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Controlled Continuous G-Frames and Their Multipliers in Hilbert Spaces

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ABSTRACT. In this paper, we introduce $(\mathcal{C},\mathcal{C}')$-controlled continuous g-Bessel families and their multipliers in Hilbert spaces and investigate some of their properties. We show that under some conditions sum of two $(\mathcal{C},\mathcal{C}')$-controlled continuous g-frames is a $(\mathcal{C},\mathcal{C}')$-controlled continuous g-frame. Also, we investigate when a $(\mathcal{C},\mathcal{C}')$-controlled continuous g-Bessel multiplier is a p-Schatten class operator.

1. Introduction and Preliminaries

In 1952, Duffin and Schaeffer [8] introduced the concept of discrete frames in Hilbert spaces. Weighted and controlled frames have been introduced in [9]. In [6, 9], controlled and weighted frames were used as tools for spherical wavelets. Balazs, Antoine and Grybos [4] investigated relations of weighted frames and controlled frames. Also, they showed that controlled frames are equivalent to standard frames, and the concept of controlled frame gives us a generalized way to check the frame condition, while offering a numerical advantage in the sense of preconditioning.

Throughout this paper, $\mathcal{H}$ is a complex Hilbert space and the set of all bounded operators on $\mathcal{H}$ will be denoted by $B(\mathcal{H})$. We say that $T \in B(\mathcal{H})$ is *positive* (respectively non-negative), if $\langle Tf, f\rangle > 0$ for all $f \neq 0$ (respectively $\langle Tf, f\rangle \geq 0$ for all $f \in \mathcal{H}$).

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Let $B^+(\mathcal{H})$ be the set of all positive operators in $B(\mathcal{H})$. The set of all bounded invertible operators on $\mathcal{H}$ is denoted by $GL(\mathcal{H})$, and the set of all positive elements of $GL(\mathcal{H})$ will be showed by $GL^+(\mathcal{H})$.

Controlled and weighted continuous frames introduced in [5] and here we recall the definition of controlled and weighted continuous frame.

Definition 1.1. Let (Ω, μ) be a measure space with a positive measure μ and $\mathcal{C} \in GL(\mathcal{H})$. A $\mathcal{C}$-controlled continuous frame is a map $F : \Omega \to \mathcal{H}$ such that there exists $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega \langle f, F(\omega)\rangle \langle \mathcal{C}F(\omega), f\rangle \, d\mu \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

Definition 1.2. Let (Ω, μ) be a measure space with a positive measure μ and $m : \Omega \to \mathbb{R}^+$. The mapping $F : \Omega \to \mathcal{H}$ is called a weighted continuous frame with respect to (Ω, μ) and m, if

(1) F is weakly measurable and m is measurable;
(2) there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_\Omega m(\omega) |\langle f, F(\omega)\rangle|^2 d\mu \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The concept of g-frame as a natural generalization of frame introduced by Sun in [12].

Definition 1.3. Let $\mathcal{H}$ be a Hilbert space and $\{\mathcal{K}_i\}_{i\in I}$ be a sequence of Hilbert spaces. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

We call A, B the lower and upper frame bounds, respectively. If the right hand inequality of (1.1) holds for all $f \in \mathcal{H}$ then Λ is called a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$.

If $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is a g-Bessel sequence, the operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f,$$

is a bounded operator and if Λ is a g-frame for $\mathcal{H}$, then S_Λ is a bounded invertible positive operator and $A_\Lambda I \leq S_\Lambda \leq B_\Lambda I$. The operator S_Λ is called the *g-frame operator* of Λ.

Controlled g-frame as a generalization of controlled frame introduced by Rahimi and Fereydooni [11].

Definition 1.4. Let $\mathcal{C}, \mathcal{C}' \in GL^+(\mathcal{H})$ and $\{\mathcal{K}_i\}_{i\in I}$ be a sequence of Hilbert spaces. $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is called a $(\mathcal{C}, \mathcal{C}')$-controlled

g-frame for $\mathcal{H}$, if Λ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ and there exists constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \left\langle \Lambda_i \mathcal{C} f, \Lambda_i \mathcal{C}' f\right\rangle \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

If $\mathcal{C}' = I$ then Λ is called a $\mathcal{C}$-controlled g-frame.

In 2007, P. Balazs [3] introduced Bessel and frames multipliers for Hilbert spaces.

