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On the Structure of Metric-like Spaces

Amin Hosseini[1]* and Ajda Fošner[2]

Abstract. The main purpose of this paper is to introduce several concepts of the metric-like spaces. For instance, we define concepts such as equal-like points, cluster points and completely separate points. Furthermore, this paper is an attempt to present compatibility definitions for the distance between a point and a subset of a metric-like space and also for the distance between two subsets of a metric-like space. In this study, we define the diameter of a subset of a metric-like space, and then we provide a definition for bounded subsets of a metric-like space. In line with the aforementioned issues, various examples are provided to better understand this space.

1. Introduction

The notion of distance is fundamental in mathematics and there exist many generalizations of the concept of distance in the literature (see [4]). One such generalization is the partial metric which was introduced by Matthews (see [6]). It differs from a metric in that points are allowed to have non-zero "self-distances" (i.e., $d(x,x) \geq 0$), and the triangle inequality is modified to account for positive self-distances. O'Neill [7] extended Matthews' definition to partial metrics with "negative distances". Before describing the major points of the paper, let us recall some basic definitions and set the notations which we use in the sequel.

Definition 1.1. A mapping $p : X \times X \to \mathbb{R}^+$, where X is a non-empty set, is said to be a partial metric on X if for any $x, y, z \in X$, the following four conditions hold true:

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* Corresponding author.

- $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$;
- $p(x, x) \leq p(x, y)$;
- $p(x, y) = p(y, x)$;
- $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a *partial metric space.* A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. A sequence $\{x_n\}$ of elements of X is called *p-Cauchy* if the limit $\lim_{m,n\to\infty} p(x_n, x_m)$ exists and is finite. A partial metric space (X, p) is called complete if for each *p-Cauchy* sequence $\{x_n\}$, there is some $x \in X$ such that

$$\begin{aligned}\lim_{n\to\infty} p(x_n, x) &= p(x, x)\\ &= \lim_{m,n\to\infty} p(x_n, x_m).\end{aligned}$$

A handy example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For more examples of partial metric spaces see [1, 3, 5] and references therein.

In 2012, A. Amini-Harandi [2] introduced a new generalization of a partial metric space which is called a *metric-like space*. Here, we state the concept of a metric-like space.

Definition 1.2. A mapping $D : X \times X \to \mathbb{R}^+$, where X is a non-empty set, is said to be a metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

- $D(x, y) = 0 \Rightarrow x = y$;
- $D(x, y) = D(y, x)$;
- $D(x, y) \leq D(x, z) + D(z, y)$.

The pair (X, D) is then called a metric-like space. It is evident that the concept of a metric-like space is a generalization of the concept of a partial metric space. Partial metrics are used in computer sciences (see [3] and references therein). For this reason, working on this topic can be very useful in practical applications. Since metric-likes are, indeed, generalizations of partial metrics, knowing them can therefore provide us more applicable fields. In fact, this is our motivation to study the metric-like spaces. Each metric-like D on X generates a topology τ_D on X whose base is the family of open D-balls. An open D-ball, with center x and radius $r > 0$, is the set

$$B_D(x, r) = \{y \in X : |D(x, y) - D(x, x)| < r\}, \text{ for all } x \in X,\ r > 0.$$

It is clear that a sequence $\{x_n\}$ in the metric-like space (X, D) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} D(x_n, x) = D(x, x)$. A sequence $\{x_n\}$ of elements of X is called D-Cauchy if the limit $\lim_{n,m\to\infty} D(x_n, x_m)$

exists and is finite. A metric-like space (X, D) is called complete if for each D-Cauchy sequence $\{x_n\}$, there is some $x \in X$ such that

$$\lim_{n\to\infty} D(x_n, x) = D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m).$$

For more details see [2]. Note that every partial metric space is a metric-like space, but the converse is not true in general. For example, let $X = \mathbb{R}$, and let $D(x, y) = \max\{|x - 5|, |y - 5|\}$ for all $x, y \in \mathbb{R}$. So (X, D) is a metric-like space, but since $D(0, 0) \nleq D(1, 2)$, then (X, D) is not a partial metric space. In this article, we focus on the structure of metric-like spaces. For instance, we introduce some concepts such as equal-like points, completely separate points, distance between a point and a subset of a metric-like space, and distance between two subsets of a metric-like space. Additionally, we obtain several results for metric-like spaces.

