epsilon^{k,\mathbb{S}}(A+\beta I) = V_\epsilon^{k,\mathbb{S}}(A) + \beta$ for all $\beta \in \mathbb{C}$;*

(v) *If $\mathbb{S}^t = \mathbb{S}$, then $V_\epsilon^{k,\mathbb{S}}(A^t) = V_\epsilon^{k,\mathbb{S}}(A)$. Also, if $\mathbb{S}^* = \mathbb{S}$, then $V_\epsilon^{k,\mathbb{S}}(A^*) = \overline{V_\epsilon^{k,\mathbb{S}}(A)}$.*

Proof. To prove (i), let $z \in V_\epsilon^{k,\mathbb{S}}(U^*AU)$. By (2.3), there exists $E \in \mathbb{S}$ with $\|E\| \leq \epsilon$ such that $z \in V^k(U^*AU + E) = V^k(A + UEU^*)$. Since $\|UEU^*\| \leq \epsilon$ and $UEU^* \in \mathbb{S}$, we see that $z \in V_\epsilon^{k,\mathbb{S}}(A)$. This shows that $V_\epsilon^{k,\mathbb{S}}(U^*AU) \subseteq V_\epsilon^{k,\mathbb{S}}(A)$. For the converse, by the above case, we have $V_\epsilon^{k,\mathbb{S}}(A) = V_\epsilon^{k,\mathbb{S}}(U(U^*AU)U^*) \subseteq V_\epsilon^{k,\mathbb{S}}(U^*AU)$. So, the proof of (i) is complete.

The assertion in (ii) follows directly from (1.2) and (2.3). To prove (iii), we use (2.3). The equality $\alpha\mathbb{S} = \mathbb{S}$ implies that $\mathbb{S} = \alpha^{-1}\mathbb{S}$. Hence, Proposition 1.1 (i) reveals that

$$\begin{aligned} V_\epsilon^{k,\mathbb{S}}(\alpha A) &= \bigcup_{E\in\mathbb{S}\|E\|\leq\epsilon} V^k(\alpha A + E) \\ &= \alpha \bigcup_{E\in\mathbb{S}\|E\|\leq\epsilon} V^k(A + \alpha^{-1}E) \\ &= \alpha \bigcup_{E'\in\mathbb{S}\|E'\|\leq\epsilon/|\alpha|} V^k(A + E') \end{aligned}$$

$$= \alpha V^{k,\mathbb{S}}_{\epsilon/|\alpha|}(A).$$

So, the assertion in (iii) holds. Using (2.3) and Proposition 1.1 (ii), we obtain the assertion (iv). Finally, to prove (v), by (2.3), we have

$$\begin{aligned} V^{k,\mathbb{S}}_{\epsilon}(A^t) &= \bigcup_{E\in\mathbb{S}\|E\|\leq\epsilon} V^k(A^t+E) \\ &= \bigcup_{E\in\mathbb{S}\|E\|\leq\epsilon} V^k(A+E^t)^t \\ &= \bigcup_{E\in\mathbb{S}\|E\|\leq\epsilon} V^k(A+E) \\ &= V^{k,\mathbb{S}}_{\epsilon}(A). \end{aligned}$$

The second assertion also holds in a similar way. So, the proof is complete. □

In the rest of this section, we use the following notations. The sets $\mathcal{N}$, $\mathcal{H}$, and $\mathsf{skew}\mathcal{H}$ denote the set of all $n\times n$ normal, Hermitian, and skew-Hermitian matrices, respectively. Also, $\mathcal{S}$ and $\mathsf{skew}\mathcal{S}$ denote the set of all $n\times n$ real symmetric and real skew-symmetric matrices, respectively.

Theorem 2.9. *Let $A\in\mathbb{M}_n$, $\epsilon\geq 0$ and $0\neq\alpha\in\mathbb{C}$. Then the following assertions are true:*

(i) *If $\mathbb{S}\in\{\mathcal{N},\mathcal{H},\mathsf{skew}\mathcal{H}\}$, then $V^{k,\mathbb{S}}_{\epsilon}(U^*AU)=V^{k,\mathbb{S}}_{\epsilon}(A)$ for any unitary matrix $U\in\mathbb{M}_n$;*

(ii) *If $\mathbb{S}\in\{\mathcal{N},\mathcal{H},\mathsf{skew}\mathcal{H},\mathcal{S},\mathsf{skew}\mathcal{S}\}$, then $V^{k,\mathbb{S}}_{\epsilon}(A^t)=V^{k,\mathbb{S}}_{\epsilon}(A)$ and $V^{k,\mathbb{S}}_{\epsilon}(A^*)=\overline{V^{k,\mathbb{S}}_{\epsilon}(A)}$;*