Definition 1.5. : Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $\{\phi_i\}_{i\in I} \subset \mathcal{H}_1$ and $\{\psi_i\}_{i\in I} \subset \mathcal{H}_2$ be Bessel sequences. Fix $m = \{m_i\}_{i\in I} \in \ell^\infty$. The operator $M_{m,\{\phi_i\},\{\psi_i\}} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by

$$M_{m,\{\phi_i\},\{\psi_i\}} f := \sum_{i\in I} m_i \langle f, \phi_i\rangle \psi_i,$$

is called the *Bessel multiplier* for the Bessel sequences $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$. The sequence m is called the symbol of $M_{m,\{\phi_i\},\{\psi_i\}}$.

Continuous g-frame in Hilbert spaces as a common generalization of g-frame and continuous frame defined by Abdollahpour and Faroughi [2].

In the following, we suppose that (Ω, μ) is a measure space with positive measure μ and $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ is a family of Hilbert spaces. We say that $F \in \Pi_{\omega\in\Omega}\mathcal{K}_\omega$ is strongly measurable if F as a mapping of Ω into $\bigoplus_{\omega\in\Omega} \mathcal{K}_\omega$ is measurable.

Definition 1.6. A family of operators $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a *continuous g-frame* with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ if

(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,

(ii) there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$A_\Lambda ||f||^2 \leq \int_\Omega ||\Lambda_\omega f||^2 d\mu_\omega \leq B_\Lambda ||f||^2, \quad f \in \mathcal{H}. \tag{1.2}$$

$\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is called a continuous g-Bessel family with bound B_Λ, if the right hand inequality in (1.2) holds for all $f \in \mathcal{H}$.

Authors of this paper, introduced the concept of continuous g-Bessel multipliers [1].

Definition 1.7. Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and $m \in L^\infty(\Omega, \mu)$, the operator $M_{m,\Lambda,\Phi} : \mathcal{H} \to \mathcal{H}$ defined by

$$\langle M_{m,\Lambda,\Phi} f, g\rangle = \int_\Omega m(\omega) \langle \Lambda^*_\omega \Phi_\omega f, g\rangle \, d\mu_\omega, \quad f, g \in \mathcal{H},$$

is called *continuous g-Bessel multiplier* of Λ, Φ and m.

2. Controlled Continuous g-frames

In this section, we intend to introduce controlled continuous g-frames in Hilbert spaces. In the following result, we provide a sufficient and necessary condition, under which a combination of two members of $GL^+(\mathcal{H})$ also belongs to $GL^+(\mathcal{H})$.

Proposition 2.1. *Let $T, \mathcal{C} \in GL^+(\mathcal{H})$. Then $T\mathcal{C} \in GL^+(\mathcal{H})$ if and only if $T\mathcal{C} = \mathcal{C}T$.*

Proof. For all $f \in \mathcal{H}$ we have

$$\begin{aligned} 4\langle T\mathcal{C}f, f\rangle &= \langle T(\mathcal{C}f + f), \mathcal{C}f + f\rangle - \langle T(\mathcal{C}f - f), \mathcal{C}f - f\rangle \\ &\quad i\langle T(\mathcal{C}f + if), \mathcal{C}f + if\rangle - i\langle T(\mathcal{C}f - if), \mathcal{C}f - if\rangle . \end{aligned}$$

Since $T\mathcal{C}$ is positive, $\langle T\mathcal{C}f, f\rangle \in \mathbb{R}$, for all $f \in \mathcal{H}$ and

$$\begin{aligned} 4\langle T\mathcal{C}f, f\rangle &= \langle T(\mathcal{C}f + f), \mathcal{C}f + f\rangle - \langle T(\mathcal{C}f - f), \mathcal{C}f - f\rangle \\ &= 2\langle T\mathcal{C}f, f\rangle + 2\langle \mathcal{C}Tf, f\rangle , \end{aligned}$$

therefore $\langle T\mathcal{C}f, f\rangle = \langle \mathcal{C}Tf, f\rangle$ for all $f \in \mathcal{H}$. Thus $T\mathcal{C} = \mathcal{C}T$.

Conversely, if $T\mathcal{C} = \mathcal{C}T$ then $\mathcal{C}^{-1}$ commutes with $\mathcal{C}T\mathcal{C}$ and $\mathcal{C}^2$. Also $\mathcal{C}T\mathcal{C}$, $\mathcal{C}^2$ and $\mathcal{C}^{-1}$ are self-adjoint. By Proposition 2.4 in [4], there is $m > 0$ such that $T - mI \geq 0$, then

$$\left\langle \left(\mathcal{C}T\mathcal{C} - m\mathcal{C}^2\right) f, f\right\rangle = \langle (T - mI)\mathcal{C}f, \mathcal{C}f\rangle \geq 0, \quad f \in \mathcal{H}.$$

Therefore $\mathcal{C}T\mathcal{C} \geq m\mathcal{C}^2$. By Theorem A.6.5 in [7] we can conclude $\mathcal{C}^{-1}\mathcal{C}T\mathcal{C} \geq m\mathcal{C}$ then $T\mathcal{C} \geq m\mathcal{C} > 0$.