2. Main Results

We begin with several examples of metric-like spaces.

Example 2.1. Let X be a normed space. Then, $D : X \times X \to \mathbb{R}$ defined by $D(x, y) = \|x\| + \|y\|$ is a metric-like on X.

Example 2.2. Let X be a C^*-algebra, and let

$$D(x, y) = \begin{cases} \|x - y\|, & x, y > 0, \\ \|x\| + \|y\|, & \text{otherwise.} \end{cases}$$

Then, a straightforward verification shows that D is a metric-like on X.

Example 2.3. Let (X, d) be a metric space, and let $D(x, y) = 1+d(x, y)$ for all $x, y \in X$. Then, D is a metric-like on X.

A metric-like D is called *non-Archimedean* if instead of axiom $D(x, y) \leq D(x, z) + D(z, y)$ for all $x, y, z \in X$, it satisfies the following better inequality:

$$D(x, y) \leq \max\{D(x, z), D(z, y)\}, \text{ for all } x, y, z \in X.$$

For instance, let $d : X \times X \to \mathbb{R}$ be a non-Archimedean metric. It means that d satisfies

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \text{ for all } x, y, z \in X.$$

Hence, $D(x, y) = 1 + d(x, y)$ is a *non-Archimedean metric-like* on X.

Example 2.4. Let (X, d) be a metric space, and let c be an arbitrary fixed element of X. Then, $D(x, y) = \max\{d(x, c), d(c, y)\}$ $(x, y \in X)$ is a metric-like on X.

Example 2.5. Let X be a non-empty set, and let c be an arbitrary fixed element of X. Then,

$$D(x,y)=\begin{cases} 0, & (x,y)=(c,c),\\ 1, & (x,y)\neq(c,c),\end{cases}$$

is a metric-like on X. Open balls in this space are as follows:
$B_D(x_0,r)=X$, whenever $x_0\neq c$, and in the case that $x_0=c$, we have

$$B_D(c,r)=\begin{cases} \{c\}, & 0<r\leq 1,\\ X, & r>1.\end{cases}$$

Remark 2.6. Let (X,D) be a metric-like space, and let c be an arbitrary fixed element of X. It is easy to check that

$$d(x,y)=\begin{cases} 0, & x=y,\\ D(x,y), & x\neq y,\end{cases}$$

is a metric on X. Moreover, $d_1(x,y)=|D(x,c)-D(c,y)|$ is a pseudometric on X.

Proposition 2.7. *Let (X,D) be a metric-like space, and let x_0 be an arbitrary element of X. Then $D(x,x_0)\geq\frac{1}{2}D(x_0,x_0)$ for all $x\in X$.*

Proof. To obtain a contradiction, assume that there exists an element $a\in X$ such that $D(a,x_0)<\frac{1}{2}D(x_0,x_0)$. We therefore have

$$\begin{aligned} D(x_0,x_0)&\leq D(x_0,a)+D(a,x_0)\\ &<\frac{1}{2}D(x_0,x_0)+\frac{1}{2}D(x_0,x_0)\\ &=D(x_0,x_0),\end{aligned}$$

which is a contradiction. This contradiction shows that $D(x,x_0)\geq\frac{1}{2}D(x_0,x_0)$ for all $x\in X$. □

It follows immediately from the above proposition that if (X,D) is a metric-like space and x_0 is an arbitrary element of X, then

$$\left\{x\in X: D(x,x_0)<\frac{1}{2}D(x_0,x_0)\right\}=\emptyset.$$

Moreover, we have

$$D(x,y)\geq\max\left\{\frac{1}{2}D(x,x),\frac{1}{2}D(y,y)\right\},\text{ for all } x,y\in X.$$

One can easily prove that if $\{D_n\}$ is a sequence of metric-likes on X, then

$$D(x,y)=\sum_{n=1}^{\infty}\frac{2^{-n}D_n(x,y)}{1+D_n(x,y)},$$

is a metric-like on X.

