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Theory of Hybrid Fractional Differential Equations with Complex Order

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Abstract. We develop the theory of hybrid fractional differential equations with the complex order $\theta \in \mathbb{C}$, $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}$, in Caputo sense. Using Dhage's type fixed point theorem for the product of abstract nonlinear operators in Banach algebra; one of the operators is $\mathfrak{D}$- Lipschitzian and the other one is completely continuous, we prove the existence of mild solutions of initial value problems for hybrid fractional differential equations. Finally, an application to solve one-variable linear fractional Schrödinger equation with complex order is given.

1. Introduction

Fractional differential equations (FDEs) are employed in several fields, consisting of physics, mechanics, chemistry, engineering etc.. There has been an enormous improvement in ordinary differential equations concerning to fractional order derivative; see the monographs of Hilfer [11], Kilbas [12] and Podlubny [17]. Specifically, many works have been concerned to the initial value problems for nonlinear fractional differential equations, for instance, see [5, 13].

The topic of FDEs, which attracted a growing interest for some time, specially, on the subject of the complex order in fractional calculus, had been quickly developed in the latest years. E.R. Love [14] started the research on fractional derivatives of imaginary order. The idea is to complete the basic definitions of fractional integrals and derivatives by defining derivatives of purely imaginary orders. A use for a derivative of

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complex order in the fractional calculus was studied in [18]. A belief of fractional operator of complex order is proposed by Samko et al. [19]. On this course, numerous notions of fractional derivative of complex order were discussed [3, 4]. For instance, C.M.A. Pinto [16] introduced two approximations of the complex order Van der Pol oscillator. In [15], the authors investigated the existence of solutions of boundary value problems (BVPs) with complex order. Most recently, Vivek et al. studied the existence and stability results for pantograph equations [21] and integro-differential equations [20] with nonlocal conditions involving complex order.

Another attractive class of problems connects to hybrid fractional differential equations (HFDEs). For further study on this topic, one can refer to [1, 2, 10, 22]. The following hybrid differential of first order

$$\begin{cases} \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J := [0,T), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

was revised by Dhage et al. [8], under the assumptions $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$, and $g \in C(J \times \mathbb{R}, \mathbb{R})$. In [22], Zhao et al. looked after the fractional version of the problem (1.1), i.e.,

$$\begin{cases} D^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J, \ \alpha \in (0,1), \\ x(0) = 0, \end{cases} \tag{1.2}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, and a fixed point theorem in Banach algebras was the main implementation in this work.

Inspired by above works, we improve the concept of HFDEs with complex order. In this paper, we consider the following HFDE:

$$\begin{cases} D^{\theta}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J := [0,T], \\ x(0) = x_0 \end{cases} \tag{1.3}$$

where $D^{\theta}_{0^+}$ is the Caputo fractional derivative of order $\theta \in \mathbb{C}$, $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}$, and we consider $f \in C(J \times \mathbb{C}, \mathbb{C} - \{0\})$, and $g \in C(J \times \mathbb{C}, \mathbb{C})$. It is easy to see that the equation (1.3) is equivalent to the following integral system:

$$x(t) = f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} g(s,x(s))ds\right). \tag{1.4}$$

The paper is planned as follows. In Section 2, we recall some of the initial results required for the development of the paper. Section 3 includes existence results for the problem (1.3). Finally, in Section 4, we will study the existence of mild solution of an important equation in the complex space.

2. Fundamental Concepts

In this section, we give a few fixed point theorems used for mixed operator equations. We also offer a few definitions and properties of fractional calculus theory.

2.1. **Review on Dhage's Mixed Fixed Point Theorems.** In several areas of scientific disciplines, mathematical physics, mechanics and population dynamics, problems are modelled by mathematical equations which can be reduced to nonlinear equations of the form:

$$AxBx = x, \quad x \in S,$$

where S is closed, convex and bounded subset of a Banach algebra X, and A, B are two operators. A useful tool to deal with such issues is the celebrated fixed point theorem due to Dhage in 1988:

Theorem 2.1 ([7])**.** *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B : S \to X$ be two operators such that*

(a) *A is Lipschitzian with Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$x = AxBx$ for all $x \in S$.*

Then the operator equation $AxBx = x$ has a solution in S, whenever $\alpha M < 1$, where $M := \|B(S)\|$.