□

It is proved in [2], if $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame then there is a unique positive invertible operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle \, d\mu_\omega,$$

and $A_\Lambda I \leq S_\Lambda \leq B_\Lambda I$. Thus by Proposition 2.4 in [4], we have $S_\Lambda \in GL^+(\mathcal{H})$. The operator S_Λ is called the *continuous g-frame operator* of Λ.

The following proposition shows that under which conditions we can produce a new continuous g-frame and we omit its proof.

Proposition 2.2. *Let $\mathcal{C} \in B(\mathcal{H})$ and let the family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$. Then $\Lambda\mathcal{C} = \{\Lambda_\omega\mathcal{C} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ if only if there is $\alpha > 0$,*

$$\|\mathcal{C}f\|^2 \geq \alpha\|f\|^2, \quad f \in \mathcal{H}.$$

Definition 2.3. Let $\mathcal{C}, \mathcal{C}' \in GL^+(\mathcal{H})$. $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is called a $(\mathcal{C}, \mathcal{C}')$- *controlled continuous g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$, if Λ is a continuous g-Bessel family and there exist constants $A_{\Lambda\mathcal{C}\mathcal{C}'} > 0$ and $B_{\Lambda\mathcal{C}\mathcal{C}'} < \infty$ such that

$$A_{\Lambda\mathcal{C}\mathcal{C}'}||f||^2 \leq \int_\Omega \left\langle \Lambda_\omega \mathcal{C} f, \Lambda_\omega \mathcal{C}' f \right\rangle d\mu_\omega \leq B_{\Lambda\mathcal{C}\mathcal{C}'}||f||^2, \quad f \in \mathcal{H}. \tag{2.1}$$

$A_{\Lambda\mathcal{C}\mathcal{C}'}$ and $B_{\Lambda\mathcal{C}\mathcal{C}'}$ are called the *controlled continuous g-frame bounds*. If $\mathcal{C}' = \mathbf{I}$, then we call Λ a $\mathcal{C}$-controlled continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$. If the right hand inequality of (2.1) holds for all $f \in \mathcal{H}$ then Λ is called a $(\mathcal{C}, \mathcal{C}')$-controlled continuous g-Bessel family with bound $B_{\Lambda\mathcal{C}\mathcal{C}'}$. If Λ is a $(\mathcal{C}, \mathcal{C})$-controlled g-frame then we use the notations $A_{\Lambda\mathcal{C}}$ and $B_{\Lambda\mathcal{C}}$ for bounds of Λ instead of $A_{\Lambda\mathcal{C}\mathcal{C}}$ and $B_{\Lambda\mathcal{C}\mathcal{C}}$.

Proposition 2.4. *Let $\mathcal{C}, \mathcal{C}' \in GL^+(\mathcal{H})$ and $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a $(\mathcal{C}, \mathcal{C}')$- controlled continuous g-frame for $\mathcal{H}$ with controlled continuous g-frame bounds $A_{\Lambda\mathcal{C}\mathcal{C}'}, B_{\Lambda\mathcal{C}\mathcal{C}'}$. Then there exists a unique positive and invertible operator $S_{\Lambda\mathcal{C}\mathcal{C}'} : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$,*

$$\langle S_{\Lambda\mathcal{C}\mathcal{C}'} f, g\rangle = \int_\Omega \left\langle \mathcal{C}' \Lambda_\omega^* \Lambda_\omega \mathcal{C} f, g \right\rangle d\mu_\omega, \tag{2.2}$$

and $A_{\Lambda\mathcal{C}\mathcal{C}'} I \leq S_{\Lambda\mathcal{C}\mathcal{C}'} \leq B_{\Lambda\mathcal{C}\mathcal{C}'} I$.

Proof. The mapping

$$\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \qquad \sigma(f, g) = \int_\Omega \left\langle \mathcal{C}' \Lambda_\omega^* \Lambda_\omega \mathcal{C} f, g \right\rangle d\mu_\omega,$$

is a bounded sesqlinear form, since by Cauchy-Schwartz's inequality, we have

$$\begin{aligned} |\sigma(f, g)| &\leq \left(\int_\Omega \|\Lambda_\omega \mathcal{C} f\|^2 d\mu_\omega \right)^{1/2} \left(\int_\Omega \|\Lambda_\omega \mathcal{C}' g\|^2 d\mu_\omega \right)^{1/2} \\ &\leq B_\Lambda \|\mathcal{C}\| \|\mathcal{C}'\| \|f\| \|g\|, \end{aligned}$$

for all $f, g \in \mathcal{H}$.

