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A Generalization of the Meir-Keeler Condensing Operators and its Application to Solvability of a System of Nonlinear Functional Integral Equations of Volterra Type

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Abstract. In this paper, we generalize the Meir-Keeler condensing operators via a concept of the class of operators $O(f;.)$, that was given by Altun and Turkoglu [4], and apply this extension to obtain some tripled fixed point theorems. As an application of this extension, we analyze the existence of solution for a system of nonlinear functional integral equations of Volterra type. Finally, we present an example to show the effectiveness of our results. We use the technique of measure of noncompactness to obtain our results.

1. Introduction

The theory of measure of noncompactness (MNC) is an important branch of nonlinear functional analysis. This concept was introduced by Kuratowski [13] in 1930. Since then, many authors applied this notions for studying and solving of integral equations (see, for example, [1, 2, 7, 9, 11, 17]). Moreover, the study of existence of solutions for a systems of integral and differential equations of Volterra type has been considered in many papers (see for instance [6, 15, 16] and the references therein). Darbo's fixed point theorem [10] which ensures the existence of fixed point is an essential application of this measure, since it extends both Schauder fixed point and Banach contraction principle.

In 1969, Meir and Keeler [14] obtained an interesting fixed-point theorem, which is a generalization of the Banach contraction principle.

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Samet [18] obtained some coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces.

Arab et al. introduced a new measure of noncompactness on Banach space $BC(\Omega)$, consisting of all bounded continuous functions on unbounded set Ω of $\mathbb{R}^n$ and applied it to study the existence of solutions for a class of nonlinear functional integral equations of Volterra type [5].

Aghajani et al. proved some fixed point theorems for Meir- Keeler condensing operators on the Banach space, then discussed the solvability of a nonlinear functional integral equation in [3].

The aim of this work is to generalize the Meir- Keeler condensing operators via the concept of the class of operators $O(f;.)$ and apply our extension to obtain some tripled fixed point theorems. Furthermore, we study the existence of solutions for the system

$$\begin{cases} u_1(x) = f\left(x, u_1(x), u_2(x), u_3(x), \int_{\Lambda(x)} g\left(x, y, u_1(y), u_2(y), u_3(y)\right) dy\right), \\ u_2(x) = f\left(x, u_2(x), u_1(x), u_3(x), \int_{\Lambda(x)} g\left(x, y, u_2(y), u_1(y), u_3(y)\right) dy\right), \\ u_3(x) = f\left(x, u_3(x), u_2(x), u_1(x), \int_{\Lambda(x)} g\left(x, y, u_3(y), u_2(y), u_1(y)\right) dy\right), \end{cases} \tag{1.1}$$

where $x \in \Omega$ (Ω is an unbounded subset of $\mathbb{R}^n$) and f, g, u, and Λ, are continuous functions satisfy some certain conditions, specified later. Also, we show that Eq.(1.1) has one solution that belongs to $\{BC(\Omega)\}^3$.

The structure of this paper is as follows. In Section 2, some definitions and concepts are recalled. Sections 3 and 4 are devoted to extend the Meir-Keeler condensing operators and prove some tripled fixed point theorems. In section 5, as an application for the main results, we present an existence theorem. Finally, in section 6 an example is given to illustrate our results.

2. Preliminaries

In this section, we provide some basic definitions and facts which will be used in our main results. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ and $(E, \|.\|)$ be a real Banach space with the zero element 0. We write $\overline{B}(x, r)$ to denote the closed ball centered at x with radius r. If X is a nonempty subset of E then the symbols $\overline{X}$ and $\operatorname{conv} X$ denote the closure and closed convex hull of X, respectively. Moreover,

$\mathfrak{M}_E$ is the family of a nonempty bounded subset of E, and $\mathfrak{N}_E$ denotes its subfamily consisting of all relatively compact sets.

Definition 2.1 ([8]). A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;
2° $X \subset Y \quad \Rightarrow \quad \mu(X) \leq \mu(Y)$;
3° $\mu(\overline{X}) = \mu(X)$;
4° $\mu(\operatorname{conv} X) = \mu(X)$;
5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
6° If $\{X_n\}$ is a sequence of closed subsets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

The following concept of $O(f;.)$ was given by Altun and Turkoglu [4].

Let $F([0,\infty))$ be the class of all functions $f : [0,\infty) \to [0,\infty)$ and let Θ be the class of all operators

$$O(\bullet;\cdot) : F([0,\infty)) \to F([0,\infty)),\ f \to O(f;\cdot),$$

satisfying the following conditions:

(i) $O(f;t) > 0$ for $t > 0$ and $O(f;0) = 0$;
(ii) $O(f;t) \leq O(f;s)$ for $t \leq s$;
(iii) $\lim_{n\to\infty} O(f;t_n) = O(f;\lim_{n\to\infty} t_n)$;
(iv) $O(f;\max\{t,s\}) = \max\{O(f;t), O(f;s)\}$ for some $f \in F([0,\infty))$,

Definition 2.2 ([12]). A tripled (x,y,z) of a mapping $T : X \times X \times X \to X$ is called a tripled fixed point if

$$T(x,y,z) = x, \qquad T(y,x,z) = y, \qquad T(z,y,x) = z.$$

The following theorems are basic for our studies.