Most recently, Dhage has shown his attention on hypothesis (c) of above theorem and established the subsequent version under weaker conditions. Before stating the other fixed point theorem, we provide a helpful definition.

Definition 2.2. Let X be a Banach space. A mapping $T : X \to X$ is called $\mathfrak{D}$- Lipschitzian if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \phi\left(\|x - y\|\right),$$

for all $x, y \in X$, where $\phi(0) = 0$. If ϕ is not necessarily nondecreasing and satisfies $\phi(r) < r$, for $r > 0$, the mapping T is called a nonlinear contraction with a contraction function ϕ.

Dhage in [6] replaced the hypothesis (a) by $\mathfrak{D}$- Lipschitzian contraction condition for operator A and relaxed the hypothesis (c) with the following condition

$$x = AxBy \quad \Rightarrow \quad x \in S, \text{ for all } y \in S.$$

Theorem 2.3 ([6])**.** *Let S be a closed, convex and bounded subset of the Banach algebra X and let $A : X \to X$, $B : S \to X$ be two operators such that*

(a) *A is $\mathfrak{D}$-Lipschitzian with $\mathfrak{D}$-function ϕ,*
(b) *B is completely continuous, and*
(c) *$x = AxBy \Rightarrow x \in S$, for all $y \in S$.*

Then the operator equation $AxBx = x$ has a solution in S, whenever $M\phi(r) < r$ for $0 < r$, where $M := \|B(S)\|$.

The main fixed point theorem used in the rest of the paper is offered by Dhage in [6] as follows.

Theorem 2.4. *Let $\mathfrak{B}_r(0)$ and $\overline{\mathfrak{B}_r(0)}$ be open and closed balls in a Banach algebra X centered at a origin 0 of radius r, for some real number $r > 0$, and let $A, B : \overline{\mathfrak{B}_r(0)} \to X$ be two operators satisfying the following*

(a) *A is Lipschitz with constant γ;*
(b) *B is continuous and compact;*
(c) *$\gamma M < 1$, where*

$$M := \left\| B(\overline{\mathfrak{B}_r(0)}) \right\| = \sup \left\{ \|B(x)\| : \ x \in \overline{\mathfrak{B}_r(0)} \right\}.$$

Then, either

(i) *the equation $AxBx = x$ has a solution in $\overline{\mathfrak{B}_r(0)}$, or*
(ii) *there is an element $x \in X$ such that $\|x\| = r$ satisfying $\lambda AxBx = x$, for some $0 < \lambda < 1$.*

2.2. **Review on Fractional Calculus.** In this subsection, we present some definitions and theorems utilized in the sequel.

Definition 2.5 ([17])**.** The Riemann-Liouville fractional integral of order $\theta \in \mathbb{C}$, $(\mathrm{Re}(\theta) > 0)$ of a function f is

$$I^{\theta}_{0^+} f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s) ds.$$

Definition 2.6 ([17])**.** For a function f given on a interval J, the Caputo fractional-order $\theta \in \mathbb{C}$, $(\mathrm{Re}(\theta) > 0)$ of f, is defined by

$$(D^{\theta}_{0^+} f)(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^{(n)}(s) ds,$$

where $n = [\mathrm{Re}(\theta)] + 1$ and $[\mathrm{Re}(\theta)]$ denotes the integer part of the real number $\mathrm{Re}(\theta)$.

Definition 2.7 ([12])**.** The Stirling asymptotic formula of the Gamma function for $z \in \mathbb{C}$ is as follows:

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{z-1}{2}} e^{-z} \left[1 + O\left(\frac{1}{z}\right) \right], \quad (|\arg(z)| < \pi; |z| \to \infty), \tag{2.1}$$

and its result for $|\Gamma(u + iv)|$, $(u, v \in \mathbb{R})$ is

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}} |v|^{u-\frac{1}{2}} e^{-u-\pi|v|/2} \left[1 + O\left(\frac{1}{v}\right) \right], \quad (v \to \infty). \tag{2.2}$$

3. Main Results

In this section, our intention is to examine the existence of mild solutions to the problem (1.3).