Therefore by Theorem 2.3.6 in [10], there exists a unique operator $S_{\Lambda\mathcal{C}\mathcal{C}'}$ such that (2.2) holds for all $f, g \in \mathcal{H}$ and $\|S_{\Lambda\mathcal{C}\mathcal{C}'}\| \leq B_\Lambda \|\mathcal{C}\| \|\mathcal{C}'\|$. Then for all $f \in \mathcal{H}$ we have

$$\langle S_{\Lambda\mathcal{C}\mathcal{C}'} f, f\rangle = \int_\Omega \left\langle \Lambda_\omega \mathcal{C} f, \Lambda_\omega \mathcal{C}' f \right\rangle d\mu_\omega,$$

and $A_{\Lambda\mathcal{C}\mathcal{C}'} I \leq S_{\Lambda\mathcal{C}\mathcal{C}'} \leq B_{\Lambda\mathcal{C}\mathcal{C}'} I$. Therefore

$$\left\| I - \frac{1}{B_{\Lambda\mathcal{C}\mathcal{C}'}} S_{\Lambda\mathcal{C}\mathcal{C}'} \right\| \leq 1 - \frac{A_{\Lambda\mathcal{C}\mathcal{C}'}}{B_{\Lambda\mathcal{C}\mathcal{C}'}} < 1,$$

so $S_{\Lambda\mathcal{C}\mathcal{C}'}$ is an invertible operator. □

The operator $S_{\Lambda\mathcal{C}\mathcal{C}'}$ is called the $(\mathcal{C},\mathcal{C}')$-controlled continuous g-frame operator of $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{K}_\omega : \omega \in \Omega\}$.

Now, we intend to prove a proposition that shows any continuous g-frame is a controlled continuous g-frame.

Proposition 2.5. *Let $\mathcal{C} \in GL^+(\mathcal{H})$. The family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{K}_\omega : \omega \in \Omega\}$ is a continuous g-frame if and only if Λ is a $(\mathcal{C},\mathcal{C})$-controlled continuous g-frame.*

Proof. Let Λ be a $(\mathcal{C},\mathcal{C})$-controlled continuous g-frame. Then for all $f \in \mathcal{H}$,

$$A_{\Lambda\mathcal{C}}\|f\|^2 \leq \int_\Omega \|\Lambda_\omega \mathcal{C} f\|^2 d\mu_\omega \leq B_{\Lambda\mathcal{C}}\|f\|^2.$$

Therefore,

$$\begin{aligned} A_{\Lambda\mathcal{C}}\|f\|^2 &= A_{\Lambda\mathcal{C}}\|\mathcal{C}\mathcal{C}^{-1}f\|^2 \\ &\leq A_{\Lambda\mathcal{C}}\|\mathcal{C}\|^2\|\mathcal{C}^{-1}f\|^2 \\ &\leq \|\mathcal{C}\|^2 \int_\Omega \|\Lambda_\omega \mathcal{C}\mathcal{C}^{-1} f\|^2 d\mu_\omega \\ &= \|\mathcal{C}\|^2 \int_\Omega \|\Lambda_\omega f\|^2 d\mu_\omega, \end{aligned}$$

for all $f \in \mathcal{H}$. On the other hand,

$$\begin{aligned} \int_\Omega \|\Lambda_\omega f\|^2 f d\mu_\omega &= \int_\Omega \|\Lambda_\omega \mathcal{C}\mathcal{C}^{-1} f\|^2 d\mu_\omega \\ &\leq B_{\Lambda\mathcal{C}}\|\mathcal{C}^{-1}f\|^2 \\ &\leq B_{\Lambda\mathcal{C}}\|\mathcal{C}^{-1}\|^2\|f\|^2, \end{aligned}$$

for all $f \in \mathcal{H}$. Therefore, Λ is a continuous g-frame with bounds $A_{\Lambda\mathcal{C}}\|\mathcal{C}\|^{-2}$ and $B_{\Lambda\mathcal{C}}\|\mathcal{C}^{-1}\|^2$.