Note that in metric-like spaces the limit of a convergent sequence is not necessarily unique. For example, suppose that $X = \mathbb{R}$ and $D(x, y) = \max\{|x|, |y|\}$ for each $x, y \in X$. Putting $x_n = \frac{1}{n}$, we have $\lim_{n\to\infty} D(\frac{1}{n}, 1) = \lim_{n\to\infty} \max\{\frac{1}{n}, 1\} = 1 = D(1, 1)$, which means that the sequence $\{\frac{1}{n}\}$ converses to 1, i.e. $\frac{1}{n} \to 1$. Moreover, we have $\lim_{n\to\infty} D(\frac{1}{n}, 2) = \lim_{n\to\infty} \max\{\frac{1}{n}, 2\} = 2 = D(2, 2)$, and consequently, $\frac{1}{n} \to 2$ as well. This demonstrates that the sequence $\{\frac{1}{n}\}$ converges to two distinct points.

The above example leads us to the next definition.

Definition 2.8. (equal-like points) Let (X, D) be a metric-like space. Two points x and y of X are called equal-like points if there exists a sequence $\{x_n\}$ of X converging to both x and y, i.e. $x_n \to x$ and $x_n \to y$.

According to the previous paragraph, if $X = \mathbb{R}$ and $D(x, y) = \max\{|x|, |y|\}$ for all $x, y \in \mathbb{R}$, then 1 and 2 are equal-like points.

Definition 2.9. (completely separate points) Let (X, D) be a metric-like space. Two points x and y of X are called completely separate points if the following condition holds true:

$$D(x, y) > D(x, x) + D(y, y).$$

Theorem 2.10. *Let (X, D) be a metric-like space. Then, there is no sequence converging to two completely separate points.*

Proof. Suppose that x, y are two completely separate points. To obtain a contradiction, let $\{x_n\}$ be a sequence of X converging to both x, y. Put $d_1 = \frac{1}{3}d$, where $d = D(x, y) - D(x, x) - D(y, y)$. Since $x_n \to x$ and $x_n \to y$, for $\varepsilon = d_1$ there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \text{ implies that } |D(x_n, x) - D(x, x)| < d_1,$$

and also there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \text{ implies that } |D(x_n, y) - D(y, y)| < d_1.$$

It is evident that, if $N = \max\{N_1, N_2\}$, then we arrive at

$$\max\{|D(x_N, x) - D(x, x)|, |D(x_N, y) - D(y, y)|\} < d_1.$$

We therefore have

$$\begin{aligned}
D(x, y) &= |D(x, y)| \\
&\leq |D(x, x_N) + D(x_N, y)| \\
&= |D(x, x_N) + D(x_N, y) - D(x, x) + D(x, x) - D(y, y) + D(y, y)| \\
&\leq |D(x, x_N) - D(x, x)| + |D(x_N, y) - D(y, y)| + D(x, x) + D(y, y) \\
&< d_1 + d_1 + D(x, x) + D(y, y)
\end{aligned}$$

$$\begin{aligned}
&= 2d_1 + D(x,x) + D(y,y) \\
&= \frac{2}{3}\left(D(x,y) - D(x,x) - D(y,y)\right) + D(x,x) + D(y,y) \\
&= \frac{2}{3}D(x,y) + \frac{1}{3}D(x,x) + \frac{1}{3}D(y,y),
\end{aligned}$$

which means that $D(x,y) < D(x,x) + D(y,y)$, a contradiction. This contradiction proves our theorem, completely. □

As an immediate conclusion from the above theorem, we deduce that completely separate points are not equal-like.

Theorem 2.11. *Let (X,D) be a metric-like space. Two points x_1, x_0 of X are equal-like if and only if $B_D(x_0,r)\bigcap B_D(x_1,r) \neq \emptyset$ for any $r \in \mathbb{R}^+$.*

Proof. Suppose that x_0, x_1 are two equal-like points of X. Hence, there exists a sequence $\{x_n\}$ of X such that $x_n \to x_0$ and $x_n \to x_1$. To obtain a contradiction, assume there exists a positive number r such that $B_D(x_0,r)\bigcap B_D(x_1,r) = \emptyset$. Since $x_n \to x_0$ and $x_n \to x_1$, for $\varepsilon = r$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \text{ implies that } x_n \in B_D(x_0,r),$$

and also

$$n \geq N_2 \text{ implies that } x_n \in B_D(x_1,r).$$

Considering $N = \max\{N_1, N_2\}$, we have $x_N \in B_D(x_0,r)\bigcap B_D(x_1,r) = \emptyset$, a contradiction. Conversely, suppose that $B_D(x_0,r)\bigcap B_D(x_1,r) \neq \emptyset$ for each $r \in \mathbb{R}^+$. Our task is to show that there is a sequence $\{x_n\} \subseteq X$ converging to both x_0 and x_1. Put $r_n = \frac{1}{n}$. So according to our assumption, for any $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$x_n \in B_D\left(x_0, \frac{1}{n}\right)\bigcap B_D\left(x_1, \frac{1}{n}\right).$$