Theorem 2.3 (Schauder [1]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Then every compact and continuous map $T : C \to C$ has at least one fixed point.*

Definition 2.4 ([14]). Let (X,d) be a metric space. Then, a mapping T on X is said to be a Meir-Keeler contraction (MKC, for short) if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq d(x,y) < \epsilon + \delta \quad \Rightarrow \quad d(Tx,Ty) < \epsilon,$$

for all $x, y \in X$.

Theorem 2.5 (Meir and Keeler [14])**.** *Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

Definition 2.6 ([3])**.** Let C be a nonempty subset of a Banach space E and μ an arbitrary measure of noncompactness on E. We say that the operator $T : C \to C$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \mu(X) < \epsilon + \delta \quad \Rightarrow \quad \mu(T(X)) < \epsilon,$$

for any bounded subset X of C.

Obviously, a MKC defined by Definition 2.4 is a Meir-Keeler condensing operator, if we take $\mu(X) = diamX$ as a measure of noncompactness.

Definition 2.7 ([14])**.** A function $\lambda : [0, \infty) \to [0, \infty)$ will be called an L-function if $\lambda(0) = 0$, and for every $s > 0$ there exists $u > s$ such that $\lambda(t) \leq s$ for $t \in [s, u]$.

Note that every L-function satisfies $\lambda(s) \leq s \ \forall s > 0$.

Definition 2.8 ([3])**.** We say that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly L-function if $\psi(0) = 0$, $\psi(s) > 0$ for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\psi(t) < s$, for all $t \in [s, s + \delta]$.

3. Main Results

In this section, we generalize and extend the Meir-Keeler condensing operator via the concept of the class of $O(f; .)$.

Definition 3.1. Let C be a nonempty subset of a Banach space E and μ an arbitrary measure of noncompactness on E. We say that the operator $T : C \to C$ is an Extended Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq O(f; \mu(X)) < \epsilon + \delta \quad \Rightarrow \quad O(f; \mu(T(X))) < \epsilon,$$

for any bounded subset X of C and $O(\bullet; \cdot) \in \Theta$.

Remark 3.2. The Definition 2.6 is followed if $O(f; t) = t$ and $f = I$ (identity map) in Definition 3.1.

Theorem 3.3. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Extended Meir-Keeler condensing operator, then T has at least one fixed point in C.*

Proof. By induction, we construct a sequence $\{C_n\}$ such that $C_0 = C$ and $C_n = \text{conv}(TC_{n-1})$ for $n \geq 1$. We have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots.$$

If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$ then C_N is relatively compact. In this case, Theorem 2.3 implies that T has a fixed point. Now, we suppose that $\mu(C_n) \neq 0$ for $n \geq 0$. Define $\epsilon_n = O(f;\mu(C_n))$ and $\delta_n = \delta(\epsilon_n) > 0$. By the definition of C_n and $\epsilon_n < \epsilon_n + \delta_n$ we have

$$\begin{aligned} \epsilon_{n+1} &= O(f;\mu(C_{n+1})) \\ &= O(f;\mu(\operatorname{conv} TC_n)) \\ &= O(f;\mu(TC_n)) \\ &\leq O(f;\mu(C_n)) \\ &= \epsilon_n. \end{aligned} \tag{3.1}$$

Hence $\{\epsilon_n\}$ is a positive non-increasing sequence of real numbers and there exists $r \geq 0$ such that $\epsilon_n \to r$ as $n \to \infty$. We show that $r = 0$. If $r \neq 0$, then there exists N_0 such that $n > N_0$ implies $r \leq \epsilon_n < r + \delta_r$, and by the definition of Meir-Keeler condensing operator, we get $\epsilon_{n+1} < r$, which is a contradiction, so $r = 0$. Therefore, by letting $n \to \infty$ in (3.1) we infer that

$$\lim_{n\to\infty} \left[O\left(f;\mu(C_{n+1})\right)\right] = 0.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty}[O(f;\mu(C_{n+1}))] &= [O(f;\lim_{n\to\infty}\mu(C_{n+1}))] \\ &= 0, \end{aligned}$$

we know that $O(f;0) = 0$. Thus, $\lim_{n\to\infty}\mu(C_n) = 0$. Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, 3, \ldots$ then from 6° of definition MNC, $C_\infty = \cap_{n=1}^{\infty} C_n$ is a nonempty, closed and convex set, invariant under T and belongs to $\ker \mu$. Consequently, from Theorem 2.3, we deduce that T has at least a fixed point. □

Theorem 3.4. *Let C be a nonempty and bounded subset of a Banach space E, μ an arbitrary measure of noncompactness on E and $T : C \to C$ be a continuous operator that*

$$O\left(f;\mu(TX)\right) < \psi\left(O(f;\mu(X))\right), \tag{3.2}$$

for all X of $\mathfrak{M}_E$ with $\mu(X) \neq 0$, $O(\bullet;\cdot) \in \Theta$ and ψ is a strictly L-function. Then T is an Extended Meir-Keeler condensing operator.