Conversely, assume that Λ is a continuous g-frame. Then

$$A_\Lambda\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu_\omega \leq B_\Lambda\|f\|^2, \quad f \in \mathcal{H}.$$

So

$$\begin{aligned} \int_\Omega \|\Lambda_\omega \mathcal{C} f\|^2 d\mu_\omega &\leq B_\Lambda\|\mathcal{C}f\|^2 \\ &\leq B_\Lambda\|\mathcal{C}\|^2\|f\|^2, \quad f \in \mathcal{H}, \end{aligned}$$

and

$$\begin{aligned} A_\Lambda\|f\|^2 &= A_\Lambda\|\mathcal{C}^{-1}\mathcal{C}f\|^2 \\ &\leq A_\Lambda\|\mathcal{C}^{-1}\|^2\|\mathcal{C}f\|^2 \\ &\leq \|\mathcal{C}^{-1}\|^2\int_\Omega \|\Lambda_\omega\mathcal{C}f\|^2 f\mu_\omega, \quad f\in\mathcal{H}. \end{aligned}$$

Thus Λ is a $(\mathcal{C},\mathcal{C})$-controlled continuous g-frame with bounds

$$A_\Lambda\|\mathcal{C}^{-1}\|^{-2}, B_\Lambda\|\mathcal{C}\|^2.$$

□

Proposition 2.6. *Let $\mathcal{C},\mathcal{C}'\in B(\mathcal{H})$ and $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$. Then the operator*

$$\langle S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}f,g\rangle = \int_\Omega \left\langle \Lambda_\omega\mathcal{C}f, \Phi_\omega\mathcal{C}'g\right\rangle d\mu_\omega, \quad f,g\in\mathcal{H},$$

is a well-defined bounded operator.

Proof. Let $f,g\in\mathcal{H}$. By Cauchy-Schwartz's inequality, we have

$$\begin{aligned} |\langle S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}f,g\rangle| &= \left|\int_\Omega \left\langle \mathcal{C}'\Phi_\omega^*\Lambda_\omega\mathcal{C}f,g\right\rangle d\mu_\omega\right| \\ &\leq \int_\Omega \left|\left\langle \Lambda_\omega\mathcal{C}f,\Phi_\omega\mathcal{C}'g\right\rangle\right| d\mu_\omega \\ &\leq \int_\Omega \|\Lambda_\omega\mathcal{C}f\|\|\Phi_\omega\mathcal{C}'g\|d\mu_\omega \\ &\leq \left(\int_\Omega \|\Lambda_\omega\mathcal{C}f\|^2 d\mu_\omega\right)^{\frac{1}{2}}\left(\int_\Omega \|\Phi_\omega\mathcal{C}'g\|^2 d\mu_\omega\right)^{\frac{1}{2}} \\ &\leq \sqrt{B_\Lambda}\|\mathcal{C}f\|\sqrt{B_\Phi}\|\mathcal{C}'g\| \\ &= \sqrt{B_\Lambda B_\Phi}\|\mathcal{C}\|\|\mathcal{C}'\|\|f\|\|g\|. \end{aligned}$$

So

$$\|S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}\| \leq \sqrt{B_\Lambda B_\Phi}\|\mathcal{C}\|\|\mathcal{C}'\|,$$

therefore $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}$ is bounded and well-defined operator. □

Corollary 2.7. *Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C},\mathcal{C})$-controlled continuous g-Bessel family and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C}',\mathcal{C}')$-controlled continuous g-Bessel family with bounds $B_{\Lambda\mathcal{C}}$ and $B_{\Phi\mathcal{C}'}$, respectively. The operator $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}:\mathcal{H}\to\mathcal{H}$ defined weakly by*

$$\langle S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}f,g\rangle = \int_\Omega \left\langle \mathcal{C}'\Phi_\omega^*\Lambda_\omega\mathcal{C}f,g\right\rangle d\mu_\omega, \quad f,g\in\mathcal{H},$$

is a well-defined bounded operator and $\|S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}\|\leq\sqrt{B_{\Lambda\mathcal{C}}B_{\Phi\mathcal{C}'}}$.

Proof. Let $f, g \in \mathcal{H}$. Then by Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
|\langle S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}f, g\rangle| &= \left|\int_\Omega \left\langle \mathcal{C}'\Phi_\omega^*\Lambda_\omega\mathcal{C}f, g\right\rangle d\mu_\omega\right| \\
&\leq \int_\Omega \left|\left\langle \Lambda_\omega\mathcal{C}f, \Phi_\omega\mathcal{C}'g\right\rangle\right| d\mu_\omega \\
&\leq \int_\Omega \|\Lambda_\omega\mathcal{C}f\|\|\Phi_\omega\mathcal{C}'g\| d\mu_\omega \\
&\leq \left(\int_\Omega \|\Lambda_\omega\mathcal{C}f\|^2 d\mu_\omega\right)^{\frac{1}{2}} \left(\int_\Omega \|\Phi_\omega\mathcal{C}'g\|^2 d\mu_\omega\right)^{\frac{1}{2}} \\
&\leq \sqrt{B_{\Lambda\mathcal{C}}}\|f\|\sqrt{B_{\Phi\mathcal{C}'}}\|g\| \\
&= \sqrt{B_{\Lambda\mathcal{C}}B_{\Phi\mathcal{C}'}}\|f\|\|g\|.
\end{aligned}$$