We know that the real numbers satisfy the Archimedean property. According to the Archimedean property, for a given positive number ε there exists a natural number $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Thus, for any $n \geq N$, we have

$$\begin{aligned}
x_n &\in B_D\left(x_0, \frac{1}{n}\right)\bigcap B_D\left(x_1, \frac{1}{n}\right) \\
&\subseteq B_D\left(x_0, \frac{1}{N}\right)\bigcap B_D\left(x_1, \frac{1}{N}\right) \\
&\subseteq B_D\left(x_0, \varepsilon\right)\bigcap B_D\left(x_1, \varepsilon\right).
\end{aligned}$$

It means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies that } x_n \in B_D\left(x_0, \varepsilon\right), \tag{2.1}$$

and also for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies that } x_n \in B_D(x_1, \varepsilon). \tag{2.2}$$

We conclude from (2.1) and (2.2) that the sequence $\{x_n\}$ converges to both x_0 and x_1, i.e. $x_n \to x_0$, $x_n \to x_1$. It means that x_0 and x_1 are equal-like points. □

An immediate corollary is:

Corollary 2.12. *Let (X, D) be a metric-like space. Two points x_0, x_1 of X are not equal-like if and only if there exists a positive number r such that $B_D(x_0, r) \bigcap B_D(x_1, r) = \emptyset$.*

It is clear that if x, y are two completely separate points of a metric-like space (X, D), then there exists a positive number r such that $B_D(x, r) \bigcap B_D(y, r) = \emptyset$. It suffices to assume that

$$r = \frac{1}{2}\left(D(x, y) - D(x, x) - D(y, y)\right).$$

Below, we show that the converse of the above mentioned statement is not true in general. To see this, let $X = \mathbb{R}$, $D(x, y) = |x| + |y|$, $x_0 = 1$ and $x_1 = 5$. Obviously, $D(x_0, x_1) < D(x_0, x_0) + D(x_1, x_1)$, which means that the points x_0, x_1 are not completely separate points. We claim that $B_D(x_0, 1) \bigcap B_D(x_1, 1) = \emptyset$. To show the claim, we have

$$\begin{aligned} B_D(x_0, 1) = B_D(1, 1) &= \{x \in \mathbb{R} : |D(x, 1) - D(1, 1)| < 1\} \\ &= \{x \in \mathbb{R} : ||x| - 1| < 1\} \\ &= \{x \in \mathbb{R} : -1 < |x| - 1 < 1\} \\ &= \{x \in \mathbb{R} : 0 < |x| < 2\} \\ &= (-2, 0) \cup (0, 2). \end{aligned}$$

Moreover,

$$\begin{aligned} B_D(x_1, 1) = B_D(5, 1) &= \{x \in \mathbb{R} : |D(x, 5) - D(5, 5)| < 1\} \\ &= \{x \in \mathbb{R} : ||x| - 5| < 1\} \\ &= \{x \in \mathbb{R} : -1 < |x| - 5 < 1\} \\ &= \{x \in \mathbb{R} : 4 < |x| < 6\} \\ &= (4, 6) \cup (-6, -4). \end{aligned}$$

Therefore, we have $B_D(1, 1) \bigcap B_D(5, 1) = \emptyset$.

In every metric-like space (X, D), we have the following simple statements:

(i) If x_0, x_1 are completely separate points, then there exists a positive number r such that $B_D(x_0, r) \bigcap B_D(x_1, r) = \emptyset$.
(ii) Two points x_0, x_1 are not equal-like if and only if there exists a positive number r such that $B_D(x_0, r) \bigcap B_D(x_1, r) = \emptyset$.
(iii) If x_0, x_1 are completely separate points, then they are not equal-like points.