Lemma 3.1. *Any function satisfying* (1.3) *with* $g \in L^1(J, \mathbb{C})$ *will also satisfy the integral equation* (1.4). *Moreover, if the function* $x \to \frac{x}{f(0,x)}$ *is injective, and* $I^\theta g(t, x(t))$ *is an absolutely continuous function, then the contrary is true.*

Proof. Assume that $x(t)$ satisfies (1.3). Then, $\left(\frac{x(t)}{f(t,x(t))}\right)$ is absolutely continuous, then we get that $D^\theta_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)$ exists and is Lebesgue integrable on J. Applying the fractional integration I^θ to both sides of (1.3) we get (1.4). Contrary is also true. The proof is corresponding to Lemma 7 in [10]. Hence, we skip the proof. □

Definition 3.2. (1) The function $x \in C(J, \mathbb{C})$ is called a mild solution of the HFDEs with complex order (1.3) if it satisfies the integral equation (1.4).

(2) The function $x \in Ac(J, \mathbb{C})$, the space of absolutely continuous complex-valued functions defined on J, is called a strong solution of (1.3) if
   (a) the function $t \to \left(\frac{x}{f(t,x)}\right)$ is absolutely continuous for each $x \in \mathbb{C}$, and
   (b) x satisfies (1.3).

Assume the following conditions:

(C1) There exists a constant $\gamma \geq 0$ such that $|f(t,x) - f(t,y)| \leq \gamma |x - y|$ for all $t \in J$ and $x, y \in \mathbb{C}$.

(C2) There exists a function $h \in L^1(J, \mathbb{C})$ such that $|g(t,x)| \leq h(t)$, for all $t \in J$ and $x \in \mathbb{C}$.

(C3) $\gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right) < 1$.

(C4) There exists $r > 0$ such that

$$r > \frac{F_0\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)}{1 - \gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)},$$

where $F_0 = \sup_{t \in J}|f(t,0)|$.

Along the paper, we take $\mathfrak{B}_r(0)$ to be an open ball centered at the origin and of radius $r > 0$ in the Banach algebra $X = C(J, \mathbb{C})$ (the Banach space of continuous complex-valued functions defined on the interval J equipped with the sup-norm, $\|x\| = \sup_{s \in J}|x(s)|$, and with multiplication property defined by $(xy)(s) = x(s)y(s)$, for $s \in J$). It is easy to see that there is a mild solution to the problem (1.3), which is equivalent to the operator equation (1.4)

$$Ax(t)Bx(t) = x(t), \quad t \in J,$$

where $A, B : \overline{\mathfrak{B}_r(0)} \to X$ are defined by

$$\begin{aligned} Ax(t) &= f(t, x(t)), \\ Bx(t) &= \frac{x_0}{f(0, x_0)} + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} g(s, x(s)) ds. \end{aligned}$$

Lemma 3.3. *The operator A is Lipschitz on X.*

Proof. Let $x, y \in X$ and $t \in J$; then by (C1) we get

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(t)) - f(t, y(t))| \\ &\leq \gamma \, |x(t) - y(t)| \\ &\leq \gamma \, \|x - y\| \, . \end{aligned}$$

Taking the supremum over $t \in J$, we get that A is Lipschitz on X with Lipschitz constant γ. □

Lemma 3.4. *The operator B is continuous operator on $\overline{\mathfrak{B}_r(0)}$.*

Proof. Let $\{x_n\}$ be a convergent sequence in $\overline{\mathfrak{B}_r(0)}$ converging to $x \in \overline{\mathfrak{B}_r(0)}$. Then, by the Lebesgue dominated converging theorem,

$$\begin{aligned} \lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \left(\frac{x_0}{f(0, x_0)} + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} g(s, x_n(s)) ds \right) \\ &= \frac{x_0}{f(0, x_0)} + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \lim_{n\to\infty} g(s, x_n(s)) ds \\ &= \frac{x_0}{f(0, x_0)} + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} g(s, x(s)) ds \\ &= Bx(t), \end{aligned}$$

for all $t \in J$, we can conclude the continuity of the operator B. □

Lemma 3.5. *The operator B is a compact operator on $\overline{\mathfrak{B}_r(0)}$.*