(iii) *If $\mathbb{S}\in\{\mathcal{S},\mathsf{skew}\mathcal{S}\}$ and $\alpha\in\mathbb{R}$, or $\mathbb{S}=\mathcal{N}$ and $\alpha\in\mathbb{C}$, then $V^{k,\mathbb{S}}_{\epsilon}(\alpha A)=\alpha V^{k,\mathbb{S}}_{\epsilon/|\alpha|}(A)$;*

(iv) *$V^{k,\mathcal{H}}_{\epsilon}(\alpha A)=\alpha V^{k,\mathcal{H}}_{\epsilon/|\alpha|}(A)$ for all $\alpha\in\mathbb{R}$, and $V^{k,\mathcal{H}}_{\epsilon}(\alpha A)=\alpha V^{k,\mathsf{skew}\mathcal{H}}_{\epsilon/|\alpha|}(A)$ for all $\alpha\in i\mathbb{R}$;*

(v) *$V^{k,\mathsf{skew}\mathcal{H}}_{\epsilon}(\alpha A)=\alpha V^{k,\mathsf{skew}\mathcal{H}}_{\epsilon/|\alpha|}(A)$ for all $\alpha\in\mathbb{R}$, and $V^{k,\mathsf{skew}\mathcal{H}}_{\epsilon}(\alpha A)=\alpha V^{k,\mathcal{H}}_{\epsilon/|\alpha|}(A)$ for all $\alpha\in i\mathbb{R}$.*

Proof. First we prove assertions (i) and (ii) in the case that $\mathbb{S}=\mathcal{N}$. The other cases in these parts are proved in the same manner.

To prove (i), we show $U^*\mathcal{N}U=\mathcal{N}$ for any unitary matrix $U\in\mathbb{M}_n$. Let $A\in U^*\mathcal{N}U$. Then there exists $B\in\mathcal{N}$ such that $A=U^*BU$. Since $U^*BU\in\mathcal{N}$, so we have $A\in\mathcal{N}$. The converse also holds by setting $B:=UAU^*$. Using Theorem 2.8 (i), the result follows.

To prove (ii), let $A\in\mathcal{N}$. Then we have

$$(A^t)^*A^t=(A^*)^tA^t=(AA^*)^t=(A^*A)^t=A^t(A^*)^t=A^t(A^t)^*.$$

This shows that $A^t \in \mathcal{N}$ and hence $\mathcal{N}^t = \mathcal{N}$. Similarly, $\mathcal{N}^* = \mathcal{N}$. So, using Theorem 2.8 (v), assertion (ii) also holds.

The assertion in (iii) follows from Theorem 2.8 (iii) and the fact that $\alpha\mathbb{S} = \mathbb{S}$, where $\mathbb{S} \in \{\mathcal{N}, \mathcal{S}, \mathsf{skew}\mathcal{S}\}$, and $\alpha \in \mathbb{C}$.

To prove the assertions in (iv) and (v), we show

$$V_\epsilon^{k,\mathcal{H}}(\alpha A) = \alpha V_{\epsilon/|\alpha|}^{k,\mathsf{skew}\mathcal{H}}(A),$$

for all $\alpha \in i\mathbb{R}$. The proof of the other cases in these assertions are similar. For this purpose, let $z \in V_\epsilon^{k,\mathcal{H}}(\alpha A)$, where $\alpha \in i\mathbb{R}$. There exists $E \in \mathcal{H}$ with $\|E\| \leq \epsilon$ such that $z \in V^k(\alpha A + E)$. So, $z \in V^k(\alpha(A + \alpha^{-1}E))$. It is clear that $\alpha^{-1}E \in \mathsf{skew}\mathcal{H}$ and $\|\alpha^{-1}E\| \leq \epsilon/|\alpha|$. Therefore, using Proposition 1.1 (i) and (2.3), we have $z \in \alpha V_{\epsilon/|\alpha|}^{k,\mathsf{skew}\mathcal{H}}(A)$. For the converse, let $z \in \alpha V_{\epsilon/|\alpha|}^{k,\mathsf{skew}\mathcal{H}}(A)$ and $\alpha \in i\mathbb{R}$. Then there exists $E \in \mathsf{skew}\mathcal{H}$ with $\|E\| \leq \epsilon/|\alpha|$ such that $z \in \alpha V^k(A + E)$. So, using Proposition 1.1 (i), $z \in V^k(\alpha A + \alpha E)$. It is clear that $\alpha E \in \mathcal{H}$ and $\|\alpha E\| \leq \epsilon$. Therefore, by using (2.3), $z \in \alpha V_{\epsilon/|\alpha|}^{k,\mathsf{skew}\mathcal{H}}(A)$ and so the proof is complete. □

In the following, using Proposition 1.2 (ii), we recall and refine the results that can be found in [6].