Definition 2.13. (cluster points) Let (X, D) be a metric-like space, and let $\mathcal{A}$ be a subset of X. A point $x_0 \in X$ is said to be a cluster point of $\mathcal{A}$ whenever for every $\varepsilon > 0$ there exists $a \in \mathcal{A}$ such that $|D(a, x_0) - D(x_0, x_0)| < \varepsilon$.

The set of all *cluster points* of $\mathcal{A}$ is called the closure of $\mathcal{A}$ and denoted by $\overline{\mathcal{A}}$. Clearly, $x_0 \in \overline{\mathcal{A}}$ if and only if $B_D(x_0, \varepsilon) \cap \mathcal{A} \neq \emptyset$ for every $\varepsilon > 0$.

In the following, we establish a theorem to present a necessary and sufficient condition for cluster points in the metric-like spaces.

Theorem 2.14. *Let (X, D) be a metric-like space, and let $\mathcal{A}$ be a subset of X. Then $x_0 \in \overline{\mathcal{A}}$ if and only if there exists a sequence $\{a_n\} \subseteq \mathcal{A}$ converging to x_0.*

Proof. Suppose that $x_0 \in \overline{\mathcal{A}}$. So for each $\varepsilon_n = \frac{1}{n}$ $(n \in \mathbb{N})$, there is an element $a_n \in \mathcal{A}$ such that $|D(a_n, x_0) - D(x_0, x_0)| < \frac{1}{n}$, which means that $\lim_{n\to\infty} D(a_n, x_0) = D(x_0, x_0)$. Consequently the sequence $\{a_n\} \subseteq \mathcal{A}$ converges to x_0. Conversely, assume that $\{a_n\} \subseteq \mathcal{A}$ is a sequence converging to x_0. Now we want to show that $x_0 \in \overline{\mathcal{A}}$. Let ε be an arbitrary positive number. Therefore, there exists a natural number N such that for any $n \geq N$, we have $a_n \in B_D(x_0, \varepsilon)$, which means that $\mathcal{A} \cap B_D(x_0, \varepsilon) \neq \emptyset$. Since $\varepsilon > 0$ is arbitrary, $x_0 \in \overline{\mathcal{A}}$ is achieved. $\square$

Example 2.15. Let $X = \mathbb{R}$, $\mathcal{A} = (-1, 1)$, and let x_0 be an arbitrary real number. If $D(x, y) = \max\{|x|, |y|\}$ for any $x, y \in X$, then

$$\begin{aligned} \lim_{n\to\infty} D\left(\frac{1}{n}, x_0\right) &= \lim_{n\to\infty} \max\left\{\frac{1}{n}, |x_0|\right\} \\ &= |x_0| \\ &= D(x_0, x_0), \end{aligned}$$

which means that $\frac{1}{n} \to x_0$. Since $\left\{\frac{1}{n}\right\} \subseteq \mathcal{A}$ and also x_0 is an arbitrary element of $\mathbb{R}$, $\overline{\mathcal{A}} = \mathbb{R}$.

In the following, we define the "distance" between a point and a subset of a metric-like space.

Definition 2.16. Let (X, D) be a metric-like space, and let $\mathcal{A}$ be a non-empty subset of X. The distance between a point $x_0 \in X$ and $\mathcal{A}$ is defined as follows:

$$d(x_0, \mathcal{A}) := \inf \{ |D(x_0, a) - D(x_0, x_0)| : a \in \mathcal{A} \}.$$

For example, let $X = \mathbb{R}$, $D(x, y) = \max\{|x|, |y|\}$ for any $x, y \in X$, and let $\mathcal{A} = \{n + \frac{1}{n} \ : \ n \in \mathbb{N}\}$. Then, we have

$$\begin{aligned} d(2, \mathcal{A}) &= \inf \left\{ \left| \max \left\{ n + \frac{1}{n}, 2 \right\} - \max \{2, 2\} \right| : n \in \mathbb{N} \right\} \\ &= \inf \left\{ \left| n + \frac{1}{n} - 2 \right| : n \in \mathbb{N} \right\} \\ &= 0. \end{aligned}$$

As another example in this regard, let $X = \mathbb{R}$, $D(x, y) = |x| + |y|$, $\mathcal{A} = (-1, 1)$, and let $x_0 = 4$. Then, we have

$$\begin{aligned} d(4, \mathcal{A}) &= \inf \{ |D(4, 4) - D(4, a)| : a \in \mathcal{A} \} \\ &= \inf \{ |4 - |a|| : a \in \mathcal{A} \} \\ &= 3. \end{aligned}$$

The next theorem demonstrates the relationship between distance from a subset of a metric-like space and its closure.