Proof. Let $O\left(f;\mu(X)\right) := \epsilon > 0$ and there exist $\delta > 0$, t such that $\epsilon \leq t < \epsilon + \delta$. By the assumptions, $\psi(\epsilon) < \epsilon$. If X is a subset of C such that

$$\epsilon \leq O(f;\mu(X)) < \epsilon + \delta(\epsilon),$$

then by applying (3.2), we infer that

$$O\left(f;\mu(TX)\right) < \psi\left(O(f;\mu(X))\right)$$

$$= \psi(\epsilon)$$
$$< \epsilon.$$

Therefore, T is an Extended Meir-Keeler condensing operator. □

As a consequence of Theorems 3.3 and 3.4 we obtain the following corollary.

Corollary 3.5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be a continuous operator such that*

$$O\left(f; \mu(TX)\right) < \psi\left(O(f; \mu(X))\right),$$

for all X of $\mathfrak{M}_E$ with $\mu(X) \neq 0$, $O(\bullet; \cdot) \in \Theta$ and ψ is a strictly L-function. Then, T has at least one fixed point.

4. Tripled Fixed Point Results

first we define the notion of a three variate Extended Meir-Keeler condensing operator.

Definition 4.1. Let C be a nonempty subset of a Banach space E and μ an arbitrary measure of noncompactness on E. We say that $T : C \times C \times C \to C$ is an Extended Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq O\left(f; \max\left\{\mu(X_1), \mu(X_2), \mu(X_3)\right\}\right) < \epsilon + \delta, \tag{4.1}$$

where,

$$O\left(f; \mu(T(X_1 \times X_2 \times X_3))\right) < \epsilon,$$

for any bounded subset X of C and $O(\bullet; \cdot) \in \Theta$.

Theorem 4.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and μ an arbitrary measure of noncompactness on E. If $T : C \times C \times C \to C$ is a continuous Extended Meir-Keeler condensing operator, then T has at least one tripled fixed point.*

Proof. We know that

$$\widetilde{\mu}(X) := \max\left\{\mu(X_1), \mu(X_2), \mu(X_3)\right\},$$

for any bounded subset $X \subset E \times E \times E$, defines a measure of noncompactness on $E \times E \times E$, where X_i, $(i = 1, 2, 3)$ denote the natural projections of X. Also, the operator $G : C \times C \times C \to C \times C \times C$, given by

$$G(x, y, z) := \left(T(x, y, z), T(y, x, z), T(z, x, y)\right),$$

is clearly continuous on $C\times C\times C$. Now, we claim that G satisfies all the conditions of Theorem 3.3. To prove this claim, let $\epsilon>0$ and $\delta(\epsilon)>0$ be as in Definition 4.1. If X is a bounded subset of $C\times C\times C$ such that

$$\epsilon\leq O\left(f;\widetilde{\mu}(X)\right)<\epsilon+\delta(\epsilon),$$

then

$$\epsilon\leq O\left(f;\max\{\mu(X_1),\mu(X_2),\mu(X_3)\}\right)<\epsilon+\delta(\epsilon),$$

where $X_i, i=1,2,3$ denote the natural projections of X. By condition (iv) of $O(f;.)$ and (4.1), we have

(4.2)

$$\begin{aligned}O\left(f;\widetilde{\mu}(G(X))\right)\leq{}& O\Big(f;\widetilde{\mu}(T(X_1\times X_2\times X_3)\times T(X_2\times X_1\times X_3)\\&\times T(X_3\times X_2\times X_1))\Big)\\&=O\Big(f;\max\left\{\mu(T(X_1\times X_2\times X_3)),\mu(T(X_2\times X_1\times X_3))\right.\\&\left.,\mu(T(X_3\times X_2\times X_1))\right\}\Big)\\&<\epsilon.\end{aligned}$$

Therefore, from Theorem 3.3, G has at least one fixed point in $C\times C\times C$. ☐

Now, we present and prove a tripled fixed point theorem.

Theorem 4.3. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E, μ an arbitrary measure of noncompactness on E, and ψ a strictly L-function. Suppose that $T:C\times C\times C\to C$ is a continuous operator satisfying*

(4.3)

$$O\left(f;\mu(T(X_1\times X_2\times X_3)\right)<\frac{1}{3}O\left(f;\psi(\mu(X_1)+\mu(X_2)+\mu(X_3)\right),$$

for any subset X_1,X_2,X_3 of C and for any bounded subset X of C and $O(\bullet;\cdot)\in\Theta$. Then, T has at least one tripled fixed point.