Proof. Let x be arbitrary in $\overline{\mathfrak{B}_r(0)}$. For any complex number $\theta \in \mathbb{C}$, $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}$, we have the following relation

$$\begin{aligned} |(t-s)^{\theta-1}| &= |e^{(\theta-1)\ln(t-s)}| \\ &= |e^{((m-1)+i\alpha)\ln(t-s)}| \\ &= |e^{(m-1)\ln(t-s)} e^{i\alpha\ln(t-s)}| \\ &= e^{(m-1)\ln(t-s)} |e^{i\alpha\ln(t-s)}| \\ &= (t-s)^{m-1} |\cos(\alpha\ln(t-s)) + i\sin(\alpha\ln(t-s))| \end{aligned}$$

$$= (t-s)^{m-1}.$$

Now, by the condition (C2) and by means of Young's convolution inequality we get

$$\begin{aligned}
|Bx(t)| &\leq \left|\frac{x_0}{f(0,x_0)}\right| + \left|\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} g(s,x(s))ds\right| \\
&\leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right| |g(s,x(s))|\, ds \\
&\leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{1}{|\Gamma(\theta)|}\int_0^t (t-s)^{m-1} |g(s,x(s))|\, ds \\
&\leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{1}{|\Gamma(\theta)|}\int_0^t (t-s)^{m-1} |h(s)|\, ds \\
&\leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\,|\Gamma(\theta)|} \|h\|_{L^1},
\end{aligned}$$

which by taking the supremum over t gives

$$\|Bx\| \leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\,|\Gamma(\theta)|} \|h\|_{L^1}, \quad \forall\, x \in \overline{\mathfrak{B}_r(0)},$$

which verifies that $\overline{\mathfrak{B}_r(0)}$ is a uniformly bounded set in X.

Now, we verify that $\overline{\mathfrak{B}_r(0)}$ is an equicontinuous set in X. For $0 \leq t_1 \leq t_2 \leq T$ we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left|\frac{1}{\Gamma(\theta)}\int_0^{t_1} (t_1-s)^{\theta-1} g(s,x(s))ds - \frac{1}{\Gamma(\theta)}\int_0^{t_2} (t_2-s)^{\theta-1} g(s,x(s))ds\right| \\
&\leq \frac{\|h\|_{L^1}}{|\Gamma(\theta)|}\left|\int_0^{t_1} \left[(t_1-s)^{\theta-1} - (t_2-s)^{\theta-1}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{\theta-1} ds\right|.
\end{aligned}$$

Thus, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |Bx(t_1) - Bx(t_2)| < \epsilon, \quad t_1, t_2 \in J, \quad x \in \overline{\mathfrak{B}_r(0)}.$$

This confirms that $\overline{\mathfrak{B}_r(0)}$ is an equicontinuous set in X. Due to Arzela-Ascoli Theorem, we obtain that the operator B is a compact operator. □

Now, we are going to verify the main result of this section.

Theorem 3.6. *Under the above conditions (C1)-(C4), the equation* (1.3) *has a mild solution on* J.

Proof. Let $A, B : \overline{\mathfrak{B}_r(0)} \to X$ be the operators defined in the beginning of this section. We know that A is Lipschitz, and B is continuous and compact (see Lemmas 3.3 - 3.5). Set

$$M := \left\|B\left(\overline{\mathfrak{B}_r(0)}\right)\right\|$$

$$= \sup\left\{\|B(x)\| : x \in \overline{\mathfrak{B}_r(0)}\right\}.$$

By using Lemma 3.5, we get

$$M \leq \left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\left|\Gamma(\theta)\right|}\|h\|_{L^1}.$$

From the condition (C3), we deduce that

$$\gamma M \leq \gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\left|\Gamma(\theta)\right|}\|h\|_{L^1}\right) < 1.$$

It remains to confirm that the conclusion (ii) of Theorem 2.4 is not possible. Let $x \in X$ and $\lambda \in (0,1)$ be such that $\|x\| = r$ and $x = \lambda AxBx$. It follows that

$$\begin{aligned}
|x(t)| &= \lambda\left|f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} g(s,x(s))ds\right)\right| \\
&\leq \lambda\left|f(t,x(t))\right|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right|\left|g(s,x(s))\right| ds\right) \\
&\leq \lambda\left|f(t,x(t)) - f(t,0) + f(t,0)\right| \\
&\quad\times\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{1}{|\Gamma(\theta)|}\int_0^t (t-s)^{m-1}\left|g(s,x(s))\right| ds\right) \\
&\leq \lambda\left(\gamma\left|x(t)\right| + F_0\right)\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\left|\Gamma(\theta)\right|}\|h\|_{L^1}\right) \\
&\leq \frac{F_0\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)}{1-\lambda\gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)}.
\end{aligned}$$