Theorem 2.17. *Let (X, D) be a metric-like space, and let $\mathcal{A}$ be a non-empty subset of X. Then, $\overline{\mathcal{A}} = \{x \in X : d(x, \mathcal{A}) = 0\}$.*

Proof. First, we show that $\overline{\mathcal{A}} \subseteq \{x \in X : d(x, \mathcal{A}) = 0\}$. We have the following expressions:

$$\begin{aligned} x_0 \in \overline{\mathcal{A}} \quad &\Rightarrow \quad \forall \varepsilon > 0, B_D(x_0, \varepsilon) \cap \mathcal{A} \neq \emptyset, \\ &\Rightarrow \quad \forall \varepsilon > 0, \exists a_0 \in \mathcal{A} : |D(x_0, x_0) - D(x_0, a_0)| < \varepsilon, \\ &\Rightarrow \quad \forall \varepsilon > 0, d(x_0, \mathcal{A}) < \varepsilon, \\ &\Rightarrow \quad d(x_0, \mathcal{A}) = 0, \end{aligned}$$

which implies that $\overline{\mathcal{A}} \subseteq \{x \in X : d(x, \mathcal{A}) = 0\}$. Conversely, we show that if $d(x_0, \mathcal{A}) = 0$, then $x_0 \in \overline{\mathcal{A}}$. For each $\varepsilon_n = \frac{1}{n}$ $(n \in \mathbb{N})$, there exists an element $a_n \in \mathcal{A}$ such that $|D(x_0, x_0) - D(x_0, a_n)| < \frac{1}{n}$. Hence, we can get a sequence $\{a_n\}$ of $\mathcal{A}$ converging to x_0. Now Theorem 2.14 implies that $x_0 \in \overline{\mathcal{A}}$. So, $\{x \in X : d(x, \mathcal{A}) = 0\} \subseteq \overline{\mathcal{A}}$. This yields the desired result. □

Besides, one can easily prove the following proposition:

$$d(x_0, \mathcal{A}) = 0 \Leftrightarrow \forall\ \varepsilon > 0\ \exists\ a \in \mathcal{A} \text{ such that } |D(x_0, x_0) - D(x_0, a)| < \varepsilon.$$

Remark 2.18. Let $X = \mathbb{R}$, $\mathcal{A} = (-1, 1)$, and let $D(x, y) = \max\{|x|, |y|\}$ for any $x, y \in \mathbb{R}$. We know that $\overline{\mathcal{A}} = \mathbb{R}$. In the following, it will be shown that $\{x \in \mathbb{R} : d(x, \mathcal{A}) = 0\} = \mathbb{R}$. Suppose that $x_0 \in \mathbb{R}$ with $|x_0| \geq 1$.

Then for every $a \in \mathcal{A}$, $D(a, x_0) = \max\{|a|, |x_0|\} = |x_0|$. In this case we have

$$\begin{aligned} d(x_0, \mathcal{A}) &= \inf\{|D(x_0, x_0) - D(x_0, a)| : a \in \mathcal{A}\} \\ &= \inf\{||x_0| - |x_0|| : a \in \mathcal{A}\} \\ &= 0. \end{aligned}$$

Obviously, $d(x_0, \mathcal{A}) = 0$ for any $x_0 \in \mathcal{A}$. Therefore, it is observed that

$$\{x \in \mathbb{R} : d(x, \mathcal{A}) = 0\} = \mathbb{R} = \overline{\mathcal{A}}.$$

In the following, we define the distance between two non-empty subsets of a metric-like space.