Proof. First, define the mapping $G:C\times C\times C\to C\times C\times C$, given by

$$G(x,y,z):=(T(x,y,z),T(y,x,z),T(z,x,y)),$$

which is a continuous map. We know that

$$\widetilde{\mu}(X):=\left\{\mu(X_1)+\mu(X_2)+\mu(X_3)\right\},$$

for any bounded subset $X\subset E\times E\times E$, defines a measure of noncompactness on $E\times E\times E$, where X_i, $i=1,2,3$ denote the natural projections of X. Let $X\subset C\times C\times C$ be any nonempty subset. Then, by (4.3) and 2° we have

$$
\begin{aligned}
O\left(f;\widetilde{\mu}(G(X))\right) &\leq O\Big(f;\widetilde{\mu}(T(X_1\times X_2\times X_3))\times T(X_2\times X_1\times X_3) \\
&\quad \times T(X_3\times X_2\times X_1)\Big) \\
&= O\Big(f;\mu(T(X_1\times X_2\times X_3))+\mu(T(X_2\times X_1\times X_3)) \\
&\quad +\mu(T(X_3\times X_2\times X_1))\Big) \\
&< O(f;\psi\left(\mu(X_1)+\mu(X_2)+\mu(X_3)\right) \\
&\leq O\left(f;\psi(\widetilde{\mu}(X))\right).
\end{aligned} \tag{4.4}
$$

Therefore, all the conditions of Corollary 3.5 are satisfied and T has a tripled fixed point. □

If $O(f;t)=t$ and $f=I$ (identity map) we have the following corollary.

Corollary 4.4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E, μ an arbitrary measure of noncompactness on E, and ψ a strictly L-function. $T: C\times C\times C\to C$ is a continuous operator satisfying*

$$
\mu\left(T(X_1\times X_2\times X_3)\right) < \frac{1}{3}\left(\psi(\mu(X_1)+\mu(X_2)+\mu(X_3)\right), \tag{4.5}
$$

for any subset X_1,X_2,X_3 of C and for any bounded subset X of C. Then, T has at least one tripled fixed point.

5. Application

In this section, as an application of Corollary 4.4, we study the existence of solutions for a systems of integral equations (1.1) on the $BC(\Omega)$ where Ω is a nonempty and unbounded subset of $\mathbb{R}^n$ and $BC(\Omega)$ is the Banach space of all bounded continuous functions on Ω equipped with the standard norm

$$
\|f\| = \sup\left\{|f(x)| : x\in\Omega\right\}.
$$

The measure of noncompactness on $BC(\Omega)$ is as follows.

Theorem 5.1 ([5])**.** *Let $\mathcal{F}$ be a bounded subset of $BC(\Omega)$. For $f\in\mathcal{F}$, $\varepsilon>0$ and $T>0$ let*

$$
\begin{aligned}
&\omega^T(f,\varepsilon) = \sup\left\{|f(x)-f(y)| : x,y\in\bar{B}_T, \|x-y\|<\varepsilon\right\}, \\
&\omega^T(\mathcal{F},\varepsilon) = \sup\left\{\omega^T(f,\varepsilon) : f\in\mathcal{F}\right\}, \\
&\omega^T(\mathcal{F}) = \lim_{\varepsilon\to 0}\omega^T(\mathcal{F},\varepsilon),
\end{aligned}
$$

$$\omega(\mathcal{F}) = \lim_{T\to\infty} \omega^T(\mathcal{F}),$$
$$d(\mathcal{F}) = \limsup_{\|x\|\to\infty} diam\mathcal{F}(x).$$

Then $\omega_0 : \mathfrak{M}_{BC(\Omega)} \to \mathbb{R}$ *given by*

$$\omega_0(\mathcal{F}) = \omega(\mathcal{F}) + d(\mathcal{F}), \tag{5.1}$$

defines a measure of noncompactness on $BC(\Omega)$.

Definition 5.2 ([5]). Let Ω be an unbounded subset of an arbitrary Banach space X. We say $\Lambda : \Omega \to \mathfrak{M}_{\mathbb{R}^m}$ is a continuous function if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| < \delta \quad \Rightarrow \quad m(\Lambda(x) \triangle \Lambda(y)) < \varepsilon,$$

where m is a Lebesgue measure on Ω and $\triangle$ denotes the symmetric difference.

Theorem 5.3. *Assume that the following conditions are satisfied:*

(i) $\Lambda : \Omega \to \mathfrak{M}_{\mathbb{R}^m}$ *is a continuous function and* $\bigcup_{\|x\|\le T} \Lambda(x)$ *is a bounded subset of* $\mathbb{R}^m$ *for all* $T > 0$.

(ii) $f : \Omega \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a nondecreasing strictly L-function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *and a nondecreasing continuous function* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\Phi(0) = 0$ *such that*

$$|f(x, u_1, u_2, u_3, y) - f(x, v_1, v_2, v_3, z)| \le \varphi\left(\max_{1\le i\le 3} |u_i - v_i|\right) + \Phi(|y - z|).$$

(iii) $M := \sup\{|f(x, 0, 0, 0, 0)| : x \in \Omega\} < \infty$.

(iv) $g : \Omega \times \left(\bigcup_{x\in\Omega} \Lambda(x)\right) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous and*

$$D := \sup\left\{ \left| \int_{\Lambda(x)} g(x, y, u_1(y), u_2(y), u_3(y))dy \right| : x \in \Omega, \\ u_1, u_2, u_3 \in BC(\Omega) \right\} < \infty.$$

Moreover,

$$\lim_{\|x\|\to\infty} \left| \int_{\Lambda(x)} [g(x, y, u_1(y), u_2(y), u_3(y)) - g(x, y, v_1(y), v_2(y), v_3(y))]\, dy \right| = 0,$$

uniformly with respect to $u_i, v_i \in BC(\Omega)$.