So

$$\|S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}\| \leq \sqrt{B_{\Lambda\mathcal{C}}B_{\Phi\mathcal{C}'}},$$

therefore $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'}$ is a bounded and well-defined operator. □

In the spacial case for $\mathcal{C} = \mathcal{C}' = I$, the operator

$$\langle S_{\Lambda\Phi}f, g\rangle = \int_\Omega \langle \Phi_\omega^*\Lambda_\omega f, g\rangle \, d\mu_\omega, \quad f \in \mathcal{H},$$

is a well-defined bounded operator.

Proposition 2.8. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be continuous g-Bessel families for $\mathcal{H}$. Let $\mathcal{C}$, $\mathcal{C}'$ and $S_{\Phi\Lambda} + S_{\Lambda\Phi}$ be commutative with each others and $S_{\Phi\Lambda} + S_{\Lambda\Phi}, \mathcal{C}$ and $\mathcal{C}'$ be elements of $GL^+(\mathcal{H})$. Then $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}$ is a positive operator.*

Proof. For all $f, g \in \mathcal{H}$, we have

$$\begin{aligned}
\langle S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}f, g\rangle &= \int_\Omega \left\langle \mathcal{C}'\Phi_\omega^*\Lambda_\omega\mathcal{C}f, g\right\rangle d\mu_\omega + \int_\Omega \left\langle \mathcal{C}\Lambda_\omega^*\Phi_\omega\mathcal{C}'f, g\right\rangle d\mu_\omega \\
&= \int_\Omega \left\langle \Phi_\omega^*\Lambda_\omega\mathcal{C}f, \mathcal{C}'g\right\rangle d\mu_\omega + \int_\Omega \left\langle \Lambda_\omega^*\Phi_\omega\mathcal{C}'f, \mathcal{C}g\right\rangle d\mu_\omega \\
&= \left\langle S_{\Lambda\Phi}\mathcal{C}f, \mathcal{C}'g\right\rangle + \left\langle S_{\Phi\Lambda}\mathcal{C}'f, \mathcal{C}g\right\rangle \\
&= \left\langle \mathcal{C}'S_{\Lambda\Phi}\mathcal{C}f, g\right\rangle + \left\langle \mathcal{C}S_{\Phi\Lambda}\mathcal{C}'f, g\right\rangle \\
&= \left\langle \mathcal{C}'S_{\Lambda\Phi}\mathcal{C}f, g\right\rangle + \left\langle \mathcal{C}'S_{\Phi\Lambda}\mathcal{C}f, g\right\rangle \\
&= \left\langle \mathcal{C}'(S_{\Lambda\Phi} + S_{\Phi\Lambda})\mathcal{C}f, g\right\rangle.
\end{aligned}$$

Therefore

$$S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}} = \mathcal{C}'(S_{\Lambda\Phi} + S_{\Phi\Lambda})\mathcal{C}.$$

By Proposition 2.1, we can conclude that $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}$is positive. □

Corollary 2.9. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be* $(\mathcal{C}, \mathcal{C}')$*-controlled continuous g-frames. Let* $\mathcal{C}, \mathcal{C}', S_{\Phi\Lambda} + S_{\Lambda\Phi} \in GL^+(\mathcal{H})$ *and* $\mathcal{C}$, $\mathcal{C}'$ *and* $S_{\Phi\Lambda} + S_{\Lambda\Phi}$ *be commutative with each others. Then* $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}$ *is a positive operator.*

Corollary 2.10. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be* $(\mathcal{C}, \mathcal{C})$*-controlled continuous g-frames. Let* $\mathcal{C}, S_{\Phi\Lambda} + S_{\Lambda\Phi} \in GL^+(\mathcal{H})$*. If* $\mathcal{C}$ *and* $S_{\Phi\Lambda} + S_{\Lambda\Phi}$ *are commutative with each other then* $S_{\Lambda\mathcal{C}\Phi\mathcal{C}} + S_{\Phi\mathcal{C}\Lambda\mathcal{C}}$ *is positive operator.*

In the following proposition, we show under which conditions the sum of two $(\mathcal{C}, \mathcal{C}')$-controlled continuous g-frames is a $(\mathcal{C}, \mathcal{C}')$-controlled continuous g-frame:

Proposition 2.11. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be* $(\mathcal{C}, \mathcal{C}')$*-controlled continuous g-frames for* $\mathcal{H}$*. Let* $\mathcal{C}$, $\mathcal{C}'$ *and* $S_{\Phi\Lambda} + S_{\Lambda\Phi}$ *be commutative with each others and* $\mathcal{C}, \mathcal{C}', S_{\Phi\Lambda} + S_{\Lambda\Phi} \in GL^+(\mathcal{H})$*. Then*

$$\Lambda + \Phi = \{\Lambda_\omega + \Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\},$$

is a $(\mathcal{C}, \mathcal{C}')$*-controlled continuous g-frame for* $\mathcal{H}$*.*

Proof. Since Λ and Φ are g-Bessel families for $\mathcal{H}$ therefore $\Gamma = \Lambda + \Phi$ is a g-Bessel family for $\mathcal{H}$. By Lemma 2.8 $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}$ is positive and by proposition 2.4 in [4] there exists $m > 0$ such that $S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}} \geq mI$. Therefore

$$\begin{aligned}
\langle S_{\Gamma\mathcal{C}\mathcal{C}'} f, f\rangle &= \int_\Omega \left\langle \mathcal{C}'(\Lambda_\omega + \Phi_\omega)^*(\Lambda_\omega + \Phi_\omega)\mathcal{C} f, f\right\rangle d\mu_\omega \\
&= \langle S_{\Lambda\mathcal{C}\mathcal{C}'} f, f\rangle + \langle S_{\Phi\mathcal{C}\mathcal{C}'} f, f\rangle + \langle (S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}) f, f\rangle \\
&\geq A_{\Lambda\mathcal{C}\mathcal{C}'}\|f\|^2 + A_{\Phi\mathcal{C}\mathcal{C}'}\|f\|^2 + m\|f\|^2,
\end{aligned}$$

for all $f \in \mathcal{H}$. On the other hand,

$$\begin{aligned}
&\langle S_{\Lambda\mathcal{C}\mathcal{C}'} f, f\rangle + \langle S_{\Phi\mathcal{C}\mathcal{C}'} f, f\rangle + \langle (S_{\Lambda\mathcal{C}\Phi\mathcal{C}'} + S_{\Phi\mathcal{C}'\Lambda\mathcal{C}}) f, f\rangle \\
&\qquad \leq (B_{\Lambda\mathcal{C}\mathcal{C}'} + B_{\Phi\mathcal{C}\mathcal{C}'} + 2\sqrt{B_{\Lambda\mathcal{C}} B_{\Phi\mathcal{C}'}})\|f\|^2,
\end{aligned}$$

for all $f \in \mathcal{H}$.

□

Corollary 2.12. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Phi = \{\Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be continuous g-frames for* $\mathcal{H}$ *and* $S_{\Phi\Lambda} + S_{\Lambda\Phi}$ *be positive operator. Then*

$$\Lambda + \Phi = \{\Lambda_\omega + \Phi_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\},$$

is a continuous g-frame for $\mathcal{H}$*.*

In the following, we extend the concept of multiplier of continuous g-Bessel families and we define multiplier of $(\mathcal{C},\mathcal{C}')$-controlled continuous g-Bessel families in Hilbert spaces.

Proposition 2.13. *Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C},\mathcal{C})$ and $(\mathcal{C}',\mathcal{C}')$-controlled continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$, respectively. Let $m\in L^\infty(\Omega,\mu)$. The operator*

$$M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}:\mathcal{H}\to\mathcal{H},$$
$$\left\langle M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}f,g\right\rangle:=\int_\Omega m(\omega)\left\langle\mathcal{C}\Lambda_\omega^*\Phi_\omega\mathcal{C}'f,g\right\rangle d\mu_\omega,\quad f,g\in\mathcal{H},$$

is a well-defined bounded operator.

Proof. For any $f,g\in\mathcal{H}$ we have

$$\begin{aligned}\left|\left\langle M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}f,g\right\rangle\right| &= \left|\int_\Omega m(\omega)\left\langle\mathcal{C}\Lambda_\omega^*\Phi_\omega\mathcal{C}'f,g\right\rangle d\mu_\omega\right|\\ &\le \int_\Omega |m(\omega)||\Phi_\omega\mathcal{C}'f||\Lambda_\omega\mathcal{C}g|d\mu_\omega\\ &\le \|m\|_\infty\left(\int_\Omega\|\Phi_\omega\mathcal{C}'f\|^2d\mu_\omega\right)^{1/2}\left(\int_\Omega\|\Lambda_\omega\mathcal{C}g\|^2d\mu_\omega\right)^{1/2}\\ &\le \|m\|_\infty\sqrt{B_{\Lambda\mathcal{C}}B_{\Phi\mathcal{C}'}}\|f\|\|g\|.\end{aligned}$$