Definition 2.19. Let (X, D) be a metric-like space, and let $\mathcal{A}, \mathcal{B}$ be two non-empty subsets of X. The distance between $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

$$d(\mathcal{A}, \mathcal{B}) := \min\{\inf\{d(a, \mathcal{B}) : a \in \mathcal{A}\}, \inf\{d(b, \mathcal{A}) : b \in \mathcal{B}\}\}.$$

Example 2.20. Suppose that $X = \mathbb{R}$ and $D(x, y) = |x| + |y|$ for all $x, y \in \mathbb{R}$. We want to calculate the distance between the sets $\mathcal{A} = (-1, 1)$ and $\mathcal{B} = (3, 4)$ in the metric-like space (X, D). For an arbitrary element $a \in \mathcal{A}$, we have

$$\begin{aligned} d(a, \mathcal{B}) &= \inf\{|D(a, a) - D(a, b)| : b \in \mathcal{B}\} \\ &= \inf\{||a| - |b|| : b \in \mathcal{B}\} \\ &= 3 - |a|. \end{aligned}$$

So, $\inf\{d(a, \mathcal{B}) : a \in \mathcal{A}\} = \inf\{3 - |a| : a \in \mathcal{A}\} = 2$. Moreover, if b is an arbitrary element of $\mathcal{B}$, then

$$\begin{aligned} d(b, \mathcal{A}) &= \inf\{|D(a, b) - D(b, b)| : a \in \mathcal{A}\} \\ &= \inf\{b - |a| : a \in \mathcal{A}\} \\ &= b - 1. \end{aligned}$$

Hence, $\inf\{d(b, \mathcal{A}) : b \in \mathcal{B}\} = \inf\{b - 1 : b \in \mathcal{B}\} = 3 - 1 = 2$. So we see that $d(\mathcal{A}, \mathcal{B}) = \min\{2, 2\} = 2$.

Example 2.21. Suppose that $X = \mathbb{R}$ and $D(x, y) = \max\{|x|, |y|\}$ for all $x, y \in \mathbb{R}$. In the following, we will calculate the distance between the sets $\mathcal{A} = (-1, 1)$ and $\mathcal{B} = (3, 4)$ in the metric-like space (X, D). If a is an arbitrary element of $\mathcal{A}$, then we have

$$\begin{aligned} d(a, \mathcal{B}) &= \inf\{|D(a, a) - D(a, b)| : b \in \mathcal{B}\} \\ &= \inf\{||a| - |b|| : b \in \mathcal{B}\} \\ &= 3 - |a|. \end{aligned}$$

So $\inf\{d(a,\mathcal{B}) : a \in \mathcal{A}\} = \inf\{3-|a| : a \in \mathcal{A}\} = 2$. If b is an arbitrary element of $\mathcal{B}$, then we have

$$\begin{aligned} d(b,\mathcal{A}) &= \inf\{|D(a,b)-D(b,b)| : a \in \mathcal{A}\} \\ &= \inf\{b-b : a \in \mathcal{A}\} \\ &= 0, \end{aligned}$$

which implies that $\inf\{d(b,\mathcal{A}) : b \in \mathcal{B}\} = 0$ Thus, $d(\mathcal{A},\mathcal{B}) = \min\{2,0\} = 0$.

The following discussion is interesting. Let $\mathcal{A} = (-1,1)$ and $\mathcal{B}$ be an arbitrary subset of $\mathbb{R}$. In the metric-like space $(\mathbb{R},D)$, whenever $D(x,y) = \max\{|x|,|y|\}$ for all $x,y \in \mathbb{R}$, we have shown that $\mathbb{R} = \overline{\mathcal{A}} = \{x \in \mathbb{R} : d(x,\mathcal{A}) = 0\}$ and so $\inf\{d(b,\mathcal{A}) : b \in \mathcal{B}\} = 0$. From this fact, we infer that

$$\begin{aligned} d(\mathcal{A},\mathcal{B}) &= \min\{\inf\{d(a,\mathcal{B}) : a \in \mathcal{A}\},\ \inf\{d(b,\mathcal{A}) : b \in \mathcal{B}\}\} \\ &= \min\{\inf\{d(a,\mathcal{B}) : a \in \mathcal{A}\},\ 0\} \\ &= 0. \end{aligned}$$

Definition 2.22. Let (X,D) be a metric-like space, and let $\mathcal{A}$ be a subset of X. We define the diameter of $\mathcal{A}$ as follows:

$$\operatorname{diam}(\mathcal{A}) := \sup\{|D(x,y)-D(x,x)|, |D(x,y)-D(y,y)| : x,y \in \mathcal{A}\}.$$

A subset $\mathcal{A} \subseteq X$ is said to be bounded whenever $\operatorname{diam}(\mathcal{A}) < \infty$.