(v) *There exists a positive solution* r_0 *to the inequality*

$$\frac{1}{3}\varphi(3r) + M + \Phi(D) \le r.$$

Then the system of integral equations (1.1) *has at least one solution in the space* $\{BC(\Omega)\}^3$.

Proof. Let us consider the operator

$$F : BC(\Omega) \times BC(\Omega) \times BC(\Omega) \to BC(\Omega),$$

by the formula

(5.2)

$$F(u_1, u_2, u_3)(x) = f\left(x, u_1(x), u_2(x), u_3(x), \int_{\Lambda(x)} g(x, y, u_1(y), u_2(y), u_3(y))dy\right).$$

We observe that for any $x \in \Omega$ the function $F(x)$ is continuous and for arbitrary fixed $x \in \Omega$, by applying the assumptions (i) $-$ (v) we have

$$\begin{aligned}
&|F(u_1, u_2, u_3)(x)| \\
&\quad\leq \left| f((x, u_1(x), u_2(x), u_3(x), \int_{\Lambda(x)} g(x, y, u_1(y), u_2(y), u_3(y))dy) \right. \\
&\qquad \left. - f(x, 0, 0, 0, 0) \right| + |f(x, 0, 0, 0, 0)| \\
&\quad\leq \varphi\left(\max_{1\leq i\leq 3} |u_i - v_i|\right) + \Phi\left(\left|\int_{\Lambda(x)} g(x, y, u_1(y), u_2(y), u_3(y))dy\right|\right) \\
&\qquad + |f(x, 0, 0, 0, 0)| \\
&\quad\leq \varphi\left(F \max\{\|u_1\|, \|u_2\|, \|u_3\|\}\right) + M + \Phi(D).
\end{aligned}$$

Therefore,

$$\|F(u_1, u_2, u_3)\| \leq \varphi\left(F \max\{\|u_1\|, \|u_2\|, \|u_3\|\}\right) + M + \Phi(D), \tag{5.3}$$

and $F(u_1, u_2, u_3) \in BC(\Omega)$ for any $(u_1, u_2, u_3) \in (BC(\Omega))^3$. Due to Inequality (5.3) and using (v), the function F maps $(\bar{B}_{r_0})^3$ into $(\bar{B}_{r_0})$. Now, we prove that the operator F is a continuous operator on $(\bar{B}_{r_0})^3$. Let us fix arbitrarily $\varepsilon > 0$ and take $(u_1, u_2, u_3), (v_1, v_2, v_3) \in (\bar{B}_{r_0})^3$ such that

$$\max\{\|u_1 - v_1\|, \|u_2 - v_2\|, \|u_3 - v_3\|\} < \varepsilon.$$

Then, we have

$$\begin{aligned}
&|F(u_1, u_2, u_3)(x) - F(v_1, v_2, v_3)(x)| \\
&\quad\leq \left| f(x, u_1(x), u_2(x), u_3(x), \int_{\Lambda(x)} g(x, y, u_1(y), u_2(y), u_3(y))dy) \right. \\
&\qquad \left. -f(x, v_1(x), v_2(x), v_3(x), \int_{\Lambda(x)} g(x, y, v_1(y), v_2(y), v_3(y))dy) \right| \\
&\quad\leq \varphi\left(\max |u_1(x) - v_1(x)|, |u_2(x) - v_2(x)|, |u_3(x) - v_3(x)|\right)
\end{aligned}$$

$$+\Phi\left(\int_{\Lambda(x)} |g(x,y,u_1(y),u_2(y),u_3(y)))dy) - g(x,y,v_1(y),v_2(y),v_3(y))|\,dy\right).$$

By applying condition (v) we choose $T>0$ such that for $\|x\|>T$ the following inequality holds

$$\int_{\Lambda(x)} \big|g\big(x,y,u_1(y),u_2(y),u_3(y)\big) - g\big(x,y,v_1(y),v_2(y),v_3(y)\big)\big|\,dy < \varepsilon,$$

and we infer

$$|F(u_1,u_2,u_3)(x) - F(v_1,v_2,v_3)(x)| \le \varphi(\varepsilon) + \Phi(\varepsilon). \tag{5.4}$$

If $\|x\| \le T$, then

$$|F(u_1,u_2,u_3)(x) - F(v_1,v_2,v_3)(x)| \le \varphi(\varepsilon) + \Phi(\Lambda_T \vartheta_T(\varepsilon)), \tag{5.5}$$

where

$$\Lambda_T = \sup\{m(\Lambda(x)) : \|x\| \le T\},$$

and

$$\vartheta_T(\varepsilon) = \sup\Big\{|g(x,y,u_1,u_2,u_3) - g(x,y,v_1,v_2,v_3)| : x \in \bar{B}_T \\ , y \in \overline{\bigcup_{\|x\|\le T} \Lambda(x)}, u_i, v_i \in [-r_0, r_0], |u_i - v_i| \le \varepsilon, i = 1,2,3\Big\}.$$

By using the continuity of g on the compact set

$$\bar{B}_T \times \overline{\bigcup_{\|x\|\le T} \Lambda(x)} \times [-r_0,r_0] \times [-r_0,r_0] \times [-r_0,r_0],$$

we have $\vartheta_T(\varepsilon) \to 0$ as $\varepsilon \to 0$, and we conclude that $\Phi(\Lambda_T\vartheta_T(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Thus, from (5.4) and (5.5) we infer that F is a continuous function on $(BC(\Omega))^3$.