Lemma 2.10. *Let $A \in \mathbb{M}_n$ and $\epsilon \geq 0$.*

(i) *If $\mathbb{S} \in \{\mathcal{H}, \mathcal{S}\}$ and $A \in \mathbb{S}$, then*

$$\sigma_\epsilon^{\mathbb{S}}(A) = \sigma_\epsilon(A) \cap \mathbb{R};$$

(ii) *If $\mathbb{S} = \mathsf{skew}\mathcal{H}$ and A is skew Hermitian, then*

$$\sigma_\epsilon^{\mathbb{S}}(A) = \sigma_\epsilon(A) \cap i\mathbb{R}; \tag{2.4}$$

(iii) *Let $\mathbb{S} = \mathsf{skew}\mathcal{S}$ and A be real skew symmetric. If A has even dimension or it has odd dimension without the simple eigenvalue zero, then* (2.4) *holds.*

Proof. The assertion in (ii) follows from the fact that $A = -i(iA)$, and using the first assertion and Theorem 2.9 (v). □

Theorem 2.11. *Let $A \in \mathbb{M}_n$ and $\epsilon \geq 0$. Then the following assertions are true:*

(i) *If $\mathbb{S} \in \{\mathcal{H}, \mathcal{S}\}$ and $A \in \mathbb{S}$, then*

$$V_\epsilon^{k,\mathbb{S}}(A) = V^k(A) + [-\epsilon, \epsilon] \subseteq V_\epsilon^k(A) \cap \mathbb{R}.$$

The equality holds if $k = 1$ or n;

(ii) *If* $\mathbb{S} = \mathsf{skew}\mathcal{H}$ *and* A *is skew Hermitian, then*

$$V_\epsilon^{k,\mathbb{S}}(A) = V^k(A) + i[-\epsilon, \epsilon] \subseteq V_\epsilon^k(A) \cap i\mathbb{R}. \tag{2.5}$$

The equality holds if $k = 1$ *or* n*;*

(iii) *Let* $\mathbb{S} = \mathsf{skew}\mathcal{S}$ *and* A *be real skew symmetric. If* A *has even dimension or it has odd dimension without the simple eigenvalue zero, then* (2.5) *holds. Also, the equality in* (2.5) *holds if* $k = 1$ *or* n*.*

Proof. We just prove the assertion in (i) for the case that $\mathbb{S} = \mathcal{H}$. The other cases and the other parts can be investigate by a similar way. Since the eigenvalues of Hermitian matrices are real, by using Propositions 1.1 (iv) and 1.2, relation (2.3) and Lemma 2.10, when $k > 1$, we obtain that

$$\begin{aligned}
V_\epsilon^{k,\mathcal{H}}(A) &= \bigcup_{E \in \mathcal{H} \|E\| \leq \epsilon} V^k(A+E) \\
&= \bigcup_{E \in \mathcal{H} \|E\| \leq \epsilon} \sigma(A+E) \\
&= \sigma_\epsilon^{\mathcal{H}}(A) = \sigma_\epsilon(A) \cap \mathbb{R} \\
&= [\sigma(A) + D(0, \epsilon)] \cap \mathbb{R} \\
&= \sigma(A) + [-\epsilon, \epsilon] \\
&= V^k(A) + [-\epsilon, \epsilon] \\
&\subseteq V_\epsilon^k(A) \cap \mathbb{R}.
\end{aligned}$$

Now, let $k = 1$. Since the numerical range of a Hermitian matrix is a subset of the real axis, we conclude that

$$\begin{aligned}
W_\epsilon^{\mathcal{H}}(A) &= \bigcup_{E \in \mathcal{H} \|E\| \leq \epsilon} W(A+E) \\
&\subseteq W_\epsilon(A) \cap \mathbb{R} \\
&= [W(A) + D(0, \epsilon)] \cap \mathbb{R} \\
&= W(A) + [-\epsilon, \epsilon].
\end{aligned}$$

So, it will be enough to show that $W(A) + [-\epsilon, \epsilon] \subseteq W_\epsilon^{\mathcal{H}}(A)$. For this purpose, let $z \in W(A) + [-\epsilon, \epsilon]$ is given. Then there exist $\lambda \in W(A)$ and $\xi \in \mathbb{R}$ with $|\xi| \leq \epsilon$ such that $z = \lambda + \xi$. By setting $E = \xi I$, we see that $z \in W(A + \xi I)$ and so, the proof is complete. □

Example 2.12. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

$\epsilon = 2$ and $k > 1$. Since $\sigma(A) = \left\{2, \pm 2\sqrt{2}, 10\right\}$, using Theorem 2.11($i$) and Proposition 1.1(iv), we have

$$V_2^{k,\mathcal{H}}(A) = V_2^{k,\mathcal{S}}(A) = [-2\sqrt{2}-2, 0] \cup [2\sqrt{2}-2, 2\sqrt{2}+2] \cup [8, 12].$$