Example 2.23. Let $X = \mathbb{R}$, $D(x,y) = |x|+|y|$ for all $x,y \in \mathbb{R}$, and let $\mathcal{A} = (3,5)$. In this case, we have

$$\begin{aligned} \operatorname{diam}(\mathcal{A}) &= \sup\{|D(x,y)-D(x,x)|, |D(x,y)-D(y,y)| : x,y \in \mathcal{A}\} \\ &= \sup\{||x|-|y||, ||x|-|y|| : x,y \in \mathcal{A}\} \\ &= \sup\{|x-y| : x,y \in \mathcal{A}\} \\ &= 2. \end{aligned}$$

Example 2.24. Let $X = \mathbb{R}$, $D(x,y) = \max\{|x|,|y|\}$ for all $x,y \in \mathbb{R}$, and let $\mathcal{A} = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. Hence, we have

$$\begin{aligned} \operatorname{diam}(\mathcal{A}) = \sup\Bigg\{ & \left|D\left(\frac{m}{m+1},\frac{n}{n+1}\right) - D\left(\frac{m}{m+1},\frac{m}{m+1}\right)\right|, \\ & \left|D\left(\frac{m}{m+1},\frac{n}{n+1}\right) - D\left(\frac{n}{n+1},\frac{n}{n+1}\right)\right| : m,n \in \mathbb{N}, m \geq n\Bigg\} \\ = \sup\Bigg\{ & \left|\max\left\{\frac{m}{m+1},\frac{n}{n+1}\right\} - \frac{m}{m+1}\right|, \end{aligned}$$

$$\left|\max\left\{\frac{m}{m+1},\frac{n}{n+1}\right\}-\frac{n}{n+1}\right| : m,n\in\mathbb{N}, m\geq n\right\}$$
$$= \sup\left\{\left|\frac{m}{m+1}-\frac{m}{m+1}\right|,\left|\frac{m}{m+1}-\frac{n}{n+1}\right| : m,n\in\mathbb{N}, m\geq n\right\}$$
$$= \sup\left\{\left|\frac{m}{m+1}-\frac{n}{n+1}\right| : m,n\in\mathbb{N}, m\geq n\right\}$$
$$= 1-\frac{1}{2}$$
$$= \frac{1}{2}.$$

Sequences and convergence play an essential role in metric-like spaces. The following theorem shows that in a metric-like space every convergent sequence is also bounded.

Theorem 2.25. *Suppose that (X, D) is a metric-like space and $\{x_n\}$ is a sequence converging to $x \in X$. In this case, the set $\mathcal{A} = \{x_n : n \in \mathbb{N}\}$ is a bounded subset of X.*

Proof. For $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies that } |D(x_n, x) - D(x, x)| < 1.$$

We obtain from the above statement that $D(x_n, x) < 1 + D(x, x)$ for all $n \geq N$. So for all $m, n \geq N$, we have

$$\begin{aligned} |D(x_n, x_m) - D(x_n, x_n)| &\leq D(x_m, x_n) + D(x_n, x_n) \\ &\leq D(x_m, x) + D(x, x_n) + D(x_n, x) + D(x, x_n) \\ &< 4\Big(1 + D(x, x)\Big) < \infty. \end{aligned}$$

Using a reasoning like above, we get that

$$|D(x_n, x_m) - D(x_m, x_m)| < 4\,(1 + D(x, x)) < \infty,$$

for all $m, n \geq N$. So $\operatorname{diam}(\mathcal{A}) \leq 4\,(1 + D(x, x)) < \infty$, which means that $\mathcal{A}$ is a bounded subset of the metric-like space (X, D). Thereby, we get the required result. □

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[1] Department of Mathematics, Kashmar Higher Education Institute, Kashmar, Iran.

E-mail address: hosseini.amin82@gmail.com, a.hosseini@kashmar.ac.ir

[2] University of Primorska, Cankarjeva 5, SI-6000 Koper, Slovenia.

E-mail address: ajda.fosner@fm-kp.si