Now, we prove that F satisfies condition (4.5) of Corollary 4.4. To do this aim, let X_1, X_2 and X_3 be nonempty and bounded subsets of $\bar{B}_{r_0}$, and assume that $T>0$ and $\varepsilon>0$ are arbitrary constants. Let $x_1, x_2 \in \bar{B}_T$, with $\|x_2 - x_1\| \le \varepsilon$ and $u_1,u_2,u_3 \in X_1 \times X_2 \times X_3$. Then we have

(5.6)

$$|F(u_1,u_2,u_3)(x_1) - F(u_1,u_2,u_3)(x_2)| \\ \le \Bigg| f(x_1,u_1(x_1),u_2(x_1),u_3(x_1), \int_{\Lambda(x_1)} g(x_1,y,u_1(y),u_2(y),u_3(y))dy) \\ - f(x_2,u_1(x_1),u_2(x_1),u_3(x_1), \int_{\Lambda(x_1)} g(x_1,y,u_1(y),u_2(y),u_3(y))dy)\Bigg|$$

$$+\left|f(x_2,u_1(x_1),u_2(x_1),u_3(x_1),\int_{\Lambda(x_1)}g(x_1,y,u_1(y),u_2(y),u_3(y))dy)\right.$$
$$\left.-f(x_2,u_1(x_2),u_2(x_2),u_3(x_2)),\int_{\Lambda(x_1)}g(x_1,y,u_1(y),u_2(y),u_3(y))dy)\right|$$
$$+\left|f(x_2,u_1(x_2),u_2(x_2),u_3(x_2),\int_{\Lambda(x_1)}g(x_1,y,u_1(y),u_2(y),u_3(y))dy)\right.$$
$$\left.-f(x_2,u_1(x_2),u_2(x_2),u_3(x_2)),\int_{\Lambda(x_2)}g(x_1,y,u_1(y),u_2(y),u_3(y))dy)\right|$$
$$+\left|f(x_2,u_1(x_2),u_2(x_2),u_3(x_2),\int_{\Lambda(x_2)}g(x_1,y,u_1(y),u_2(y),u_3(y)))dy)\right.$$
$$\left.-f(x_2,u_1(x_2),u_2(x_2),u_3(x_2),\int_{\Lambda(x_2)}g(x_2,y,u_1(y),u_2(y),u_3(y))))dy)\right|$$
$$\leq \omega_{r_0}^T(f,\varepsilon)+\frac{1}{3}\varphi\left[\omega^T(u_1,\varepsilon)+\omega^T(u_2,\varepsilon)+\omega^T(u_3,\varepsilon)\right]$$
$$+U_{r_0}^T\omega^T(\Lambda,\varepsilon)+\Lambda_T\omega_{r_0}^T(g,\varepsilon),$$

where

$$\Lambda_T=\sup\{m(\Lambda(x)):\|x\|\leq T\}.$$
$$U_{r_0}^T=\sup\Big\{|g(x,y,u_1,u_2,u_3)|:\|x\|\leq T,\ y\in\overline{\bigcup_{\|x\|\leq T}\Lambda(x)},u_1,u_2,u_3\in[-r_0,r_0]\Big\},$$
$$\omega_{r_0}^T(f,\varepsilon)=\sup\Big\{|f(x_1,u_1,u_2,u_3,v)-f(x_2,u_1,u_2,u_3,v)|:x_1,x_2\in\bar{B}_T,\ \|x_2-x_1\|\leq\varepsilon,|u_i|\leq r_0,|v|<\Lambda_TU_{r_0}^T,i=1,2,3\Big\},$$
$${\omega_{r_0}}^T(g,\varepsilon)=\sup\Big\{|g(x_1,y,u_1,u_2,u_3)-g(x_2,y,u_1,u_2,u_3)|:x_1,x_2\in\bar{B}_T,\ y\in\overline{\bigcup_{\|x\|\leq T}\Lambda(x)},\|x_2-x_1\|\leq\varepsilon,|u_i|\leq r_0,i=1,2,3\Big\},$$
$$\omega^T(\Lambda,\varepsilon)=\sup\left\{m(\Lambda(x_1)\triangle\Lambda(x_2)):x_1,x_2\in\bar{B}_T,\|x_2-x_1\|\leq\varepsilon\right\}.$$