This shows that $\|M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}\|\le\|m\|_\infty\sqrt{B_{\Lambda\mathcal{C}}B_{\Phi\mathcal{C}'}}$ and so $M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}$ is well-defined and bounded. □

Definition 2.14. Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C},\mathcal{C})$ and $(\mathcal{C}',\mathcal{C}')$-controlled continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$, respectively. Let $m\in L^\infty(\Omega,\mu)$. The operator

$$M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}:\mathcal{H}\to\mathcal{H},$$
$$\left\langle M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}f,g\right\rangle:=\int_\Omega m(\omega)\left\langle\mathcal{C}\Lambda_\omega^*\Phi_\omega\mathcal{C}'f,g\right\rangle d\mu_\omega,\quad f,g\in\mathcal{H},$$

is called the $(\mathcal{C},\mathcal{C}')$-controlled continuous g-Bessel multiplier of Λ,Φ and m.

Proposition 2.15. *Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C},\mathcal{C})$ and $(\mathcal{C}',\mathcal{C}')$-controlled continuous g-frames, respectively. Then*

$$\mathcal{C}^{-1}M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}\mathcal{C}'^{-1}=M_{m,\Lambda,\Phi}.$$

Proof. By Proposition 2.5, Λ and Φ are continuous g-frames. We have

$$\begin{aligned}\left\langle M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}f,g\right\rangle &= \int_\Omega m(\omega)\left\langle \mathcal{C}\Lambda_\omega^*\Phi_\omega\mathcal{C}'f,g\right\rangle d\mu_\omega\\ &= \int_\Omega m(\omega)\left\langle \Lambda_\omega^*\Phi_\omega\mathcal{C}'f,\mathcal{C}g\right\rangle d\mu_\omega\\ &= \left\langle M_{m,\Lambda,\Phi}\mathcal{C}'f,\mathcal{C}g\right\rangle\\ &= \left\langle \mathcal{C}M_{m,\Lambda,\Phi}\mathcal{C}'f,g\right\rangle,\end{aligned}$$

for all $f,g\in\mathcal{H}$. So

$$\mathcal{C}M_{m,\Lambda,\Phi}\mathcal{C}' = M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}.$$

□

If $\mathcal{H}$ is a Hilbert space then $K(\mathcal{H})$, -the set of all compact operators in $\mathcal{H}$-, is a closed ideal of $B(\mathcal{H})$. We say $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ is the norm bounded if there is a constant $M>0$ such that $\|\Lambda_\omega\|\le M$ for every $\omega\in\Omega$. Let $m:\Omega\to\mathbb{C}$ be a bounded measurable function. We say m has support of a finite measure, if there exists a subset $K\subseteq\Omega$ with $\mu(K)<\infty$ such that $m(\omega)=0$ for almost every $\omega\in\Omega\backslash K$.

Theorem 2.16. *Let $dim(\mathcal{K}_\omega)<\infty$ for all $\omega\in\Omega$. Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be $(\mathcal{C},\mathcal{C})$ and $(\mathcal{C}',\mathcal{C}')$-controlled g-Bessel families, respectively. Let Λ or Φ be norm bounded families and $m:\Omega\to\mathbb{C}$ be a bounded measurable function with support of a finite measure. Then $M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}$ is a compact operator.*

Proof. By Proposition 2.15 we have $\mathcal{C}^{-1}M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}\mathcal{C}'^{-1}=M_{m,\Lambda,\Phi}$, therefore

$$M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}=\mathcal{C}M_{m,\Lambda,\Phi}\mathcal{C}'.$$

By Theorem 3.6 in [1], $M_{m,\Lambda,\Phi}\in K(\mathcal{H})$, thus $M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}$ is a compact operator. □

Theorem 2.17. *Let $M>0$ be such that $dim(\mathcal{K}_\omega)\le M$, for all $\omega\in\Omega$. Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ and $\Phi=\{\Phi_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be norm bounded $(\mathcal{C},\mathcal{C})$ and $(\mathcal{C}',\mathcal{C}')$-controlled g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ respectively, and $m\in L^\infty(\Omega,\mu)$. Then $M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}}$ is a Schatten p-class operator.*

Proof. By Theorem 3.10 in [1], $M_{m,\Lambda,\Phi}$ is a Schatten p-class operator. Since $S_p(\mathcal{H})$, -the set of Schatten p-class operators-, is a closed ideal of $B(\mathcal{H})$, $M_{m,\Lambda\mathcal{C},\Phi\mathcal{C}'}=\mathcal{C}M_{m,\Lambda,\Phi}\mathcal{C}'$ is a Schatten p-class operator. □

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