Taking supremum over t and using (C4) and $0 < \lambda < 1$ we get

$$\begin{aligned}
\|x\| &\leq \frac{F_0\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)}{1-\lambda\gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)} \\
&< \frac{F_0\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)}{1-\gamma\left(\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m|\Gamma(\theta)|}\|h\|_{L^1}\right)} \\
&< r
\end{aligned}$$

which contradicts $\|x\| = r$; thus the conclusion (ii) of Theorem 2.4 is impossible; hence the operator $AxBx = x$ has a solution in $\overline{\mathfrak{B}_r(0)}$. Accordingly, problem (1.3) has a mild solution on J which completes the proof of our problem. □

4. Application in an Important Equation

Consider the following equation

$$\begin{cases} D^{\beta}u(t) = wAu(t), \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $\beta \in \mathbb{C}$ $(0 < \mathrm{Re}(\beta) \leq 1)$, w and u_0 are constants belonging to $\overline{\mathfrak{B}_1(0)}$, and A is a bounded linear operator on $X := C([0,1], \overline{\mathfrak{B}_1(0)})$. This equation is a special case of (1.3) with $\theta := \beta$, $f(t,s) := 1$, $g(t,s) := wAs$. Obviously, the assumptions (C1)-(C3) are satisfied using $\gamma := 0$ and $h(t) := \|A\|$. Now, we are going to ensure the condition (C4). We know that $F_0 := \sup_{[0,1]} |f(t,0)| = 1$ and

$$\left|\frac{x_0}{f(0,x_0)}\right| + \frac{T^m}{m\,|\Gamma(\theta)|}\,\|h\|_{L^1} \leq |x_0| + \frac{\|A\|}{\mathrm{Re}(\beta)\,|\Gamma(\beta)|}.$$

Clearly, there exists $r > 0$ such that $r > |x_0| + \frac{\|A\|}{\mathrm{Re}(\beta)|\Gamma(\beta)|}$. These show that the equation (4.1) has a mild solution on $[0,1]$. In practical view, in equation (4.1), if we replace β by a positive real number, D^{β} by $\frac{\partial^{\beta}}{\partial t^{\beta}}$ and w by $(-i)^{\beta}$, the following one-dimensional fractional Schrödinger equation is obtained:

$$\begin{cases} \frac{\partial^{\beta}u(t)}{\partial t^{\beta}} = (-i)^{\beta}Au(t), \quad (\beta > 0), \\ u(0) = u_0, \end{cases} \tag{4.2}$$

in which $(-1)^{\beta} = e^{\frac{-i\beta\pi}{2}}$. The Schrödinger equation is a basic mathematical equation in quantum mechanics, that describes the changes over time of a physical system in which quantum effects, such as wave-particle duality, are significant. Gorka et al. [9] proved existence and uniqueness of equation (4.2), where A is a positive self-adjoint operator on a Hilbert space H. They also introduced an example of equation (4.1) with the operator $A := -\Delta$, the Laplace operator, on $L^2(\mathbb{R})$. A special case of equation (4.1) can be presented as

$$\begin{cases} D^{\beta}u(t) = wu(t), \\ u(0) = u_0, \end{cases} \tag{4.3}$$

where $0 < \beta \leq 1$ and $w, u_0 \in \overline{\mathfrak{B}_1(0)}$. After taking Laplace transformation of equation (4.3), we can determined the solution $u(t)$ in terms of the Mittag-Leffler function in the form

$$\overline{u}(s) = \frac{u_0}{(s^{\beta} - w)}, \tag{4.4}$$

where $\overline{u}(s)$ is the Laplace transform of u. Recall that the Laplace transform of Caputo's fractional derivative gives an interesting formula as

$$\mathfrak{L}\left[D_{0^+}^{\alpha}u(t)\right] = s^{\alpha}\overline{u}(s) - \sum_{k=0}^{n-1} u^{(k)}(0)s^{\alpha-k-1}.$$

By calculating the inverse Laplace transform of both sides of (4.4), the unique solution of equation (4.3) is given by

$$u(t) = u_0 E_{\beta}(wt^{\beta}),$$

where $E_{\beta}(z)$ is the Mittag-Leffler function, that is

$$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}, \quad (\beta > 0,\ z \in \mathbb{C}).$$

Moreover, the uniqueness of the solution of equation (4.3) follows by the uniqueness theorem for the Laplace transform.

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