3. Additional Results

The spectral, pseudospectral, numerical and pseudonumerical radius of A, for $\epsilon \geq 0$, are defined and denoted, respectively by

$$\rho(A) = \sup\left\{|\lambda| : \lambda \in \sigma(A)\right\},$$
$$\rho_\epsilon(A) = \sup\left\{|\lambda| : \lambda \in \sigma_\epsilon(A)\right\},$$
$$r(A) = \sup\left\{|\lambda| : \lambda \in W(A)\right\},$$
$$r_\epsilon(A) = \sup\left\{|\lambda| : \lambda \in W_\epsilon(A)\right\}.$$

Like (pseudo)spectral radius and (pseudo)numerical radius, we can define the pseudopolynomial numerical radius of matrices.

Definition 3.1. The $\epsilon-$pseudopolynomial radius (pseudopolynomial numerical radius for short) of order k of $A \in \mathbb{M}_n$ is defined and denoted by

$$r_\epsilon^k(A) = \sup\left\{|\lambda| : \lambda \in V_\epsilon^k(A)\right\}.$$

Now, in view of Theorem 2.2, we have the following result.

Proposition 3.2. *Let $A, B \in \mathbb{M}_n$ and $\delta = \|A - B\|$. Then for all $\epsilon \geq 0$,*

$$r_\epsilon^k(A) \leq r_{\epsilon+\delta}^k(B).$$

In particular, if $\epsilon \geq \delta$, then

$$r_{\epsilon-\delta}^k(B) \leq r_\epsilon^k(A) \leq r_{\epsilon+\delta}^k(B).$$

Also, in view of Theorem 3.4 in [4], we can easily verify the following properties of $r_\epsilon^k(.)$.

Theorem 3.3. *Let $\epsilon \geq 0$ and $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) $r_\epsilon^k(U^*AU) = r_\epsilon^k(A)$, *where $U \in \mathbb{M}_n$ is unitary;*
(ii) $\rho(A) + \epsilon \leq \rho_\epsilon(A) = r_\epsilon^m(A) \leq \cdots \leq r_\epsilon^{k+1}(A) \leq r_\epsilon^k(A) \leq \cdots \leq r_\epsilon^1(A) = r_\epsilon(A) = r(A) + \epsilon$, *where $m \geq n$;*
(iii) $r_\epsilon^k(\alpha A + \beta I) \leq |\alpha| r_{\epsilon/|\alpha|}^k + |\beta|$, *where α, β are two arbitrary complex numbers with $\alpha \neq 0$;*
(iv) $r_\epsilon^k(A) = r_\epsilon^k(A^*) = r_\epsilon^k(A^T)$;
(v) $r_\epsilon^k(A) = |\lambda| + \epsilon$ *if and only if $A = \lambda I$, where $\lambda \in \mathbb{C}$;*
(vi) *If $A = A_1 \oplus A_2$ with $A_i \in \mathbb{M}_{n_i}$ ($n_1 + n_2 = n$), then*

$$r_\epsilon^k(A) \geq \max\left\{r_\epsilon^k(A_1), r_\epsilon^k(A_2)\right\}.$$

Remark 3.4. Since the spectral radius is not a norm, $r^k_\epsilon(.)$ is not also a norm on $\mathbb{M}_n$. Moreover, from [4, Theorem 3.9] and Theorem 3.3 (*ii*), for all $\epsilon \geq 0$ and $1 \leq k \leq n$, we have

$$\rho(A) \leq r^k(A) \leq r^k_\epsilon(A) - \epsilon \leq r(A),$$

which is a refinement of the famous inequality $\rho(A) \leq r(A)$.

Since $\rho_\epsilon(A) \leq r^k_\epsilon(A)$, for all $A \in \mathbb{M}_n$, by the same techniques as used in the proof of the [5, Lemma 3.1 and Theorems 3.2, 3.3], we obtain the following two properties.

Theorem 3.5. *Let $A, B \in \mathbb{M}_n$ be such that $AB = BA$. Then for $\epsilon \geq 0$,*

(i) $r^k_\epsilon(A + B) \leq r^k_\epsilon(A) + r^k_\epsilon(B)$;

(ii) *If A, B are nonscalar matrices and $0 < \epsilon \leq r^k_\epsilon(A) + r^k_\epsilon(B) - 1$, then*

$$r^k_\epsilon(AB) \leq r^k_\epsilon(A) r^k_\epsilon(B).$$

In particular, if $r(A) + r(B) \geq 1$, then $r(AB) \leq r(A)r(B)$.

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