Since (u_1,u_2,u_3) is an arbitrary element of $X_1\times X_2\times X_3$ in (5.6), we have

$$\omega^T\left(F(X_1\times X_2\times X_3),\varepsilon\right)$$
$$<\omega_{r_0}^T(f,\varepsilon)+\frac{1}{3}\varphi\left[\omega^T(X_1,\varepsilon)+\omega^T(X_2,\varepsilon)+\omega^T(X_3,\varepsilon)\right]$$

$$+ U_{r_0}^T \omega^T(\Lambda, \varepsilon) + \Lambda_T \omega_{r_0}^T(g, \varepsilon),$$

and by the uniform continuity of f, g and Λ on the compact sets

$$\bar{B}_T \times [-r_0, r_0] \times [-r_0, r_0] \times [-r_0, r_0] \times \left[-\Lambda_T U_{r_0}^T, \Lambda_T U_{r_0}^T\right],$$

$$\bar{B}_T \times \left(\overline{\bigcup_{\|x\| \leq T} \Lambda(x)}\right) \times [-r_0, r_0] \times [-r_0, r_0] \times [-r_0, r_0],$$

and $\bar{B}_T$ respectively, we have ${\omega_{r_0}}^T(f, \varepsilon) \to 0$, ${\omega_{r_0}}^T(g, \varepsilon) \to 0$ and $\omega^T(\Lambda, \varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, we obtain

$$\omega^T\left(F(X_1 \times X_2 \times X_3)\right) < \frac{1}{3}\varphi\left[\omega^T(X_1) + \omega^T(X_2) + \omega^T(X_3)\right],$$

and

$$\omega(F(X_1 \times X_2 \times X_3)) < \frac{1}{3}\varphi\left[\omega(X_1) + \omega(X_2) + \omega(X_3)\right]. \tag{5.7}$$

Now for all $(u_1, u_2, u_3), (v_1, v_2, v_3) \in X_1 \times X_2 \times X_3$ and $x \in \Omega$ we have

$$\begin{aligned}
&|F(u_1, u_2, u_3)(x) - F(v_1, v_2, v_3)(x)| \\
&\quad < \frac{1}{3}\varphi\left[|u_1(x) - v_1(x)| + |u_2(x) - v_2(x)| + |u_3(x) - v_3(x)|\right] \\
&\qquad + \Phi\left(\int_{\Lambda(x)} \left[g(x, y, u_1(y), u_2(y), u_3(y)) - g(x, y, v_1(y), v_2(y), v_3(y))\right] dy\right).
\end{aligned}$$

Therefore,

(5.8)

$$\begin{aligned}
&\operatorname{diam}(F(X_1 \times X_2 \times X_3)(x)) \\
&\quad < \frac{1}{3}\varphi\left[\operatorname{diam}(X_1(x)) + \operatorname{diam}(X_2(x)) + \operatorname{diam}(X_3(x))\right] \\
&\qquad + \Phi\left(\int_{\Lambda(x)} \left[g(x, y, u_1(y), u_2(y), u_3(y)) - g(x, y, v_1(y), v_2(y), v_3(y))\right] dy\right).
\end{aligned}$$

By take $\|x\| \to \infty$ in the inequality (5.8), then apply (iii) and (v) we infer that

(5.9)

$$\begin{aligned}
&\limsup_{\|x\| \to \infty} \operatorname{diam} F(X_1 \times X_2 \times X_3)(x) \\
&\quad < \frac{1}{3}\varphi\left[\limsup_{\|x\| \to \infty} \operatorname{diam}(X_1(x)) + \limsup_{\|x\| \to \infty} \operatorname{diam}(X_2(x)) + \limsup_{\|x\| \to \infty} \operatorname{diam}(X_3(x))\right].
\end{aligned}$$

Now, combining (5.7) and (5.9) we get

(5.10)

$$d(F(X_1 \times X_2 \times X_3)) + \omega(F(X_1 \times X_2 \times X_3))$$

$$< \frac{1}{3}\varphi[\omega(X_1)+\omega(X_2)+\omega(X_3)] + \frac{1}{3}\varphi[d(X_1)+d(X_2)+d(X_3)].$$

Since φ is a concave function, (5.10) implies

(5.11)

$$\begin{aligned} &d(F(X_1\times X_2\times X_3))+\omega(F(X_1\times X_2\times X_3)) \\ &\quad < \frac{1}{3}\varphi([d(X_1)+\omega(X_1)]) + \frac{1}{3}\varphi([d(X_2)+\omega(X_2)]) + \frac{1}{3}\varphi([d(X_3)+\omega(X_3)]). \end{aligned}$$

Finally, since μ is defined by

$$\mu(X) = \omega(F) + d(F),$$

we get

$$\mu(F(X_1\times X_2\times X_3)) < \frac{1}{3}\varphi(\mu(X_1)+\mu(X_2)+\mu(X_3)).$$

Thus from Corollary 4.4 we obtain that the operator F has a tripled fixed point and thus the system of functional integral equations (1.1) has at least one solution in $(BC(\Omega))^3$. □

6. Example

Example 6.1. Consider the following system of integral equations

(6.1)

$$\begin{cases} x(t) = \frac{1}{2}e^{-t^2} + \frac{\arctan x(t)+\sin y(t)}{4\pi+t^4} + \frac{\ln(1+|z(t)|)}{2\pi+t^2} \\ \qquad + \int_0^t \frac{s^2|\cos x(s)| + \sqrt{e^s(1+x^2(s))(1+\sin^2 y(s))(1+\cos^2 z(s))}}{e^t(1+x^2(s))(1+\sin^2 y(s))(1+\cos^2 z(s))} ds, \\ y(t) = \frac{1}{2}e^{-t^2} + \frac{\arctan y(t)+\sin x(t)}{4\pi+t^4} + \frac{\ln(1+|z(t)|)}{2\pi+t^2} \\ \qquad + \int_0^t \frac{s^2|\cos y(s)| + \sqrt{e^s(1+y^2(s))(1+\sin^2 x(s))(1+\cos^2 z(s))}}{e^t(1+y^2(s))(1+\sin^2 x(s))(1+\cos^2 z(s))} ds, \\ z(t) = \frac{1}{2}e^{-t^2} + \frac{\arctan z(t)+\sin y(t)}{4\pi+t^4} + \frac{\ln(1+|x(t)|)}{2\pi+t^2} \\ \qquad + \int_0^t \frac{s^2|\cos z(s))| + \sqrt{e^s(1+z^2(s))(1+\sin^2 y(s))(1+\cos^2 x(s))}}{e^t(1+z^2(s))(1+\sin^2 y(s))(1+\cos^2 x(s))} ds. \end{cases}$$

We observe that this system of integral equations (6.1) is a special case of (1.1) with

$\Lambda(t) = [0,t]$,

$\Phi(t) = t$,

$$\varphi(t) = \frac{\max\{\sin t, \arctan t, \ln 1 + t\}}{4},$$

$$f(t, x, y, z, p) = \frac{1}{2}e^{-t^2} + \frac{\arctan x + \sin\ y}{4\pi + t^4} + \frac{\ln(1 + |z|)}{4\pi + t^2} + p,$$

$$g(t, s, x, y, z) = \frac{s^2|\cos x| + \sqrt{e^s(1 + x^2)(1 + \sin^2 y)(1 + \cos^2 z)}}{e^t(1 + x^2)(1 + \sin^2 y)(1 + \cos^2 z)}.$$

To solve this system, we need to verify the conditions (i) - (v) of Theorem 5.3.
Condition (i) is clearly evident.

Now we have

(6.2)

$$\begin{aligned}
&|f(t, x, y, z, m) - f(t, u, v, w, n)| \\
&\quad\leq \frac{|\arctan x - \arctan u| + |\sin\ y - \sin\ w|}{4\pi + t^4} + \frac{\ln\left(\frac{1+|z|}{1+|w|}\right)}{4\pi + t^2} + |m - n| \\
&\quad\leq \frac{\arctan|x - u|}{4\pi} + \frac{\sin|y - w|}{4\pi} + \frac{\ln(1 + |z - w|)}{4\pi} + |m - n| \\
&\quad\leq \varphi(\max\{|x - u|, |y - v|, |z - w|\}) + \Phi(|m - n|).
\end{aligned}$$

Obviously the function φ is a strictly L-function and concave on $\mathbb{R}_+$ and if we define $\Phi(t) = t$ so we can find that f and Φ satisfy condition (ii) of Theorem 5.3. Also,

$$\begin{aligned}
M &= \sup\{|f(t, 0, 0, 0, 0)| : t \in \mathbb{R}\} \\
&= \sup\left\{\frac{1}{2}e^{-t^2} : t \in \mathbb{R}\right\} \\
&< 0.2.
\end{aligned}$$

So, condition (iii) of Theorem 5.3 is valid. Moreover, g is continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$ and

$$\begin{aligned}
D = \sup&\left\{\left|\int_0^{|t|} \frac{s^2|\cos x(s))| + \sqrt{e^s(1 + x^2(s))(1 + \sin^2 y(s))(1 + \cos^2 z(s))}}{e^t(1 + x^2(s))(1 + \sin^2 y(s))(1 + \cos^2 z(s))} ds\right|\right. \\
&\left. : t, s \in \mathbb{R}_+, x, y, z \in BC(\mathbb{R}_+)\right\} \\
&< \sup \frac{s^2}{e^t} < 0.5.
\end{aligned}$$

Then,

$$\lim_{|t|\to\infty}\left|\int_0^t \frac{s^2\,|\cos x| + \sqrt{e^s(1+x^2)(1+\sin^2 y)(1+\cos^2 z)}}{e^t(1+x^2)(1+\sin^2 y)(1+\cos^2 z)}\right.$$

$$\left. - \frac{s^2\,|\cos u| + \sqrt{e^s(1+u^2)(1+\sin^2 v)(1+\cos^2 w)}}{e^t(1+u^2)(1+\sin^2 v)(1+\cos^2 w)} ds\right|$$

$$\leq \lim_{|t|\to\infty}\left|\int_0^t \frac{s^2}{e^t} ds\right| = 0.$$

Furthermore, it is easy to see that each number $r \geq 3$ satisfies the inequality in condition (v), i.e.,

$$\frac{1}{3}\varphi(3r) + M + \Phi(D) \leq \ln(1+r) + 0.7 \leq r.$$

Consequently, all the conditions of Theorem 5.3 are satisfied. Hence system of integral equations (6.1) has at least one solution which belongs to the space $BC(\mathbb{R})^3$.

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